TSUKUBA J. MATH.
Vol. 32 No. 1 (2008), 179–196

ON p-MAPS AND M-MAPS

By

Yun-Feng Bai and Takuo Miwa

Abstract. We introduce new notions of p-maps and M-maps, and investigate some of their basic properties, which are extensions of corresponding properties of p-spaces and M-spaces.

1. Introduction

In this paper, we introduce new notions of p-maps and M-maps, and in sections 3 and 4 investigate some basic properties of these maps and their relationships with Čech-complete maps ([2]) and k-maps ([10], [2]). p-Maps and M-maps are respectively extensions of p-spaces ([1]) and M-spaces ([11], [12]) to the notions of continuous maps. Further, in section 5 we investigate these maps in the realm of paracompact maps ([4]) and in section 6 their relations with metrizable type (MT-)maps ([6]) is studied.

This branch of General Topology is now known as General Topology of Continuous Maps or Fibrewise General Topology. For an arbitrary topological space B one considers the category TOP_B, the objects of which are continuous maps into the space B, and for the objects $f : X \to B$ and $g : Y \to B$, a *morphism* from f into g is a continuous map $\lambda : X \to Y$ with the property $f = g \circ \lambda$. This is denoted by $\lambda : f \to g$. A morphism $\lambda : f \to g$ is said to be onto, closed, perfect, quasi-perfect, if respectively, such is the map $\lambda : X \to Y$. An object $f : X \to B$ of TOP_B is called a *projection*, and X or (X, f) is called a *fibrewise space*. We also call a morphism $\lambda : f \to g$ a *fibrewise map* when we write $\lambda : (X, f) \to (Y, g)$ or $\lambda : X \to Y$.

We note that the fibrewise category TOP_B is a generalization of the topological category TOP (of topological spaces and continuous maps as mor-

2000 *Mathematics Subject Classification*. Primary 55R70, 54C10, Secondary 54E18.
Key words and phrases. Fibrewise topology, p-map, M-map, Čech-complete map, k-map, compactification of map, T_2-compactifiable map.
Received July 12, 2007.
Revised December 13, 2007.

phisms), since the category TOP is isomorphic to the particular case of TOP_B in which the space B is a singleton set.

Throughout this paper, we assume that all spaces are topological spaces, and all maps and projections are continuous. For other terminology and notations undefined in this paper, one can consult [7] about TOP, and [10] and [4], [5], [6] about TOP_B.

2. Preliminaries

In this section, we refer to the notions and notations in Fibrewise Topology, which are used in latter sections.

Let (B, τ) be a fixed topological space B with a fixed topology τ. Throughout the paper we will use the abbreviation *nbd(s)* for *neighborhood(s)*. We denote the set of all open nbds of $b \in B$ by $N(b)$ and the set of all natural numbers by $\mathbf{N}$. Note that regularity of (B, τ) is assumed in Proposition 2.12, Theorems 3.2, 3.4(2), 3.5, 3.7, 5.2 and 6.1, Corollaries 3.3, 6.2 and 6.3, and Lemma 5.4. Further, in Theorem 3.8 it is assumed that B is regular and B satisfies the first axiom of countability.

For a projection $f : X \to B$ and each point $b \in B$, the *fibre* over b is the subset $X_b = f^{-1}(b)$ of X. Also for each subset B' of B we regard $X_{B'} = f^{-1}(B')$ as a fibrewise space over B' with the projection determined by f. For a filter (base) $\mathscr{F}$ in X, we denote by $f_*(\mathscr{F})$ the filter generated by the set $\{f(F) \mid F \in \mathscr{F}\}$. For a fibrewise map $\lambda : (X, f) \to (Y, g)$ and a filter (base) $\mathscr{F}$ in X, we define $\lambda_*(\mathscr{F})$ in the same manner. For a filter (base) $\mathscr{G}$ in Y, we define $\lambda^*(\mathscr{G})$ as the filter generated by the set $\{\lambda^{-1}(U) \mid U \in \mathscr{G}\}$.

We begin by defining some separation axioms on maps.

Definition 2.1. A projection $f : X \to B$ is called a *T_i-map*, $i = 0, 1, 2$ (T_2 is also called *Hausdorff*), if for all $x, x' \in X$ such that $x \neq x'$ and $f(x) = f(x')$, the following condition is respectively satisfied:

(1) $i = 0$: at least one of the points x, x' has a nbd in X not containing the other point;

(2) $i = 1$: each of the points x, x' has a nbd in X not containing the other point;

(3) $i = 2$: the points x and x' have disjoint nbds in X.

Definition 2.2. (1) A T_0-map $f : X \to B$ is called *regular* if for every point $x \in X$ and every closed set F in X such that $x \notin F$, there exists a nbd $W \in N(f(x))$ such that the set $\{x\}$ and $F \cap X_W$ have disjoint nbds in X_W.

(2) A T_1-map $f : X \to B$ is called *normal* (resp. *collectionwise normal*) if for every $O \in \tau$, every closed (in X_O) disjoint sets $\{F_1, F_2\}$ (resp. closed discrete (in X_O) collection $\{F_s \mid s \in S\}$) and every $b \in O$, there exists $W \in N(b)$, $W \subset O$ such that $\{F_1 \cap X_W, F_2 \cap X_W\}$ (resp. $\{F_s \cap X_W \mid s \in S\}$) have disjoint nbds (resp. discrete pairwise disjoint nbds) in X_W.

We now give the definitions of submap, compact map [16] and locally compact map [14].

Definition 2.3. (1) The restriction of the projection $f : X \to B$ on a closed (resp. open, type G_δ, etc.) subset of the space X is called a *closed* (resp. *open, type* G_δ, etc.) *submap* of the map f.

(2) A projection $f : X \to B$ is called a *compact map* if it is perfect (i.e. it is closed and all its fibres $f^{-1}(b)$ are compact). Note that in [10], Definition 3.1, the space X is called *fibrewise compact over B*.

(3) A projection $f : X \to B$ is said to be a *locally compact map* if for each $x \in X_b$, where $b \in B$, there exists a nbd $W \in N(b)$ and a nbd $U \subset X_W$ of x such that $g : X_W \cap \bar{U} \to W$ is a compact map, where g is the restriction of f on $X_W \cap \bar{U}$.

Note that a closed submap of a (resp. locally) compact map is (resp. locally) compact, and for a (resp. locally) compact map $f : X \to B$ and every $B' \subset B$ the restriction $f \mid X_{B'} : X_{B'} \to B'$ is (resp. locally) compact.

Definition 2.4. (1) For a map $f : X \to B$, a map $c(f) : c_f X \to B$ is called a *compactification* of f if $c(f)$ is compact, X is dense in $c_f X$ and $c(f) \mid X = f$.

(2) A map $f : X \to B$ is called a T_2-*compactifiable map* if f has a compactification $c(f) : c_f X \to B$ and $c(f)$ is a T_2-map.

The following holds.

Proposition 2.5. (1) *For* $i = 0, 1, 2$, *every submap of a* T_i-*map is also a* T_i-*map. Every submap of a regular map is also regular.*

(2) *Compact* T_2-*map* $\Rightarrow$ *normal map* $\Rightarrow$ *regular map* $\Rightarrow$ T_2-*map.*

(3) ([10] *Section* 8) *Every normal map is a* T_2-*compactifiable map.*

(4) ([10] *Section* 8) *Every locally compact* T_2-*map is a* T_2-*compactifiable map.*

Definition 2.6. For the collection of fibrewise spaces $\{(X_\alpha, f_\alpha) \mid \alpha \in \Lambda\}$, the subspace $X = \{t = \{t_\alpha\} \in \prod\{X_\alpha : \alpha \in \Lambda\} : f_\alpha t_\alpha = f_\beta t_\beta \ \forall \alpha, \beta \in \Lambda\}$ of the Tychonoff product $\prod = \prod\{X_\alpha : \alpha \in \Lambda\}$ is called the *fan product* of the spaces X_α with respect to the maps f_α, $\alpha \in \Lambda$.

For the projection $pr_\alpha : \prod \to X_\alpha$ of the product $\prod$ onto the factor X_α, the restriction π_α on X will be called the projection of the fan product onto the factor X_α, $\alpha \in \Lambda$. From the definition of fan product we have that, $f_\alpha \circ \pi_\alpha = f_\beta \circ \pi_\beta$ for every $\alpha, \beta \in \Lambda$. Thus one can define a map $f : X \to B$, called the *product* of the maps f_α, $\alpha \in \Lambda$, by $f = f_\alpha \circ \pi_\alpha$, $\alpha \in \Lambda$. The fibrewise space (X, f) is called the *fibrewise product space* of $\{(X_\alpha, f_\alpha) \mid \alpha \in \Lambda\}$.

Obviously, the projections f and π_α, $\alpha \in \Lambda$, are continuous.

The following proposition holds.

