TSUKUBA J. MATH.
Vol. 26 No. 2 (2002), 339–349

S^4-FORMULA AND S^2-FORMULA FOR QUASI-TRIANGULAR BIFROBENIUS ALGEBRAS

By

Norifumi YAMATANI

Abstract. We introduce the notion of a quasi-triangular biFrobenius algeba and prove the S^2-formula and S^4-formula.

Introduction

The notion of a quasi-triangular Hopf algebra was introduced by Drinfel'd [Dr1] in 1986, playing a basic role in quantum group theory. He has further proved in [Dr2] that the square of the antipode S of a quasi-triangular Hopf algebra H is inner by the so-called Drinfel'd element u and that S^4 is inner by $g = u(Su)^{-1}$ which is group-like. He has also expressed g in terms of the modular element and the modular function when H is finite-dimensional, by using Radford's S^4-fomula [R1]. This expression was also proved by Radford [R2] independently. See also [Mo], Chapter 10.

Doi and Takeuchi [DT] have recently introduced the notion of a biFrobenius algebra as natural generalization of a finite-dimensional Hopf algebra. Among other things, they have proved an analogue of Radford's S^4-formula [DT], Theorem 3.6. It seems quite interesting and meaning to extend the notion of a quasi-triangular Hopf algebra to this new notion. In this paper, we formulate and study quasi-triangular biFrobenius algebra and proved analogues of the S^2- and S^4-formulas.

Throughout this paper we work over a field k. For any algebra A, the twist map: $H \otimes H \to H \otimes H$ $(h \otimes g \mapsto g \otimes h)$ is denoted by τ. And the dual space $A^* = \mathrm{Hom}(A, k)$ is a A-bimodule via

$$\langle x \rightharpoonup f, y \rangle = \langle f, yx \rangle, \langle f \leftharpoonup x, y \rangle = \langle f, xy \rangle \quad \text{for } x, y \in A, f \in A^*$$

For any coalgebra C, we denote the comultiplication $\Delta : C \to C \otimes C$ by $c \mapsto \sum c_1 \otimes c_2$. The dual space C^* becomes an algebra under the convolution product

Received March 12, 2001.
Revised June 18, 2001.

$$\langle f * g, c\rangle = \sum \langle f, c_1\rangle\langle g, c_2\rangle \quad \text{for } c \in C, f, g \in C^*.$$

C is a C^*-bimodule via

$$f \rightharpoonup c = \sum c_1\langle f, c_2\rangle, c \leftharpoonup f = \sum c_2\langle f, c_1\rangle \quad \text{for } c \in C, f \in C^*.$$

1. BiFrobenius Algebras

We recall some properties of biFrobenius algebras [DT] which are neccessary for our purposes.

Let A be a finite dimensional algebra and $\phi \in A^*$. The pair (A, ϕ) is called *a Frobenius algebra* if $A^* = \phi \leftharpoonup A$. The element ϕ is called *a Frobenius basis.* Since the bilinear form $A \times A \to k$, $(x, y) \mapsto \langle \phi, xy\rangle$ is non-degenerate, there exists a unique element $\sum a_i \otimes b_i$ of $A \otimes A$ which is called *the dual basis for* ϕ such that $x = \sum a_i\langle \phi, b_i y\rangle$ for ${}^\forall x \in A$. For a Frobenius algebra (A, ϕ), *the Nakayama automorphism* N is defined by

$$\langle \phi, xy\rangle = \langle \phi, yN(x)\rangle \quad \text{for } {}^\forall x, y \in A.$$

Given an augumentation $\varepsilon : A \to k$ for a Frobenius algebra (A, ϕ), we put

$$\mathrm{I}_r(A) = \{t \in A \mid tx = t\varepsilon(x), {}^\forall x \in A\},$$

which is called *the space of right integrals in* A. Taking $t \in A$ such that $\phi \leftharpoonup t = \varepsilon$, we have $\mathrm{I}_r(A) = kt$. Since xt is also a right integral for all $x \in A$, there exists $\alpha \in \mathrm{Alg}(A, k)$ such that $xt = \alpha(x)t$. We call α *the right modular function for* A.

We dualize this idea. Let C be a finite dimensional coalgebra and $t \in C$. The pair (C, t) is called *a Frobenius coalgebra* if $C = t \leftharpoonup C^*$. For a Frobenius coalgebra (C, t), there exists a unique coalgebra automorphism ${}^cN$ such that

$$\sum t_1 \otimes t_2 = \sum {}^cN(t_2) \otimes t_1.$$

This automorphism is called *the coNakayama automorphism.* We consider a Frobenius coalgebra (C, t) with group-like element 1_C. Then the dual algebra C^* has an augumentation $C^* \to k$, $f \mapsto \langle f, 1_C\rangle$. We have

$$\mathrm{I}_r(C^*) = \{\phi \in C^* \mid \sum\langle \phi, c_1\rangle c_2 = \langle \phi, c\rangle 1_C, {}^\forall c \in C\}.$$

Taking $\phi \in C^*$ such that $t \leftharpoonup \phi = 1_C$, we have $\mathrm{Ir}(C^*) = k\phi$. There exists a group-like element a such that

$$\sum x_1\langle \phi, x_2\rangle = \langle \phi, x\rangle a \quad \text{for } {}^\forall x \in C \text{ and } {}^\forall f \in C^*.$$

We call a *the* (*right*) *modular element for* C.

Let H be a finite dimensional algebra and coalgebra, $t \in H$ and $\phi \in H^*$. Suppose we have:

(BF1) The unit 1_H is a group-like element,

(BF2) The counit ε_H is an algebra map,

(BF3) (H, ϕ) is a Frobenius algebra,

(BF4) (H, t) is a Frobenius coalgebra.

Define a map $S : H \to H$ by the formula:

$$S(h) = \sum \langle \phi, t_1 h \rangle t_2 \quad \text{for } {}^\forall h \in H. \tag{1.1}$$

DEFINITION 1.1. Let H be a finite dimensional algebra and coalgebra. Let $t \in H$ and $\phi \in H^*$. Define $S : H \to H$ as above. The 4-tuple (H, ϕ, t, S) is called *a biFrobenius algebra* (or *bF-algebra*) if we have (BF1–4) and,

(BF5) S is an anti-algebra map,

(BF6) S is an anti-coalgebara map.

S is called *the antipode* of the bF-algebra H. The map S is bijective, and we denote its composit inverse by $\bar{S}$. Since $\varepsilon \circ S = \varepsilon$ and $S(1) = 1$, it follows that $\phi \leftharpoonup t = \varepsilon$ and $t \leftharpoonup \phi = 1$. Hence t and ϕ are right integrals. The dual basis for ϕ is given by $\sum \bar{S}(t_2) \otimes t_1$, and we have

$$h = \sum \bar{S}(t_2) \langle \phi, t_1 h \rangle, \quad {}^\forall h \in H, \qquad \text{([DT], 3.1 Proposition)} \tag{1.2}$$

$$\sum \langle \phi, x_1 y \rangle x_2 = \sum \langle \phi, x y_1 \rangle S(y_2), \quad {}^\forall x, y \in H. \quad \text{([DT], 3.2 Proposition)} \tag{1.3}$$

The Nakayama automorphism N has the following expression:

$$N(h) = \sum \langle \phi, h_1 t \rangle \bar{S}^2(h_2) = \overline{S^2}(h \leftharpoonup \alpha), \quad {}^\forall h \in H. \qquad \text{(ibid.)} \tag{1.4}$$

The modular function α is $*$-invertible with $\alpha^{-1} = \alpha \circ \bar{S} = \alpha \circ S$ and $\alpha^{-1} : H \to H$ is an algebra map. The map $H \to H$, $h \mapsto h \leftharpoonup \alpha$ is an algebra automorphism ([DT], 3.3 Proposition). Dually, we have that the modular element a is invertible with $a^{-1} = S(a) = \bar{S}(a)$ (ibid.).