PROPOSITION 2.7. *Let $\{(X_\alpha, f_\alpha) \mid \alpha \in \Lambda\}$ be a collection of fibrewise spaces.*

(1) *If each f_α is T_i $(i = 0, 1, 2)$, then the product f is also T_i $(i = 0, 1, 2)$.*

(2) *If each f_α is a surjective regular map, then the product f is also a regular map.*

(3) ([10] *Prop.* 3.5) *If each f_α is a compact map, then the product f is also a compact map.*

(4) *If each f_α is a T_2-compactifiable map, then the product f is also T_2-compactifiable.*

We shall conclude this section by defining the concept of paracompact map ([4], [5]), metrizable type (MT-)map ([6]), Čech-complete map ([2]), k-map ([10], [2]) and b-filters (or tied filters) ([10]).

DEFINITION 2.8. (1) A map $f : X \to B$ is said to be *paracompact* if for every point $b \in B$ and every open (in X) cover $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathcal{A}\}$ of the fibre X_b (i.e. $X_b \subset \bigcup\{U_\alpha \mid \alpha \in \mathcal{A}\}$), there exist $W \in N(b)$ and an open (in X) cover $\mathcal{V}$ of X_W such that X_W is covered by $\mathcal{U}$ and $\mathcal{V}$ is a locally finite (in X_W) refinement of $\{X_W\} \wedge \mathcal{U}$.

(2) For a map $f : X \to B$ and $b \in B$, let $\mathcal{U}$ be an open (in X) cover of X_b. The family $\mathcal{V}$ of subsets of X is said to be a *b-star refinement* of $\mathcal{U}$ if $V \cap X_b \neq \emptyset$ for every $V \in \mathcal{V}$, $X_b \subset \bigcup \mathcal{V}$ and there exists $W \in N(b)$ such that $\mathcal{U}$ covers X_W and $\{st(V, \mathcal{V}) \mid V \in \mathcal{V}\} < \mathcal{U} \wedge \{X_W\}$.

DEFINITION 2.9. (1) Let $f : X \to B$ be a map. The sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of open (in X) covers of X_b, $b \in B$, is said to be a *b-development* if for every $x \in X_b$ and every nbd $U(x)$ of x in X, there exist $i \in \mathbf{N}$ and $W \in N(b)$ such that

$x \in \mathrm{St}(x, \mathscr{W}_i \wedge \{X_W\}) \subset U(x)$. The map f is said to have an *f-development* if it has a b-development for every $b \in B$.

(2) A closed map $f : X \to B$ is said to be a *metrizable type* (*MT-*)*map* if it is collectionwise normal and has an f-development.

The following proposition was obtained in [6] and [4].

Proposition 2.10. *The following implications hold in TOP_B.*
$MT \Rightarrow$ paracompact $T_2 \Rightarrow$ collectionwise normal $\Rightarrow$ normal.

Definition 2.11. (1) Let X be a topological space, and A a subset of X. We say that the *diameter of A is less than a family $\mathscr{A} = \{A_s\}_{s \in S}$* of subsets of the space X, and we shall write $\delta(A) < \mathscr{A}$, provided that there exists an $s \in S$ such that $A \subset A_s$.

(2) ([10] Section 4.) For a fibrewise space (X, f), by a *b-filter* (or *tied filter*) on X we mean a pair $(b, \mathscr{F})$, where $b \in B$ and $\mathscr{F}$ is a filter on X such that b is a limit point of the filter $f_*(\mathscr{F})$ on B. By an *adherence point* of a b-filter $\mathscr{F}$ $(b \in B)$ on X, we mean a point of the fibre X_b which is an adherence point of $\mathscr{F}$ as a filter on X.

(3) ([2]) A T_2-compactifiable map $f : X \to B$ is said to be *Čech-complete* if for each $b \in B$, there exists a countable family $\{\mathscr{A}_n\}_{n \in \mathbf{N}}$ of open (in X) covers of X_b with the property that every b-filter $\mathscr{F}$ which contains sets of diameter less than $\mathscr{A}_n$ for every $n \in \mathbf{N}$ has an adherence point.

The following result for Čech-complete maps is proved in [2] Theorem 5.1.

Proposition 2.12. *Suppose that B is regular. For a T_2-compactifiable map $f : X \to B$, the following are equivalent*:

(1) *f is Čech-complete.*

(2) *For every T_2-compactification $f' : X' \to B$ of f and each $b \in B$, X_b is a G_δ-subset of X'_b.*

(3) *There exists a T_2-compactification $f' : X' \to B$ of f such that X_b is a G_δ-subset of X'_b for each $b \in B$.*

Finally we give the definition of k-map, see [10] Section 10 and [2] Section 6.

Definition 2.13. (1) Let $f : X \to B$ be a map. A subset H of X is said to be *quasi-open* (resp. *quasi-closed*) if the following condition is satisfied: for each

$b \in B$ and $V \in N(b)$ there exists a nbd $W \in N(b)$ with $W \subset V$ such that whenever $f|K : K \to W$ is compact, the subset $H \cap K$ is open (resp. closed) in K.

(2) Let $f : X \to B$ be a T_2-map. The map f is said to be a *k-map* if every quasi-closed subset of X is closed in X or, equivalently, if every quasi-open subset of X is open in X. (Note that in [10] X is said to be a *fibrewise compactly generated space* over B.)

3. Definition and Basic Properties of p-maps

In this section, we define a p-map and investigate some of its basic properties. The concept of p-maps is a generalization of p-spaces ([1]).

Definition 3.1. A T_2-compactifiable map $f : X \to B$ is a *p-map* if for every $b \in B$, there exists a sequence $\{\mathscr{U}_n\}_{n\in\mathbf{N}}$ of open (in X) covers of X_b with the following properties: if $x \in X_b$ and $x \in U_n \in \mathscr{U}_n$ for every $n \in \mathbf{N}$, then

(P1) $(\bigcap_{n\in\mathbf{N}} \overline{U_n}) \cap X_b$ is compact.

(P2) For every open (in X) set U with $(\bigcap_{n\in\mathbf{N}} \overline{U_n}) \cap X_b \subset U$, there exist $n_0 \in \mathbf{N}$ and $W \in N(b)$ such that $(\bigcap_{n\in\mathbf{N}} \overline{U_n}) \cap X_b \subset (\bigcap_{i\le n_0} \overline{U_i}) \cap X_W \subset U$.

For a p-map $f : X \to B$, we can characterize it by using a compactification of f as follows.

Theorem 3.2. *Suppose that B is regular. A map $f : X \to B$ is a p-map if and only if there is a T_2-compactification $f' : X' \to B$ of f such that for every $b \in B$ there is a sequence $\{\mathscr{P}_n\}_{n\in\mathbf{N}}$ of open families of X' satisfying the following conditions*:

(1) *For every $n \in \mathbf{N}$, $X_b \subset \bigcup \mathscr{P}_n$,*

(2) *For every $x \in X_b$, $\bigcap_{n\in\mathbf{N}} st(x, \mathscr{P}_n) \cap X'_b \subset X_b$.*

Proof. ["Only If" part]: If $f : X \to B$ is a p-map, there exists a sequence $\{\mathscr{U}_n\}_{n\in\mathbf{N}}$ of open (in X) covers of X_b satisfying Definition 3.1. Let $f' : X' \to B$ be a T_2-compactification of f. For every $n \in \mathbf{N}$, take a family $\mathscr{P}_n$ of open subsets of X' such that $\mathscr{P}_n \wedge \{X\} = \mathscr{U}_n$, then $X_b \subset \bigcup \mathscr{P}_n$ for every $n \in \mathbf{N}$. We shall prove that (2) holds. If not, there is $x \in X_b$ and $y \in X'_b \setminus X_b$ such that $\{x, y\} \subset P_n \in \mathscr{P}_n$ for every $n \in \mathbf{N}$. By Definition 3.1, $F = (\bigcap_{n\in\mathbf{N}} \overline{P_n \cap X}^X) \cap X_b$ is compact and since $y \notin F$, there is an open subset G of X' such that $F \subset G \subset \bar{G}^{X'} \subset X' \setminus \{y\}$, because f' is compact and B is regular. Thus there exist $n_0 \in \mathbf{N}$ and $W \in N(b)$ such that $F \subset (\bigcap_{i\le n_0} \overline{P_i \cap X}^X) \cap X_W \subset G$. Let $V = (\bigcap_{i\le n_0} P_i) \cap (X' \setminus \bar{G}^{X'}) \cap X'_W$, then $V \in N(y)$ and $V \cap X = \varnothing$ which contradicts $X' = \overline{X}$.

["If" part]: Let $f' : X' \to B$ be a T_2-compactification of f such that for every $b \in B$, there is a sequence $\{\mathscr{P}_n\}_{n\in\mathbf{N}}$ of open families of X' satisfying (1) and (2). For every $n \in \mathbf{N}$ let $\mathscr{U}_n = \{U : U$ is open in $X, U \cap X_b \neq \varnothing$ and $\overline{U}^{X'} \subset P$ for some $P \in \mathscr{P}_n\}$, then $\{\mathscr{U}_n\}_{n\in\mathbf{N}}$ is a sequence of open (in X) covers of X_b. We shall now show that if $x \in X_b$ and $x \in U_n \in \mathscr{U}_n$ for every $n \in \mathbf{N}$, then conditions (P1) and (P2) of Definition 3.1 hold.