By [DT], 3.4 Proposition, we have

$$N(h) = a^{-1} S^2(\alpha \rightharpoonup h) a \quad \text{for } {}^\forall h \in H, \tag{1.5}$$

$${}^c N(h) = \bar{S}^2(ah) = \overline{S^2}(ah) \quad \text{for } {}^\forall h \in H, \tag{1.6}$$

$$\sum \bar{S}(t_2) \otimes t_1 = \sum S(t_1) a \otimes t_2. \tag{1.7}$$

Hence

$$\sum t_1 \otimes t_2 = \sum a \bar{S}^2(t_2) \otimes t_1. \tag{1.8}$$

We have

$$S^4(h) = a(\alpha^{-1} \rightharpoonup h \leftharpoonup \alpha)a^{-1} \quad \text{for } {}^\forall h \in H. \qquad \text{([DT], 3.6 Theorem)} \tag{1.9}$$

LEMMA 1.2. *We have*

$$\sum t_1 x \otimes t_2 = \sum t_1 \otimes t_2 S(x) \quad \textit{for all } x \in H. \tag{1.10}$$

PROOF. Applying $p \in H^*$ to both sides of (1.3), we have

$$\sum\langle\phi, x_1 y\rangle\langle p, x_2\rangle = \sum\langle\phi, x y_1\rangle\langle p, S(y_2)\rangle,$$

$$\sum\langle\phi_1, x_1\rangle\langle\phi_2, y\rangle\langle p, x_2\rangle = \sum\langle\phi_1, x\rangle\langle\phi_2, y_1\rangle\langle p, S(y_2)\rangle,$$

$$\sum\langle\phi_1 * p, x\rangle\langle\phi_2, y\rangle = \sum\langle\phi_1, x\rangle\langle\phi_2 * S^*(p), y\rangle,$$

$$\sum\langle\phi_1 * p \otimes \phi_2, x \otimes y\rangle = \sum\langle\phi_1 \otimes \phi_2 * S^*(p), x \otimes y\rangle,$$

Since above formula holds for arbitrary x and y of H,

$$\sum \phi_1 * p \otimes \phi_2 = \sum \phi_1 \otimes \phi_2 * S^*(p), \quad {}^\forall p \in H^*$$

Dualizing this, we have

$$\sum t_1 x \otimes t_2 = \sum t_1 \otimes t_2 S(x), \quad {}^\forall x \in H.$$

2. Quasi-Triangular BiFrobenius Algebra

Let (H, ϕ, t, S) be a biFrobenius algebra. Let $R \in H \otimes H$ be an invertible element.

DEFINITION 2.1. The pair (H, R) is called *a quasi-triangular biFrobenius algebra* if

(QTB 0) $R^{-1} = (S \otimes id)(R) = (id \otimes \bar{S})(R)$,
(QTB 1) $(\Delta \otimes id)(R) = R^{13}R^{23}$,
(QTB 2) $(id \otimes \Delta)(R) = R^{13}R^{12}$,
(QTB 3) $\Delta^{\mathrm{OP}}(x)R = R\Delta(x)$ for all $x \in H$,

where $R^{13} = \sum R^{(1)} \otimes 1 \otimes R^{(2)}$, $R^{12} = \sum R^{(1)} \otimes R^{(2)} \otimes 1$, $R^{23} = \sum 1 \otimes R^{(1)} \otimes R^{(2)}$ with $R = \sum R^{(1)} \otimes R^{(2)}$ and $\Delta^{\mathrm{OP}} = \tau \circ \Delta$.

From definitions it follows that

$$(\varepsilon \otimes id)(R) = (id \otimes \varepsilon)(R) = 1 \quad \text{and} \quad (S \otimes S)(R) = R.$$

Indeed

$$\begin{aligned} R &= (\varepsilon \otimes id \otimes id)(\Delta \otimes id)(R) \\ &= (\varepsilon \otimes id \otimes id) R^{13} R^{23} && \text{(by (QTB1))} \\ &= \sum \varepsilon(R^{(1)}) r^{(1)} \otimes R^{(2)} r^{(2)} && \text{(where } R = r) \\ &= (1 \otimes \varepsilon(R^{(1)}) R^{(2)}) R. \end{aligned}$$

Hence

$$(\varepsilon \otimes id)(R) = 1$$

Similarly

$$(id \otimes \varepsilon)(R) = 1.$$

We have

$$\begin{aligned} (S \otimes S)(R) &= (id \otimes S)(S \otimes id)(R) = (id \otimes S)(R^{-1}) \\ &= (id \otimes S)(id \otimes \bar{S})(R) = R \end{aligned}$$

There are shown in the same way as usual quasi-triangular Hopf algebras. The next lemma is an analogue of [R2], Proposition 3.

Lemma 2.2. *Let (H, R) be a quasitriangular biFrobenius algebra with $R = \sum R^{(1)} \otimes R^{(2)}$. For any $\eta \in H^*$, set $g_\eta = \sum R^{(1)} \eta(R^{(2)})$. Then*

(a) *If $\eta \in G(H^*) = \mathrm{Alg}(H, k)$, g_η is a group-like element.*

(b) *The linear map: $H^* \to H$, $\eta \mapsto g_\eta$ is an anti-algebra map.*

(c) *$(x \leftharpoonup \eta) g_\eta = g_\eta (\eta \rightharpoonup x)$ for all $x \in H$, $\eta \in G(H^*)$.*

Proof. (a)

$$\begin{aligned} \Delta(g_\eta) &= \Delta\left(\sum R^{(1)} \eta(R^{(2)})\right) \\ &= \sum (R^{(1)})_1 \otimes (R^{(1)})_2 \eta(R^{(2)}) \\ &= \sum R^{(1)} \otimes r^{(1)} \eta(R^{(2)} r^{(1)}) && \text{(where } R = r) \\ &= \sum R^{(1)} \eta(R^{(2)}) \otimes r^{(1)} \eta(r^{(2)}) \\ &= g_\eta \otimes g_\eta. \end{aligned}$$

$$\begin{aligned} \varepsilon(g_\eta) &= \varepsilon\left(\sum R^{(1)} \eta(R^{(2)})\right) \\ &= \eta\left(\sum \varepsilon(R^{(1)}) R^{(2)}\right) = 1. \end{aligned}$$

(b) Follows from (QTB2).

(c) Applying $id \otimes \eta$ to both sides of (QTB3), we have

$$\sum x_2 R^{(1)} \langle \eta, x_1 R^{(2)} \rangle = \sum R^{(1)} x_1 \langle \eta, R^{(2)} x_2 \rangle$$
$$(x \leftharpoondown \eta) g_\eta = g_\eta (\eta \rightharpoonup x). \qquad \square$$

Denote the right modular function by $\alpha \in \mathrm{Alg}(H, k)$ and the modular element by a,

$$xt = \alpha(x)t, \sum x_1 \langle \phi, x_2 \rangle = \sum \langle \phi, x \rangle a \quad \text{for } {}^\forall x \in H.$$

LEMMA 2.3. (a) $S(g_\alpha) = g_{\alpha^{-1}} = (g_\alpha)^{-1}$.
(b) $(\alpha^{-1} \rightharpoonup x \leftharpoondown \alpha) = g_\alpha x g_\alpha^{-1}$ *for* ${}^\forall x \in H$.
(c) $S^4(x) = (a g_\alpha) x (a g_\alpha)^{-1}$ *for* ${}^\forall x \in H$.

PROOF. (a)

$$\begin{aligned} S(g_\alpha) &= \sum S(R^{(1)}) \alpha(R^{(2)}) \\ &= \sum R^{(1)} \alpha \circ \bar{S}(R^{(2)}) && \text{(by (QTB0))} \\ &= \sum R^{(1)} \alpha^{-1}(R^{(2)}) \\ &= g_{\alpha^{-1}}. \end{aligned}$$

By Lemma 2.2 (b), we have

$$g_\alpha g_{\alpha^{-1}} = g_{\alpha^{-1}\alpha} = 1,$$

hence

$$g_{\alpha^{-1}} = (g_\alpha)^{-1}.$$

(b) By lemma 2.2 (c), we have

$$(x \leftharpoondown \alpha) g_\alpha = g_\alpha (\alpha \rightharpoonup x)$$

Applying the algebra automorphism $\alpha^{-1} \rightharpoonup$, we have

$$(\alpha^{-1} \rightharpoonup x \leftharpoondown \alpha)(\alpha^{-1} \rightharpoonup g_\alpha) = (\alpha^{-1} \rightharpoonup g_\alpha) x$$
$$(\alpha^{-1} \rightharpoonup x \leftharpoondown \alpha) g_\alpha \alpha^{-1}(g_\alpha) = \alpha^{-1}(g_\alpha) g_\alpha x$$