(P1): For every $n \in \mathbf{N}$ there is $P_n \in \mathscr{P}_n$ such that $\overline{U_n}^{X'} \subset P_n$. Thus $(\bigcap_{n\in\mathbf{N}} \overline{U}_n^X) \cap X_b \subset (\bigcap_{n\in\mathbf{N}} \overline{U}_n^{X'}) \cap X'_b = (\bigcap_{n\in\mathbf{N}} \overline{U}_n^X) \cap X_b$ because from (2), $(\bigcap_{n\in\mathbf{N}} \overline{U}_n^{X'}) \cap X'_b \subset (\bigcap_{n\in\mathbf{N}} st(x, \mathscr{P}_n)) \cap X'_b \subset X_b$. Consequently, $(\bigcap_{n\in\mathbf{N}} \overline{U}_n^X) \cap X_b$ is compact.

(P2): For every open subset U in X with $(\bigcap_{n\in\mathbf{N}} \overline{U}_n^X) \cap X_b \subset U$, take an open subset G of X' such that $U = X \cap G$. Since X'_b is compact and $\{G\} \cup \{X' \backslash \overline{U}_n^{X'} \mid n \in \mathbf{N}\}$ is an open cover of X'_b, there is $n_0 \in \mathbf{N}$ such that $X'_b \subset \bigcup_{i \le n_0} (X' \backslash \overline{U}_i^{X'}) \cup G$. Since f' is closed, there is $W \in N(b)$ such that $X'_b \subset X'_W \subset \bigcup_{i \le n_0} (X' \backslash \overline{U}_i^{X'}) \cup G$ and therefore, $(\bigcap_{n\in\mathbf{N}} \overline{U}_n^X) \cap X_b \subset (\bigcap_{i \le n_0} \overline{U}_i^X) \cap X_W \subset U$. □

Since a locally compact T_2-map $f : X \to B$ has an Alexandorff-type compactification $f' : X' \to B$ (Proposition 2.5(4)), and therefore X is open in X', we have the following.

Corollary 3.3. *If B is regular, then a locally compact T_2-map is a p-map.*

For submaps of p-maps, we have the following.

Theorem 3.4. *For a p-map $f : X \to B$, we have:*

(1) *If F is a closed subset of X, then the submap $f|F$ is a p-map.*

(2) *Suppose that B is regular. If G is a G_δ-subset of X, then the submap $f|G$ is a p-map.*

Proof. (1) Since $f : X \to B$ is a p-map, for every $b \in B$ there exists a sequence $\{\mathscr{U}_n\}_{n\in\mathbf{N}}$ of open (in X) covers of X_b satisfying (P1) and (P2) of Definition 3.1.

For every $n \in \mathbf{N}$, let $\mathscr{G}_n = \{F \cap U : U \in \mathscr{U}_n\}$, then $\{\mathscr{G}_n\}_{n\in\mathbf{N}}$ is a sequence of open covers of F_b in F. If $x \in F_b$ and $x \in G_n \in \mathscr{G}_n$ for every $n \in \mathbf{N}$, then there is an element $U_n \in \mathscr{U}_n$ with $x \in G_n = U_n \cap F \subset U_n$ for every $n \in \mathbf{N}$.

(1′) $(\bigcap_{n\in\mathbf{N}} \bar{G}_n^F) \cap F_b = (\bigcap_{n\in\mathbf{N}} \bar{G}_n^X) \cap X_b \subset (\bigcap_{n\in\mathbf{N}} \overline{U}_n^X) \cap X_b$, i.e. $(\bigcap_{n\in\mathbf{N}} \bar{G}_n^F) \cap F_b$ is closed in $(\bigcap_{n\in\mathbf{N}} \overline{U}_n^X) \cap X_b$, so that it is compact.

(2′) For every open subset G in F with $(\bigcap_{n\in\mathbf{N}} \bar{G}_n^F) \cap F_b \subset G$, take an open subset U in X with $G = U \cap F$. Let $U_0 = U \cup (X \backslash F)$, then U_0 is open in X and $(\bigcap_{n\in\mathbf{N}} \bar{U}_n^X) \cap X_b \subset U_0$. Then, there exist $n_0 \in \mathbf{N}$ and $W \in N(b)$ such that $(\bigcap_{n\in\mathbf{N}} \bar{U}_n^X) \cap X_b \subset (\bigcap_{i\le n_0} \bar{U}_i^X) \cap X_W \subset U_0$ and therefore, $(\bigcap_{n\in\mathbf{N}} \bar{G}_n^F) \cap F_b \subset (\bigcap_{i\le n_0} \bar{G}_i^F) \cap F_W \subset G$.

It follows from (1′) and (2′) that $f|F$ is a p-map.

(2) Since $f : X \to B$ is a p-map, from Theorem 3.2 there is a T_2-compactification $f' : X' \to B$ satisfying properties (1) and (2) of Theorem 3.2.

Since G is a G_δ-subset of X, there exists a sequence $\{G_n\}_{n\in\mathbf{N}}$ of open subsets in X' such that $G = (\bigcap_{n\in\mathbf{N}} G_n) \cap X$. Obviously $f' \,|\, \bar{G}^{X'} : \bar{G}^{X'} \to B$ is a T_2-compactification of $f|G$. For every $n \in \mathbf{N}$, let $\mathscr{U}_n = \{G_n \cap \bar{G}^{X'} \cap P : P \in \mathscr{P}_n\}$. Then the sequence $\{\mathscr{U}_n\}_{n\in\mathbf{N}}$ of open families of $\bar{G}^{X'}$ satisfies:

(1′) For every $n \in \mathbf{N}$, $G_b \subset \bigcup \mathscr{U}_n$,

(2′) For every $x \in G_b$, $(\bigcap_{n\in\mathbf{N}} st(x, \mathscr{U}_n)) \cap \bar{G}_b^{X'} \subset (\bigcap_{n\in\mathbf{N}}(st(x, \mathscr{P}_n) \cap G_n \cap \bar{G}^{X'})) \cap X_b' \subset (\bigcap_{n\in\mathbf{N}}(st(x, \mathscr{P}_n) \cap X_b')) \cap (\bigcap_{n\in\mathbf{N}} G_n) \cap \bar{G}^{X'} \subset X_b \cap G = G_b$.

Thus, from Theorem 3.2, $f|G$ is a p-map. □

In connection with Theorem 3.4, note that a submap of a p-map is not necessarily a p-map even when the submap is a closed and open map. For this, see [9] Example 3.23. In this example, there is a p-space X in which a subspace Y is not a p-space. It is then easy to see that the map f from X onto a singleton set B gives the necessary example.

Theorem 3.5. *Suppose that B is regular. Let $f_n : X_n \to B$ be a p-map for every $n \in \mathbf{N}$. Then the product map $f = \prod_B f_n : \prod_B X_n \to B$ is a p-map.*

Proof. Since f_n is a p-map for every $n \in \mathbf{N}$, from Theorem 3.2 there is a compactification $f_n' : X_n' \to B$ of f_n such that for every $b \in B$ there is a sequence $\{\mathscr{P}_{nm}\}_{m\in\mathbf{N}}$ of open families of X_n' satisfying:

(1) For every $m \in \mathbf{N}$, $X_{nb} \subset \bigcup \mathscr{P}_{nm}$;

(2) For every $x \in X_{nb}$, $(\bigcap_{m\in\mathbf{N}} st(x, \mathscr{P}_{nm})) \cap X_{nb}' \subset X_{nb}$.

We can assume that $\mathscr{P}_{n,m+1}$ is a refinement of $\mathscr{P}_{nm}$. Since $f' = \prod_B f_n' : \prod_B X_n' \to B$ is compact (Proposition 2.7(3)), $f' \,|\, \overline{\prod_B X_n} : \overline{\prod_B X_n} \to B$ is a compactification of f.

For every $m \in \mathbf{N}$, let $\mathscr{G}_m' = \mathscr{P}_{1m} \times_B \cdots \times_B \mathscr{P}_{mm} \times_B (\prod_B X_n')_{n>m}$ and $\mathscr{G}_m = \mathscr{G}_m' \,|\, \overline{\prod_B X_n}$, then it is easy to see that $\mathscr{G}_m$ is an open family of $\overline{\prod_B X_n}$ and $\mathscr{G}_m$ is an open cover of $(\prod_B X_n)_b$. By Theorem 3.2 we only need to prove that for every $x = (x_1, x_2, \ldots, x_n, \ldots) \in (\prod_B X_n)_b$, $\bigcap_{m\in\mathbf{N}} st(x, \mathscr{G}_m) \cap (\overline{\prod_B X_n})_b \subset (\prod_B X_n)_b$.

Assume there is a point $x' = (x_1', x_2', \dots, x_n', \dots) \in (\bigcap_{m \in \mathbf{N}} st(x, \mathscr{G}_m) \cap (\overline{\prod_B X_n})_b) \backslash (\prod_B X_n)_b$, then there is some $n \in \mathbf{N}$ such that $x_n' \notin X_{nb}$. Since $(\bigcap_{m \in \mathbf{N}} st(x_n, \mathscr{P}_{nm})) \cap X_{nb}' \subset X_{nb}$, there exists $m \in \mathbf{N}$ such that $x_n' \notin st(x_n, \mathscr{P}_{nm})$. Let $l = \max\{m, n\}$, then $x' \notin st(x, \mathscr{G}_l)$ which contradicts $x' \in \bigcap_{m \in \mathbf{N}} st(x, \mathscr{G}_m) \cap (\overline{\prod_B X_n})_b$. □

THEOREM 3.6. *Let $f : X \to B$ and $g : Y \to B$ be maps and $\lambda : f \to g$ be a perfect morphism. If g is a p-map, then f is also a p-map.*

PROOF. Since g is a p-map, for every $b \in B$ there is a sequence $\{\mathscr{V}_n\}_{n \in \mathbf{N}}$ of open covers of Y_b satisfying (P1) and (P2) of Definition 3.1.

For every $n \in \mathbf{N}$, let $\mathscr{U}_n = \{\lambda^{-1}(V) : V \in \mathscr{V}_n\}$, then $\{\mathscr{U}_n\}_{n \in \mathbf{N}}$ is a sequence of open covers of X_b. Using the properties of $\{\mathscr{V}_n\}_{n \in \mathbf{N}}$ we deduce the following properties of $\{\mathscr{U}_n\}_{n \in \mathbf{N}}$. If $x \in X_b$ and $x \in U_n \in \mathscr{U}_n$ for every $n \in \mathbf{N}$, there is a $V_n \in \mathscr{V}_n$ with $U_n = \lambda^{-1}(V_n)$ for every $n \in \mathbf{N}$.

(1′) Since $(\bigcap_{n \in \mathbf{N}} \overline{V}_n) \cap Y_b$ is compact and $(\bigcap_{n \in \mathbf{N}} \overline{U_n}) \cap X_b = (\bigcap_{n \in \mathbf{N}} \overline{\lambda^{-1}(V_n)}) \cap \lambda^{-1}(Y_b) = (\bigcap_{n \in \mathbf{N}} \lambda^{-1}(\overline{V}_n)) \cap \lambda^{-1}(Y_b) = \lambda^{-1}((\bigcap_{n \in \mathbf{N}} \overline{V}_n) \cap Y_b)$, we conclude that $(\bigcap_{n \in \mathbf{N}} \overline{U}_n) \cap X_b$ is compact from the perfectness of λ.

(2′) If U is an open subset of X with $(\bigcap_{n \in \mathbf{N}} \overline{U}_n) \cap X_b \subset U$, then $(\bigcap_{n \in \mathbf{N}} \overline{U}_n) \cap X_b = \lambda^{-1}((\bigcap_{n \in \mathbf{N}} \overline{V}_n) \cap Y_b) \subset U$ and therefore, $(\bigcap_{n \in \mathbf{N}} \overline{V_n}) \cap Y_b \subset Y \backslash \lambda(X \backslash U)$. Let $V = Y \backslash \lambda(X \backslash U)$, then V is open in Y and $(\bigcap_{n \in \mathbf{N}} \overline{V_n}) \cap Y_b \subset V$. Since g is a p-map, there exist $n_0 \in \mathbf{N}$ and $W \in N(b)$ such that $(\bigcap_{n \in \mathbf{N}} \overline{V_n}) \cap Y_b \subset (\bigcap_{i \le n_0} \overline{V_i}) \cap Y_W \subset V$. It is not difficult to see that $(\bigcap_{n \in \mathbf{N}} \overline{U}_n) \cap X_b \subset (\bigcap_{i \le n_0} \overline{U}_i) \cap X_W \subset U$.

Thus f is a p-map. □

If $f : X \to B$ is a paracompact p-map, the converse of Theorem 3.6 also holds (see Theorem 5.2).

We shall conclude this section by studying the relations of Čech-complete map, p-map and k-map, and sharpen Theorem 6.3 of [2] that a Čech-complete map is a k-map.

THEOREM 3.7. *Suppose that B is regular. If $f : X \to B$ is Čech-complete, then f is a p-map.*

PROOF. Since B is regular and f is Čech-complete, there is a T_2-compactification f' of f such that for every $b \in B$ there is a sequence $\{G_n\}_{n \in \mathbf{N}}$ of open subsets of X' such that $X_b = (\bigcap_{n \in \mathbf{N}} G_n) \cap X_b'$. Let $\mathscr{P}_n = \{G_n\}$, then $\{\mathscr{P}_n\}_{n \in \mathbf{N}}$ satisfies conditions (1) and (2) of Theorem 3.2, so that f is a p-map. □

Theorem 3.8. *Suppose that B is regular and satisfies the first axiom of countability. Then a p-map $f : X \to B$ is a k-map.*

Proof. If f is not a k-map, there is a quasi-closed subset H in X which is not closed, say $x \in \overline{H} \backslash H$. Let $b = f(x)$ and $\{W_n\}_{n \in \mathbf{N}}$ be a decreasing nbd base of b with $\overline{W}_{n+1} \subset W_n$ for every $n \in \mathbf{N}$. Since f is a p-map, there exists a sequence $\{\mathcal{G}_n\}_{n \in \mathbf{N}}$ of open (in X) covers of X_b satisfying (P1) and (P2) of Definition 3.1.

For every $n \in \mathbf{N}$ choose $U_n \in N(x)$ and $G_n \in \mathcal{G}_n$ such that $x \in U_n \subset \bar{U}_n \subset \bigcap_{i \leq n} G_i$, then $K_1 = (\bigcap_{n \in \mathbf{N}} U_n) \cap X_b = (\bigcap_{n \in \mathbf{N}} \bar{U}_n) \cap X_b \subset (\bigcap_{n \in \mathbf{N}} \bar{G}_n) \cap X_b$ is compact.

If $K_1 \cap H$ is not closed in K_1, then for every $W \in N(b)$ and every $W' \in N(b)$ with $W' \subset W$, K_1 is fibrewise compact over W' (Definition 2.3 (2)) but $K_1 \cap H$ is not closed in K_1 which contradicts the fact that H is quasi-closed. Thus, in the case that $K_1 \cap H$ is not closed in K_1, the proof is complete.

If $K_1 \cap H$ is closed in K_1, then $K_1 \cap H$ is compact and there is $V_0 \in N(x)$ with $K_1 \cap H \cap V_0 = \emptyset$. For every $n \in \mathbf{N}$ choose $V_n \in N(x)$ such that $x \in V_n \subset \overline{V}_n \subset V_{n-1}$. Let $K_2 = \bigcap_{n \in \mathbf{N}} (U_n \cap V_n \cap X_{W_n}) \cap X_b = \bigcap_{n \in \mathbf{N}} (\bar{U}_n \cap \overline{V}_n \cap X_{\overline{W}_n}) \cap X_b$, then K_2 is compact and $K_2 \cap H = \emptyset$. We first prove that $\{U_n \cap V_n \cap X_{W_n}\}_{n \in \mathbf{N}}$ is a nbd base of K_2 in X. If not, one can find a nbd U of K_2 and $x_n \in (U_n \cap V_n \cap X_{W_n}) \backslash U$ for every $n \in \mathbf{N}$. If $\overline{\{x_n\}_{n \in \mathbf{N}}} \cap X_b = \emptyset$, then $(\bigcap_{n \in \mathbf{N}} \bar{G}_n) \cap X_b \subset X \backslash \overline{\{x_n\}_{n \in \mathbf{N}}}$ and therefore, there exists $n_0 \in \mathbf{N}$ such that $(\bigcap_{n \in \mathbf{N}} \bar{G}_n) \cap X_b \subset (\bigcap_{i \leq n_0} G_i) \cap X_{W_{n_0}} \subset X \backslash \overline{\{x_n\}_{n \in \mathbf{N}}}$ which contradicts $x_n \in (\bigcap_{i \leq n_0} G_i) \cap X_{W_{n_0}}$ for every $n \geq n_0$, so $\overline{\{x_n\}_{n \in \mathbf{N}}} \cap X_b \neq \emptyset$. Since $\{x_n\}_{n \in \mathbf{N}} \cap U = \emptyset$, $\overline{\{x_n\}_{n \in \mathbf{N}}} \cap U = \emptyset$, but $\overline{\{x_n\}_{n \in \mathbf{N}}} \cap X_b \subset \bigcap_{n \in \mathbf{N}} (U_n \cap V_n \cap X_{W_n}) \cap X_b = K_2$, which is a contradiction.

For every $n \in \mathbf{N}$ take a point $x_n \in U_n \cap V_n \cap X_{W_n} \cap H$. Since $\{U_n \cap V_n \cap X_{W_n}\}_{n \in \mathbf{N}}$ is a base of K_2, $F_n = K_2 \cup \{x_i : i \geq n\}$ is compact and $F_n \cap H$ is not closed in F_n for every $n \in \mathbf{N}$. Thus, for every $W \in N(b)$, there exists $n \in \mathbf{N}$ such that $W_n \subset W$ and F_n is fibrewise compact over W_n (Definition 2.3 (2)), but $H \cap F_n$ is not closed in F_n which contradicts the fact that H is quasi-closed in X. Thus, in the case that $K_1 \cap H$ is closed in K_1, the proof is also complete. □

4. Definition and Basic Properties of M-maps

In this section, we define an M-map and investigate some of its basic properties. The concept of M-maps is a generalization of M-spaces ([11], [12]).