Since g_α is invertible, $\alpha^{-1}(g_\alpha) \neq 0$. Hence by (a),

$$(\alpha^{-1} \rightharpoonup x \leftharpoonup \alpha) = g_\alpha x g_\alpha^{-1}$$

(c)
$$\begin{aligned} S^4(x) &= a(\alpha^{-1} \rightharpoonup x \leftharpoonup \alpha)a^{-1} && \text{(by (1.9))} \\ &= a g_\alpha x g_\alpha a^{-1} \\ &= (a g_\alpha) x (a g_\alpha)^{-1} \end{aligned}$$ □

Set

$$u = \sum S(R^{(2)})R^{(1)}.$$

This element is called *the Drinfel'd element.* In quasi-triangular Hopf algebras u plays important roles. We show that u also plays the same role in quasi-triangular biFrobenius algebras.

At first we show fundamental properties of u.

Proposition 2.4. *Let (H, R) be a quasitriangular biFrobenius algebra with $R = \sum R^{(1)} \otimes R^{(2)}$. Set $u = \sum S(R^{(2)})R^{(1)}$.*

(a) *$S^2(x)u = ux$ for $^\forall x \in H$.*

(b) *u is invertible, with inverse given by $u^{-1} = \sum R^{(2)}S^2(R^{(1)})$.*

Proof. (a)

$$\begin{aligned} S^2(x)u &= \sum \langle \phi, S^2(x)S(R^{(2)})\bar{S}(t_2)\rangle t_1 R^{(1)} && \text{(by (1.2))} \\ &= \sum \langle \phi, S^2(x)S(R^{(2)})S(t_1)a\rangle t_2 R^{(1)} && \text{(by (1.7))} \\ &= \sum \langle \phi, S(t_1 R^{(2)} S(x))a\rangle t_2 R^{(1)} \\ &= \sum \langle \phi, S(R^{(2)} t_2 S(x))a\rangle R^{(1)} t_1 && \text{(by (QTB3))} \\ &= \sum \langle \phi, S(R^{(2)} t_2)a\rangle R^{(1)} t_1 x && \text{(by (1.10))} \\ &= \sum \langle \phi, S(t_1 R^{(2)})a\rangle t_2 R^{(1)} x && \text{(by (QTB3))} \\ &= \sum \langle \phi, S(R^{(2)})S(t_1)a\rangle t_2 R^{(1)} x \\ &= \sum \langle \phi, S(R^{(2)})\bar{S}(t_2)\rangle t_1 R^{(1)} x && \text{(by (1.7))} \\ &= ux && \text{(by (1.2))} \end{aligned}$$

(b) Put $u' = \sum R^{(2)}S(R^{(1)})$

$$
\begin{aligned}
uu' &= \sum uR^{(2)}S^2(R^{(1)}) \\
&= \sum S^2(R^{(2)})uS^2(R^{(1)}) && \text{(by (a))} \\
&= \sum S^2(R^{(2)})S(r^{(2)})r^{(1)}S^2(R^{(1)}) && \text{(where } R = r) \\
&= \sum S(r^{(2)}S(R^{(2)}))r^{(1)}S^2(R^{(1)}) \\
&= \sum S(r^{(2)}R^{(2)})r^{(1)}S(R^{(1)}) && \text{(since } (S\otimes S)(R) = R)) \\
&= S(1)1 = 1 && \text{(by (QTB0)}
\end{aligned}
$$

Similaly

$$u'u = 1. \qquad \square$$

LEMMA 2.5. (a) $S(u) = \sum a^{-1}S^2(\alpha \rightharpoonup R^{(2)})S(R^{(1)})$
(b) $aS(u) = \sum R^{(2)}\bar{S}(R^{(1)} \leftharpoonup \alpha)$

PROOF. (a) We have

$$
\begin{aligned}
\sum\langle\phi, t_1R^{(2)}\rangle t_2R^{(1)} &= \sum\langle\phi, t_1\rangle t_2S(R^{(2)})R^{(1)} && \text{(by (1.10))} \\
&= (t \leftharpoonup \phi)u \\
&= u. && \text{(since } t \leftharpoonup \phi = 1)
\end{aligned}
$$