Definition 4.1. A T_2-compactifiable map $f : X \to B$ is an *M-map* if for every $b \in B$ there is a sequence $\{\mathcal{U}_n\}_{n \in \mathbf{N}}$ of open (in X) covers of X_b satisfying:

(M1) If $x \in X_b$ and $x_n \in st(x, \mathscr{U}_n) \cap X_b$ for every $n \in \mathbf{N}$, then the sequence $\{x_n\}_{n \in \mathbf{N}}$ has an accumulation point in X_b,

(M2) For every $n \in \mathbf{N}$, $\mathscr{U}_{n+1}$ is a b-star refinement of $\mathscr{U}_n$.

For submaps of M-maps, we have the following.

Theorem 4.2. *For an M-map $f : X \to B$ and a closed subset F of X, $f|F$ is an M-map.*

Proof. Since $f : X \to B$ is an M-map, for every $b \in B$ there is a sequence $\{\mathscr{U}_n\}_{n \in \mathbf{N}}$ of open (in X) covers of X_b satisfying (M1) and (M2) of Definition 4.1. For every $n \in \mathbf{N}$, let $\mathscr{G}_n = \mathscr{U}_n \wedge \{F\}$. Since F is closed, $\{\mathscr{G}_n\}_{n \in \mathbf{N}}$ is a sequence of open covers of F_b which satisfies (M1) and (M2) of Definition 4.1 and therefore, $f|F$ is an M-map. □

Theorem 4.3. *For the maps $f : X \to B$ and $g : Y \to B$, if there is a quasi-perfect morphism $\lambda : f \to g$ and g is an M-map, then f is an M-map.*

Proof. Since $g : Y \to B$ is an M-map, for every $b \in B$ there is a sequence $\{\mathscr{V}_n\}_{n \in \mathbf{N}}$ of open (in Y) covers of Y_b satisfying (M1) and (M2) of Definition 4.1.

For every $n \in \mathbf{N}$, let $\mathscr{U}_n = \lambda^{-1}(\mathscr{V}_n)$, then $\{\mathscr{U}_n\}_{n \in \mathbf{N}}$ is a sequence of open (in X) covers of X_b such that $\mathscr{U}_{n+1}$ is a b-star refinement of $\mathscr{U}_n$, for every $n \in \mathbf{N}$. Let us now show that if $x \in X_b$ and $x_n \in st(x, \mathscr{U}_n) \cap X_b$ for every $n \in \mathbf{N}$, then the sequence $\{x_n\}_{n \in \mathbf{N}}$ has an accumulation point in X_b. If not, since $\lambda(x_n) \in st(\lambda(x), \mathscr{V}_n) \cap Y_b$, $\{\lambda(x_n)\}_{n \in \mathbf{N}}$ has an accumulation point $y \in Y_b$. By countable compactness of $\lambda^{-1}(y)$, we can assume that $\overline{\{x_n\}_{n \in \mathbf{N}}} \cap \lambda^{-1}(y) = \varnothing$. Since λ is closed, there exists $V \in N(y)$ such that $\{x_n\}_{n \in \mathbf{N}} \cap \lambda^{-1}(V) = \varnothing$ and therefore, $V \cap \{\lambda(x_n)\}_{n \in \mathbf{N}} = \varnothing$ which contradicts $y \in \overline{\{\lambda(x_n)\}_{n \in \mathbf{N}}}$. □

5. Paracompact p-maps and M-maps

One can note that neither of the classes of p-maps and M-maps imply the other. It is enough to consider the case when B is a singleton set and X a p-space (resp. M-space) that is not an M-space (resp. p-space). In the realm of para-compact maps, we prove in Theorem 5.1 that the notions of M-map and p-map are equivalent, which corresponds to [1] Theorem 16. Further, we prove in Theorem 5.2 that a perfect image of a paracompact p-map is also a paracompact p-map which corresponds to [8] Theorem 1.

Theorem 5.1. *A paracompact map $f : X \to B$ is an M-map if and only if it is a p-map.*

Proof. ["Only if" part]: If $f : X \to B$ is an M-map, for every $b \in B$ there is a sequence $\{\mathscr{U}_n\}_{n \in \mathbf{N}}$ of open (in X) covers of X_b satisfying (M1) and (M2) of Definition 4.1.

We shall prove that the sequence $\{\mathscr{U}_n\}_{n \in \mathbf{N}}$ satisfies the definition of p-map. Let $x \in X_b$ and $x \in U_n \in \mathscr{U}_n$ for every $n \in \mathbf{N}$. We show that (P1) and (P2) hold.

(P1) We need to show that $(\bigcap_{n \in \mathbf{N}} \overline{U_n}) \cap X_b$ is compact. Since f is paracompact, the closed subspace $(\bigcap_{n \in \mathbf{N}} \overline{U_n}) \cap X_b$ of X_b is paracompact. Next, consider a sequence $\{x_i\}_{i \in \mathbf{N}} \subset (\bigcap_{n \in \mathbf{N}} \overline{U_n}) \cap X_b$. Since $\mathscr{U}_{n+1}$ is a cover of X_b, for every $i \in \mathbf{N}$ there exists $U_{x_i} \in \mathscr{U}_{n+1}$ such that $x_i \in U_{x_i}$ and therefore, $x_i \in U_{x_i} \subset st(U_{n+1}, \mathscr{U}_{n+1})$. By (M2), there exists $U'_n \in \mathscr{U}_n$ such that $st(U_{n+1}, \mathscr{U}_{n+1}) \subset U'_n$ and hence, $\{x_i\}_{i \in \mathbf{N}} \subset U'_n$. Thus for every $n \in \mathbf{N}$ we can choose $U'_n \in \mathscr{U}_n$ such that $\{x_n, x\} \subset U'_n$ and therefore, $x_n \in st(x, \mathscr{U}_n) \cap X_b$. It follows from (M1) that $\{x_n\}_{n \in \mathbf{N}}$ has an accumulation point in X_b, so that $(\bigcap_{n \in \mathbf{N}} \overline{U_n}) \cap X_b$ is countably compact and therefore, compact.

(P2) Let U be open in X and $(\bigcap_{n \in \mathbf{N}} \overline{U_n}) \cap X_b \subset U$. We first prove that there exists $n_0 \in \mathbf{N}$ such that $(\bigcap_{n \in \mathbf{N}} \overline{U_n}) \cap X_b \subset (\bigcap_{i \le n_0} \overline{U_i}) \cap X_b \subset U$. If not, for every $n \in \mathbf{N}$ there is $x_n \in ((\bigcap_{i \le n} \overline{U_i}) \cap X_b) \backslash U$. For every $n \in \mathbf{N}$, since $\mathscr{U}_{n+1}$ is a cover of X_b, there is $U_{x_{n+1}} \in \mathscr{U}_{n+1}$ such that $x_{n+1} \in U_{x_{n+1}} \subset st(U_{n+1}, \mathscr{U}_{n+1})$. Consequently, one can find $U'_n \in \mathscr{U}_n$ such that $\{x_{n+1}, x\} \subset st(U_{n+1}, \mathscr{U}_{n+1}) \subset U'_n$, because $\mathscr{U}_{n+1}$ is a b-star refinement of $\mathscr{U}_n$. Thus $x_{n+1} \in st(x, \mathscr{U}_n)$ and $\{x_n\}_{n \in \mathbf{N}}$ has an accumulation point $x_0 \in X_b$. Then $x_0 \in \overline{\{x_i\}}_{i \ge n} \subset \overline{U_n}$ for every $n \in \mathbf{N}$ and therefore, $x_0 \in (\bigcap_{n \in \mathbf{N}} \overline{U_n}) \cap X_b \subset U$ which contradicts $\{x_n\}_{n \in \mathbf{N}} \cap U = \emptyset$.

Since $X_b \subset (X \backslash \bigcap_{i \le n_0} \overline{U_i}) \cup U$ and f is closed, there exists $W \in N(b)$ such that $X_b \subset X_W \subset (X \backslash \bigcap_{i \le n_0} \overline{U_i}) \cup U$ and therefore, $(\bigcap_{i \le n_0} \overline{U_i}) \cap X_b \subset (\bigcap_{i \le n_0} \overline{U_i}) \cap X_W \subset U$.

["If" part]: If f is a p-map, then for every $b \in B$, there exists a sequence $\{\mathscr{U}_n\}_{n \in \mathbf{N}}$ of open (in X) covers of X_b satisfying (P1) and (P2) of Definition 3.1.

Since f is paracompact, from [4] Theorem 3.12, for every $n \in \mathbf{N}$ there exists an open (in X) cover $\mathscr{G}_{n+1}$ of X_b which is a b-star-refinement of $\mathscr{G}_n \wedge \mathscr{U}_{n+1}$, where $\mathscr{G}_1 = \mathscr{U}_1$. Obviously the sequence $\{\mathscr{G}_n\}$ satisfies (M2), and we are only left to prove that $\{\mathscr{G}_n\}$ satisfies (M1). Let $x \in X_b$ and $x_n \in st(x, \mathscr{G}_n) \cap X_b$ for every $n \in \mathbf{N}$. Since $\mathscr{G}_2$ is a b-star refinement of $\mathscr{G}_1$, there is $G_1 \in \mathscr{G}_1$ such that $x_2 \in st(x, \mathscr{G}_2) \subset G_1$. Inductively, for every $n \ge 2$ there is $G_n \in \mathscr{G}_n$ such that $x_{n+1} \in st(x, \mathscr{G}_{n+1}) \subset G_n$. Then $G_{n+1} \subset G_n$ for every $n \in \mathbf{N}$, and $\{x_i\}_{i > n} \subset G_n$. For every $n \in \mathbf{N}$ there exists $U_n \in \mathscr{U}_n$ such that $G_n \subset U_n$. If $\{x_n\}_{n \in \mathbf{N}}$ has no accumulation point in X_b, then $\{k \mid x_k \in (\bigcap_{n\in\mathbf{N}} \overline{U_n}) \cap X_b\}$ is finite, so one can suppose that $\{x_n\}_{n\in\mathbf{N}} \cap ((\bigcap_{n\in\mathbf{N}} \overline{U_n}) \cap X_b) = \emptyset$. Then since $\overline{\{x_n\}_{n\in\mathbf{N}}} \cap (X_b \backslash \{x_n\}_{n\in\mathbf{N}}) = \emptyset$, $\bigcap_{n\in\mathbf{N}} \overline{U_n} \cap X_b \subset X \backslash \overline{\{x_n\}_{n\in\mathbf{N}}}$. From (P2), there is $W \in N(b)$ such that $(\bigcap_{n\in\mathbf{N}} \overline{U_n}) \cap X_b \subset (\bigcap_{i\le n_0} \overline{U_i}) \cap X_W \subset X \backslash \overline{\{x_n\}_{n\in\mathbf{N}}}$, which contradicts $\{x_i\}_{i>n_0} \subset \bigcap_{i\le n_0} \overline{U_i} \cap X_W$. Consequently, the sequence $\{x_n\}_{n\in\mathbf{N}}$ has an accumulation point in X_b. □

The last theorem of this section relates to invariance of paracompact p-maps under perfect morphisms.

Theorem 5.2. *Suppose that B is regular, $f : X \to B$ and $g : Y \to B$ are T_2-compactifiable maps, and there exists an onto perfect morphism $\lambda : f \to g$. If f is a paracompact p-map then so is g.*

To prove the theorem we need the following two lemmas.

Lemma 5.3. *Let $f : X \to B$ and $g : Y \to B$ be T_2-compactifiable maps and $f' : X' \to B$ and $g' : Y' \to B$ be T_2-compactifications of f and g, respectively. If there exists an onto perfect morphism $\lambda : f \to g$, then there exists a morphism $\lambda' : f' \to g'$ such that*

(1) *$\lambda' \mid X = \lambda$ and λ' is perfect*;

(2) *$\lambda'(X'_b \backslash X_b) \subset Y'_b \backslash Y_b$ for every $b \in B$.*

Proof. (1) Let $\mu = e \circ \lambda$ where e is the embedding of Y to Y'. Since $f' : X' \to B$ and $g' : Y' \to B$ are T_2-compactifications of f and g, $\overline{X} = X'$ and for every b-filter $\mathscr{F}$ on X which is convergent in X', the b-filter $\mu_*\mathscr{F}$ has a unique adherence point in Y'. For every $b \in B$ and every $x \in X'_b$, let $\mathscr{F}_x$ be the nbd b-filter of x in X', and let y_x be the unique adherence point of the b-filter $\mu(\mathscr{F}_x|X)$ in Y'. For every $b \in B$ and every $x \in X'_b$, let $\lambda'(x) = y_x$, then $\lambda' : X' \to Y'$ is a fibrewise continuous map and $\lambda' \mid X = \lambda$ from [10] Proposition 4.6.

For every closed subset F of X', the map $g' \mid \lambda'(F) : \lambda'(F) \to B$ is compact since $f' \mid F : F \to B$ is compact, and therefore $\lambda'(F)$ is closed in Y'. Since Y'_b is regular and $\lambda^{-1}(y)$ is closed in X'_b for every $y \in Y'_b$, $\lambda'^{-1}(y)$ is compact for every $y \in Y$, so that λ' is perfect. Consequently, the proof of (1) is complete.

(2) If there exists $b \in B$ and $x \in X'_b \backslash X_b$ such that $\lambda'(x) = y \in Y_b$, then $\lambda_*(\mathscr{F}_x|X)$ is convergent to y, where $\mathscr{F}_x$ is the nbd b-filter of x in X'. Since λ is perfect, $\mathscr{F}_x|X$ is convergent to some point $x' \in \lambda^{-1}(y)$ in X ([10] Proposition 4.3). Then x and x' are different adherence points of $\mathscr{F}_x$ in X', which contradicts the fact that f' is T_2. Thus $\lambda'(X'_b \backslash X_b) \subset Y'_b \backslash Y_b$ for every $b \in B$. □

Lemma 5.4. *Suppose that B is regular. For a paracompact T_2-map $f : X \to B$, let $f' : X' \to B$ be a T_2-compactification of f. If $\mathscr{U}$ is an open cover of X_b in X' for every $b \in B$, then there exists an open (in X') cover $\mathscr{P}$ of X_b satisfying:*

(1) *For every $x \in \bigcup \mathscr{P}$, there exists $U \in \mathscr{U}$ such that $\overline{st(x, \mathscr{P})}^{X'} \subset U$;*

(2) *For every $x \in \bigcup \mathscr{P}$, $\mathscr{P}$ is locally finite at the point x.*

Proof. Since B is regular and $\mathscr{U}$ is an open cover of X_b in X', for every $x \in X_b$ take $U_{1x} \in \mathscr{U}$ with $x \in U_{1x}$ and let U_x be an open nbd of x in X such that $x \in U_x \subset \overline{U_x}^{X'} \subset U_{1x}$. Let $\mathscr{U}_1 = \{U_x \mid x \in X_b\}$, then $\mathscr{U}_1$ is an open cover of X_b in X. Since f is paracompact, there exists an open (in X) cover $\mathscr{U}_2$ of X_b which is a b-star refinement of $\mathscr{U}_1$ in X. Then there exists $W \in N(b)$ and an open family $\mathscr{U}_3$ in X which is a locally finite (in X_W) cover of X_W and satisfies $\mathscr{U}_3 < \{X_W\} \wedge \mathscr{U}_2$. For every $V \in \mathscr{U}_3$ take an open set $U(V) \subset X'_W$ in X' such that $U(V) \cap X = V$. Let $\mathscr{U}_4 = \{U(V) \mid V \in \mathscr{U}_3\}$ and $G = \{x \in X' \mid \mathscr{U}_4$ is locally finte at $x\}$. Then G is open in X' and $X_W \subset G$ since $\overline{X} = X'$. Let $\mathscr{P} = \{G \cap U \mid U \in \mathscr{U}_4\}$ which is an open (in X') cover of X_b and satisfies (2). For every $x \in \bigcup \mathscr{P}$ let $\{P \in \mathscr{P} \mid x \in P\} = \{P_1, \dots, P_k\}$. For $i \le k$ take $U(V_i) \in \mathscr{U}_4$ such that $P_i = G \cap U(V_i)$. Then since $U(V_i) \cap U(V_j) \cap X \ne \varnothing$ for every $i, j \le k$, we have $V_i \cap V_j \ne \varnothing$ for every $i, j \le k$. Since $\mathscr{U}_3 < \{X_W\} \wedge \mathscr{U}_2$ and $\mathscr{U}_2$ is a b-star refinement of $\mathscr{U}_1$ in X, there exists $x_0 \in X_b$ and $U_{x_0} \in \mathscr{U}_1$ such that $\bigcup_{i \le k} V_i \subset U_{x_0}$. Then, $\overline{st(x, \mathscr{P})}^{X'} = \overline{st(x, \mathscr{P}) \cap X}^{X'} \subset \overline{U_{x_0}}^{X'} \subset U_{1x_0} \in \mathscr{U}$, and (1) is satisfied. $\square$

We can now prove Theorem 5.2.

Proof (Theorem 5.2). Since $f : X \to B$ is a p-map, take a T_2-compactification $f' : X' \to B$ of f such that for every $b \in B$ there exists a sequence $\{\mathscr{P}_n\}_{n \in \mathbf{N}}$ of open covers of X_b in X' satisfying:

(1) For every $n \in \mathbf{N}$, $X_b \subset \bigcup \mathscr{P}_n$;

(2) For every $x \in X_b$, $\bigcap_{n \in \mathbf{N}} st(x, \mathscr{P}_n) \cap X'_b \subset X_b$.

By Lemma 5.4 we can suppose the following.

(3) For every $n \in \mathbf{N}$ and $x \in \bigcup \mathscr{P}_n$, $\mathscr{P}_n$ is locally finite at the point x;

(4) For every $n \in \mathbf{N}$ and $x \in \bigcup \mathscr{P}_{n+1}$, there exists $P \in \mathscr{P}_n$ such that $\overline{st(x, \mathscr{P}_{n+1})}^{X'} \subset P$.

Furthermore, we show that the following (5), (6), (7) and (8) hold.