Applying S to the above result,

$$
\begin{aligned}
S(u) &= \sum\langle\phi, t_1R^{(2)}\rangle S(t_2R^{(1)}) \\
&= \sum\langle\phi, R^{(2)}t_2\rangle S(R^{(1)}t_1) && \text{(by (QTB3))} \\
&= \sum\langle\phi, R^{(2)}t_2\rangle S(t_1)S(R^{(1)}) \\
&= \sum\langle\phi, R^{(2)}t_2\rangle S(t_1)aa^{-1}S(R^{(1)}) \\
&= \sum\langle\phi, R^{(2)}t_1\rangle \bar{S}(t_2)a^{-1}S(R^{(1)}) && \text{(by (1.7))} \\
&= \sum \bar{S}^2(\langle\phi, R^{(2)}t_1\rangle S(t_2))a^{-1}S(R^{(1)}) \\
&= \sum\langle\phi, (R^{(2)})_1t\rangle \bar{S}^2((R^{(2)})_2)a^{-1}S(R^{(1)}) && \text{(by (1.3))} \\
&= \sum \bar{S}^2(R^{(2)} \leftharpoonup \alpha)a^{-1}S(R^{(1)}),
\end{aligned}
$$

hence by (1.4) and (1.5)

$$S(u) = \sum a^{-1} S^2(R^{(2)} \rightharpoonup \alpha) S(R^{(1)}).$$

(b) We have

$$\begin{aligned}
\sum \langle \phi, t_2 R^{(1)} \rangle t_1 R^{(2)} &= \sum \langle \phi, t_2 S(\bar{S}(R^{(1)})) \rangle t_1 R^{(2)} \\
&= \sum \langle \phi, t_2 \rangle t_1 \bar{S}(R^{(1)}) R^{(2)} && \text{(by (1.10))} \\
&= (\phi \rightharpoonup t) \bar{S}(S(R^{(2)}) R^{(1)}) \\
&= a\bar{S}(u) = aS(u),
\end{aligned}$$

since $a = \phi \rightharpoonup t$ and $S^2(u) = u$.

Therefore

$$\begin{aligned}
aS(u) &= \sum \langle \phi, t_2 R^{(1)} \rangle t_1 R^{(2)} \\
&= \sum \langle \phi, R^{(1)} t_1 \rangle R^{(2)} t_2 && \text{(by (QTB3))} \\
&= \sum R^{(2)} \bar{S}(\langle \phi, R^{(1)} t_1 \rangle S(t_2)) \\
&= \sum R^{(2)} \bar{S}(\langle \phi, (R^{(1)})_1 t \rangle (R^{(1)})_2) \\
&= \sum R^{(2)} \bar{S}(\alpha((R^{(1)})_1)(R^{(1)})_2) && \text{(by (1.3))} \\
&= \sum R^{(2)} \bar{S}(R^{(1)} \leftharpoonup \alpha). && \square
\end{aligned}$$

The following theorem gives a biFrobenius anologue of Drinfeld's S^4-fomula [Dr2] (see also [R2], Theorem 2 and [Mo], 10.1.13 Theorem.)

THEOREM 2.6. *Let (H, R) be a quasi-triangular biFrobenius algebra. Set $u = S(R^{(2)})R^{(1)}$ where $R = \sum R^{(1)} \otimes R^{(2)}$. Set $g_\alpha = \sum R^{(1)} \alpha(R^{(2)})$. Then*

(a) $ag_\alpha = uS(u)^{-1}$,

(b) $S^4(x) = uS(u)^{-1} x u^{-1} S(u)$ *for any* $x \in H$.

PROOF. (a) By Lemma 2.5 (a),

$$\begin{aligned}
aS(u) &= \sum S^2((R^{(1)})_1 \alpha((R^{(2)})_2)) S(R^{(1)}) \\
&= \sum S^2(r^{(2)} \alpha(R^{(2)})) S(R^{(1)} r^{(1)}) && \text{(by (QTB2))} \\
&= \sum S^2(r^{(2)})) S(r^{(1)}) S(R^{(1)} \alpha(R^{(2)})) && \text{(since } (S \otimes S)(R) = R)
\end{aligned}$$

$$= \sum S(r^{(2)})r^{(1)}S(R^{(1)}\alpha(R^{(2)})) \qquad \text{(where } R = r)$$
$$= uS(g_\alpha)$$
$$= ug_\alpha^{-1}.$$

Since a and g_α are S^4-stable, they commute with u and $S(u)$.