(5) For every $b \in B$ if $x \in X'_b \backslash X_b$, then $\bigcap_{n \in \mathbf{N}} st(x, \mathscr{P}_n) \cap X'_b \subset X'_b \backslash X_b$.

If not, there exist $x_0 \in X_b$ and $P_n \in \mathscr{P}_n$ for every $n \in \mathbf{N}$ such that $\{x, x_0\} \subset P_n$. Then $x \in \bigcap_{n \in \mathbf{N}} st(x_0, \mathscr{P}_n) \cap X'_b$, which contradicts (2).

(6) If $F \subset X_b$ is compact, then $\bigcap_{n \in \mathbf{N}} st(F, \mathscr{P}_n) \cap X'_b \subset X_b$.

If not, there exists $x \in \bigcap_{n\in\mathbf{N}} st(F,\mathscr{P}_n) \cap (X_b' \backslash X_b)$. Then for every $n \in \mathbf{N}$, there exists $P_n \in \mathscr{P}_n$ such that $x \in P_n$ and $P_n \cap F \neq \emptyset$. For every $n \in \mathbf{N}$, $F_n = F \cap \overline{st(x,\mathscr{P}_n)}^{X'}$ is compact and $F_{n+1} \subset F_n$ from (4). Therefore, there exists $x_0 \in X_b$ such that $x_0 \in \bigcap_{n\in\mathbf{N}} F_n$. However, $x_0 \in \bigcap_{n\in\mathbf{N}} \overline{st(x,\mathscr{P}_n)}^{X'} \cap X_b' = \bigcap_{n\in\mathbf{N}} st(x,\mathscr{P}_n) \cap X_b'$, which contradicts (5).

(7) If $F \subset X_b$ is compact, then $\overline{st(F,\mathscr{P}_n)}^{X'} \subset st(F,\mathscr{P}_{n-1})$ for every $n \in \mathbf{N}$.

For every $n \in \mathbf{N}$, since $\mathscr{P}_n$ is locally finite at every point of $\bigcup \mathscr{P}_n$, let $\{P \in \mathscr{P}_n \mid P \cap F \neq \emptyset\} = \{P_1, \dots, P_k\}$. By (4), for each $i \le k$ there exists $P_i' \in \mathscr{P}_{n-1}$ such that $\overline{P_i}^{X'} \subset P_i'$. Thus $\overline{st(F,\mathscr{P}_n)}^{X'} = \bigcup_{i\le k} \overline{P_i}^{X'} \subset \bigcup_{i\le k} P_i' \subset st(F,\mathscr{P}_{n-1})$.

(8) For every $b \in B$ and $n \in \mathbf{N}$, let $\mathscr{U}_n = \{st(\lambda^{-1}(y),\mathscr{P}_n) \mid y \in Y_b\}$. Then $\bigcap_{n\in\mathbf{N}} st(\lambda^{-1}(y),\mathscr{U}_n) \cap X_b' \subset X_b$ for every $y \in Y_b$.

If not, there exist $y_n \in Y_b$ and $x \in \bigcap_{n\in\mathbf{N}} st(\lambda^{-1}(y),\mathscr{U}_n) \cap (X_b' \backslash X_b)$ such that $st(\lambda^{-1}(y_n),\mathscr{P}_n) \cap \lambda^{-1}(y) \neq \emptyset$ and $x \in st(\lambda^{-1}(y_n),\mathscr{P}_n)$. Thus $st(\lambda^{-1}(y),\mathscr{P}_n) \cap \lambda^{-1}(y_n) \neq \emptyset$ and $st(x,\mathscr{P}_n) \cap \lambda^{-1}(y_n) \neq \emptyset$. Let $x_n \in st(\lambda^{-1}(y),\mathscr{P}_n) \cap \lambda^{-1}(y_n)$, $x_n' \in st(x,\mathscr{P}_n) \cap \lambda^{-1}(y_n)$ and $T_1 = (\bigcap_{n\in\mathbf{N}} \overline{st(\lambda^{-1}(y),\mathscr{P}_n)}^{X'}) \cap X_b'$. Then T_1 is compact in X_b' and $T_1 = \bigcap_{n\in\mathbf{N}} st(\lambda^{-1}(y),\mathscr{P}_n)) \cap X_b' \subset X_b$ from (6) and (7). Let $T_2 = (\bigcap_{n\in\mathbf{N}} \overline{st(x,\mathscr{P}_n)}^{X'}) \cap X_b'$. Then T_2 is also compact in X_b' and $T_2 = (\bigcap_{n\in\mathbf{N}} st(x,\mathscr{P}_n)) \cap X_b' \subset X_b' \backslash X_b$ from (4) and (5).

Let $g' : Y' \to B$ be a T_2-compactification of g and let $\lambda' : f' \to g'$ be a morphism extension of λ satisfying properties (1) and (2) of Lemma 5.3. Then, for the above subsets T_1 and T_2 we have that $\lambda'(T_1)$ and $\lambda'(T_2)$ are compact in Y_b' with $\lambda'(T_1) \cap \lambda'(T_2) = \emptyset$. Therefore, there exist nbds V_i of $\lambda_i'(T_i)$ $(i = 1, 2)$ such that $V_1 \cap V_2 = \emptyset$.

Since $y_n \in Y_b$ and $x_n \in st(\lambda^{-1}(y),\mathscr{P}_n) \cap \lambda^{-1}(y_n) \subset X_b \subset X_b'$, $\{x_n\}_{n\in\mathbf{N}} \subset X_b \subset X_b'$. Then, there exists $n_1 \in \mathbf{N}$ such that $x_n \in \lambda'^{-1}(V_1)$ for all $n \ge n_1$. Otherwise, for every $n \in \mathbf{N}$ there exists $k_n \ge n$ such that $x_{k_n} \notin \lambda'^{-1}(V_1)$. Then $\{x_{k_n}\}_{n\in\mathbf{N}} \cap \lambda'^{-1}(V_1) = \emptyset$ and therefore, $\overline{\{x_{k_n}\}}_{n\in\mathbf{N}} \cap \lambda'^{-1}(V_1) = \emptyset$. Since $\{x_{k_n}\}_{n\in\mathbf{N}} \subset \{x_n\}_{n\in\mathbf{N}} \subset X_b'$, $\{x_{k_n}\}_{n\in\mathbf{N}}$ has adherence points in X_b'. Suppose x_0 is such a point. From (7), $\overline{st(\lambda^{-1}(y),\mathscr{P}_n)}^{X'} \subset st(\lambda^{-1}(y),\mathscr{P}_{n-1})$. It follows that $\{x_{k_i}\}_{i\ge n} \subset \{x_i\}_{i\ge n} \subset st(\lambda^{-1}(y),\mathscr{P}_n)$. Consequently, $x_0 \in \overline{\{x_{k_i}\}_{i\ge n}}^{X'} \subset \overline{st(\lambda^{-1}(y),\mathscr{P}_n)}^{X'}$ and therefore, $x_0 \in \bigcap_{n\in\mathbf{N}} \overline{st(\lambda^{-1}(y),\mathscr{P}_n)}^{X'} \cap X_b' = T_1$, which contradicts $\overline{\{x_{k_n}\}}_{n\in\mathbf{N}} \cap \lambda'^{-1}(V_1) = \emptyset$.

Since $y_n \in Y_b$ and $x_n' \in st(x,\mathscr{P}_n) \cap \lambda^{-1}(y_n) \subset X_b \subset X_b'$, $\{x_n'\}_{n\in\mathbf{N}} \subset X_b \subset X_b'$. Analogous to the above one can prove that there exists $n_2 \in \mathbf{N}$ such that $x_n' \in \lambda'^{-1}(V_2)$ for every $n \ge n_2$. Let $n_0 = \max\{n_1, n_2\}$, then for every $n \ge n_0$, $x_n \in \lambda'^{-1}(V_1)$ and $x_n' \in \lambda'^{-1}(V_2)$. Thus $y_n = \lambda'(x_n) = \lambda'(x_n') \in V_1 \cap V_2$, which contradicts $V_1 \cap V_2 = \emptyset$.

Thus (8) is completely proved.

Finally, for every $n \in \mathbf{N}$ let $\mathcal{G}_n = \{G = Y' \backslash \lambda'(X' \backslash U) : U \in \mathcal{U}_n\}$. Then

(1′) For every $y \in Y_b$ and $n \in \mathbf{N}$ there exists $U \in \mathcal{U}_n$ such that $\lambda^{-1}(y) \subset U$, then $y \in G = Y' \backslash \lambda'(X' \backslash U) \in \mathcal{G}_n$ and hence, $Y_b \subset \bigcup \mathcal{G}_n$;

(2′) Since $\lambda'^{-1}(\mathcal{G}_n)$ is a refinement of $\mathcal{U}_n$, for every $y \in Y_b$,

$$\lambda'^{-1}((\textstyle\bigcap_{n \in \mathbf{N}} st(y, \mathcal{G}_n)) \cap Y'_b) = (\textstyle\bigcap_{n \in \mathbf{N}} \lambda'^{-1}(st(y, \mathcal{G}_n))) \cap X'_b$$

$$\subset (\textstyle\bigcap_{n \in \mathbf{N}} st(\lambda'^{-1}(y), \mathcal{U}_n))) \cap X'_b \subset X_b.$$

Hence $(\bigcap_{n \in \mathbf{N}} st(y, \mathcal{G}_n)) \cap Y'_b \subset Y_b$.