Hence we have

$$ag_\alpha = uS(u)^{-1}.$$

(b) Follos from (a) and Lemma 2.3 (c). □

REMARK 2.7 (cf. [R3], Proposition 3). Let (H, R) be a biFrobenius algebra. Define a map $\Phi_R : H^* \to H$ by $p \mapsto (p \otimes id)R^{21}R$.

Let ε be a counit and α the right modular function.

Then

$$\Phi_R(\varepsilon) = \Phi_R(\alpha) = 1.$$

Indeed

$$\Phi_R(\varepsilon) = \sum\langle \varepsilon, R^{(2)}r^{(1)}\rangle R^{(1)}r^{(2)} \qquad \text{(where } R = r)$$
$$= \sum \varepsilon(R^{(2)})R^{(1)}\varepsilon(r^{(1)})r^{(2)}$$
$$= 1.$$

The equality $\Phi_R(\alpha) = 1$ is shown as follows:

$$\Phi_R(\alpha) = \sum\langle \alpha, R^{(2)}r^{(1)}\rangle R^{(1)}r^{(2)}$$
$$= \sum \alpha(R^{(2)})R^{(1)}\alpha(r^{(1)})r^{(2)}$$
$$= g_\alpha \tilde{g}_\alpha$$

where $\tilde{g}_\alpha = \sum \alpha(r^{(1)})r^{(2)}$.

By Lemma 2.5 (b),

$$aS(u) = \sum R^{(2)}\bar{S}(\alpha((R^{(1)})_1)(R^{(1)})_2)$$
$$= \sum \alpha(R^{(1)})R^{(2)}r^{(2)}\bar{S}(r^{(2)}) \qquad \text{(by (QTB1))}$$
$$= \tilde{g}_\alpha u$$

On the other hand, we have proved that $aS(u) = ug_\alpha^{-1} = g_\alpha^{-1}u$ in Theorem 2.6 (a).

Hence we have $\tilde{g}_\alpha u = g_\alpha^{-1} u$. Since u is invertible, this implies $\tilde{g}_\alpha = g_\alpha^{-1}$. Therefore $\Phi_R(\alpha) = g_\alpha \tilde{g}_\alpha = 1$.

We call (H, R) *factorizable* if Φ_R is an isomorphism. It follows that factorizable qusitriangular biFrobenius algebras are unimodular.

References

[Drl] V. G. Drinfel'd, Quantum groups, in "Proceedings of the ICM, Berkeley, CA 1986", pp. 798–820.

[Dr2] V. G. Drinfeld, On almost cocommutative Hopf algebras, Leningrad Math. J. **1** (1990), 321–342.

[DT] Y. Doi and M. Takeuchi, BiFrobenius algebras, in "New Trends in Hopf Algebra Theory" Comtemp. Math. **267**, pp 67–97, Amer. Math. Soc., 2000.

[K] C. Kassel, Quantum groups, GTM **155**, Springer-Verlag, 1995.

[Ma] S. Majid, Foundationsof quantum group theory, Cambridge Univ. Press, 1995.

[Mo] S. Montgomery, Hopf algebras and their actions on rings, CBMS **82**, Amer. Math. Soc. 1993.

[R1] D. E. Radford, The order of the antipode of a finite-dimensional Hopf algebra is finite, Amer. J, Math. **98** (1976), 333–355.

[R2] D. E. Radford, On the Antipode of a Quasitriangular Hopf algebra, Journal of Algebra **151** (1992), 1–11.

[R3] D. E. Radford, On Kauffman's knot invariants arising from finite-dimensional Hopf algebras, in "Advances in Hopf algebras", L. N. in Pure and App. Math. **158**, pp. 205–266, Marcel Dekker Inc., 1994.

Institude of Mathematics
University of Tsukuba
Tsukuba 305-8571
Japan