Consequently, from (1′), (2′) and Theorem 3.2, g is a p-map. Since paracompactness is preserved by closed maps ([5] Theorem 2.11), g is a paracompact p-map. □

In connection with Theorem 5.2, note that if f is not paracompact, the result does not necessarily hold. For this, consider the case when B is the singleton set and [3] Example 2.1.

6. Metrizable Type (MT-)maps and p-maps

In this section, we investigate the relations of MT-maps with (paracompact) M-maps and some problems analogous to those encountered in the relations of metrizable spaces with (paracompact) M-spaces.

THEOREM 6.1. *Suppose that B is regular. If a T_2-compactifiable map $f : X \to B$ has an f-development, then it is a p-map.*

PROOF. Since f has an f-development, for every $b \in B$ there is a sequence $\{\mathcal{U}_n\}_{n \in \mathbf{N}}$ of open (in X) covers of X_b which is a b-development. For every $n \in \mathbf{N}$ and $x \in X_b$ take $U_x \in \mathcal{U}_n$ and $V_x \in N(x)$ such that $x \in V_x \subset \overline{V}_x \subset U_x$. Let $\mathcal{V}_n = \{V_x \mid x \in X_b\}$ and $\mathcal{V} = \{\mathcal{V}_n\}_{n \in \mathbf{N}}$. For $n \in \mathbf{N}$ and $x \in X_b$, if $x \in V_n \in \mathcal{V}_n$, there exists $U_n \in \mathcal{U}_n$ with $x \in V_n \subset \overline{V_n} \subset U_n$, so that $(\bigcap_{n \in \mathbf{N}} \overline{V_n}) \cap X_b \subset (\bigcap_{n \in \mathbf{N}} U_n) \cap X_b$. If there exists $x_0 \in (X_b \backslash \{x\}) \cap (\bigcap_{n \in \mathbf{N}} \overline{V_n})$, then $x \in X \backslash \{x_0\}$ and $x_0 \in U_n$ for every $n \in \mathbf{N}$. Since $\{\mathcal{U}_n\}_{n \in \mathbf{N}}$ is a b-development, there exists $n_0 \in \mathbf{N}$ and $W \in N(b)$ such that $st(x, \mathcal{U}_{n_0}) \cap X_W \subset X \backslash \{x_0\}$, which is a contradiction. Thus $(\bigcap_{n \in \mathbf{N}} \overline{V_n}) \cap X_b = \{x\}$ is compact. From the definition of b-development, for every open subset U of X with $(\bigcap_{n \in \mathbf{N}} \overline{V_n}) \cap X_b = \{x\} \subset U$, there exist $n \in \mathbf{N}$ and $W \in N(b)$ such that $\{x\} \in st(x, \mathcal{U}_n) \cap X_W \subset U$ and therefore, $(\bigcap_{n \in \mathbf{N}} \overline{V_n}) \cap X_b = \{x\} \in (\bigcap_{i \leq n} \overline{V_i}) \cap X_W \subset U$. Hence f is a p-map. □

COROLLARY 6.2. *If B is regular then every MT-map $f : X \to B$ is a paracompact p-map.*

Corollary 6.3. *Suppose that B is regular. Let $f: X \to B$ and $g: Y \to B$ be maps and $\lambda: f \to g$ a perfect morphism. If g is an MT-map, then f is a paracompact p-map (and therefore, an M-map).*

For two maps $f: X \to B$ and $g: Y \to B$, f is said to be (resp. *closedly*) *embeddable* to g if there exists a morphism $\lambda: f \to g$ such that $\lambda(X)$ is a (resp. closed) subspace of Y.

We now cite two problems related to (paracompact) M-maps and paracompact p-maps, that are analogous to results pertaining to (paracompact) M-spaces ([11]) and paracompact p-spaces ([13]).

Problem 6.4. Let $f: X \to B$ be an M-map (resp. paracompact M-map). Does there exist an MT-map $g: Y \to B$ and a quasi-perfect (resp. perfect) morphism $\lambda: f \to g$?

In this case, we call f the *preimage-map* of g under λ.

Problem 6.5. Let $f: X \to B$ be a paracompact p-map. Can f be closedly embeddable to a product of an MT-map and a compact map?

The next theorem is a partial answer of Problem 6.5. It follows from this theorem that if Problem 6.4 is affirmative, then so is Problem 6.5.

Theorem 6.6. *Let $f: X \to B$ be a map that is a preimage-map of an MT-map $g: Y \to B$ under a perfect morphism $\lambda: f \to g$. Then f is closedly embeddable to a product of g and a T_2-compactification $f': X' \to B$ of f.*

Proof. First, since the MT-map g is a paracompact T_2-map, it follows from [4] Proposition 4.4 that f is a paracompact T_2-map, and therefore f has a T_2-compactification $f': X' \to B$. Let $\mu = \lambda\Delta_B e: X \to Y \times_B X'$ be the map defined by $\mu(x) = (\lambda(x), e(x))$, where $e: X \to X'$ is the fibrewise embedding. Then $\mu = (\lambda \times_B id_{X'}) \circ (id_X \Delta_B e) : X \to X \times_B X' \to Y \times_B X'$ is one-to-one. We now prove that $Z = (id_X \Delta_B e)(X)$ is closed in $X \times_B X'$. Let $(x, x') \in (X \times_B X') \setminus Z$, then $e(x) \neq x'$ and $f(x) = f'(x')$. Since $f'(e(x)) = f'(x')$, there exist $U \in N(e(x))$ and $V \in N(x')$ in X' such that $U \cap V = \emptyset$. Then it is easy to see that $e^{-1}(U) \times_B V$ is a nbd of (x, x') satisfying $(e^{-1}(U) \times_B V) \cap Z = \emptyset$. Consequently, Z is closed in $X \times_B X'$. Since λ and $id_{X'}$ are perfect, $\lambda \times_B id_{X'}$ is perfect, and therefore $(\lambda \times_B id_{X'}) | Z$ is perfect. Thus $\mu(X)$ is closed in $Y \times_B X'$. $\square$

Acknowledgement

The authors would like to thank the referee for careful reading and valuable advice.

References

[1] Arhangel'skiĭ, A. V., On a class of spaces containing all metric spaces and all locally bicompact spaces, Sov. Math. Dokl., **4** (1963), 751–754.
[2] Bai, Y. F. and Miwa, T., Čech-complete maps, Glasnik Matematǎčki, **43** (63) (2008), 219–229.
[3] Balogh, Z. and Burke, D. K., Two results an spaces with a sharp base, Topology Appl., **154** (2007), 1281–1285.
[4] Buhagiar, D., Paracompact maps, Q & A in General Topology, **15** (1997), 203–223.
[5] Buhagiar, D. and Miwa, T., Covering properties on maps, Q & A in General Topology, **16** (1998), 53–66.
[6] Buhagiar, D., Miwa, T. and Pasynkov, B. A., On metrizable type (MT-)maps and spaces, Topology and its Appl., **96** (1999), 31–51.
[7] Engelking, R., General Topology, Heldermann, Berlin, rev. ed., 1989.
[8] Filippov, V. V., On the perfect image of a paracompact p-space, Dokl. Akad. Nauk. SSSR, **176** (1967); Soviet Math. Dokl., **8** (1967), 1151–1153.
[9] Gruenhage, G., Generalized metric spaces, in: (K. Kunen and J. E. Vaughan eds.) Handbook of Set-Theoretic Topology, North-Holland, (1984), 423–501.
[10] James, I. M., Fibrewise Topology, Cambridge Univ. Press, Cambridge, 1989.
[11] Morita, K., Products of normal spaces with metric spaces, Math. Ann., **154** (1964), 365–382.
[12] Morita, K., A survey of the theory of M-spaces, Gen. Topology Appl., **1** (1971), 49–55.
[13] Nagata, J., A note on M-spaces and topologically complete spaces, Proc. Japan Acad., **45** (1969), 541–543.
[14] Pasynkov, B. A., On the extension to maps of certain notions and assertions concerning spaces, in: Maps and Functors, Moscow State Univ., Moscow, (1984), 72–102. (Russian)
[15] Pasynkov, B. A., Elements of the general topology of continuous maps, in: (D. K. Musaev and B. A. Pasynkov) On Compactness and Completeness Properties of Topological Spaces, Tashkent, "FAN" Acad. of Sci. Uzbekistan Rep., (1994), 50–120. (Russian)
[16] Wainstein, I. A., On closed maps of metric spaces, Dokl. Acad. Sci. USSR **57** (1947), 319–321. (Russian)

Yun-Feng Bai
Department of Mathematics
Shimane University
Matsue 690-8504, Japan
(Capital Normal University, Beijing, China)
E-mail address: yuntingbai@126.com

Takuo Miwa
Department of Mathematics
Shimane University
Matsue 690-8504, Japan
E-mail address: miwa@riko.shimane-u.ac.jp