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ON MINIMAL SPANNING SYSTEMS OVER SEMIPERFECT RINGS

By

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A ring A is called semiperfect in case $A/\mathrm{rad}\,A$ is semisimple and idempotents lift modulo $\mathrm{rad}\,A$, or equivalently, every finitely generated right (resp. left) A-module has a projective cover, which is uniquely determined up to A-isomorphism (Cf. Bass [**4**]). The main purpose of this paper is to refine a version of Warfield [**11**] concerning Auslander-Bridger duality. (Cf. [**2**] and [**3**])

In Section 1, we first define a minimal spanning system for a finitely generated right (resp. left) A-module M $(\neq 0)$, and show that these minimal spanning systems of M have the properties analogous to bases of a finite-dimensional vector space over a field.

To more exact description of minimal spanning systems of M, in Section 2 we shall use a restricted matrix theory over A which is called the fit matrix theory, and show that any minimal spanning system of M is obtained from the one by applying finitely many times of "elementary substitutions".

Next in Section 3, for a finitely presented non-projective right (resp. left) A-module M, we shall define a relation matrix R of M, and by means of R provide characterizations of the properties that $M\in\mathrm{mod}_P A$ (resp. $\mathrm{mod}_P A^{op}$) in the sense of Auslander and Reiten [**3**] (Cf. [**2**] and [**11**]), and that M is indecomposable.

Finally in Section 4, we shall consider the following condition:

(TSF) The number of all the isomorphism classes of "top-simple" right A-modules is finite.

Then we shall show that, in case A satisfies (TSF), A has only a finite number of two-sided ideals. It should be noted that representationfinite artinian rings satisfy (TSF).

Throughout this paper, A is a semiperfect ring and $\mathrm{rad}\,A$ denotes the Jacobson radical of A, and also e, f, e_i, f_j, g_k and h_l mean always primitive (and hence local) idempotents of A.

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1. Minimal spanning systems of finitely generated modules.

An element u in a right (resp. left) A-module M is called right (resp. left) local if $u=ue$ (resp. $u=eu$) for some e. Throughout this paper we shall treat only right (or left) local elements in M. A finite sequence consisting of right (resp. left) local elements in M will be always expressed in the form of a row (resp. column) vector.

Let $M(\neq 0)$ be a finitely generated right A-module. Then, without loss of generality, we can express a projective cover of M in the form: $\bigoplus_{i=1}^{m} e_iA \xrightarrow{p} M_A$ (Cf. Mueller [**9**]).

DEFINITION. In the above, keeping the order of indices, $(p(e_1), \cdots, p(e_m))$ is called a minimal spanning system (abbreviated m. s. s.) of M_A. Here m is uniquely determined by M_A, and so we define $m=\operatorname{rank} M_A$.

This denomination is justified by the considerations below:

DEFINITION. $(u_i=u_ie_i\in M_A \mid i=1, \cdots, m)$ is called a spanning system of M_A if $M=\sum_{i=1}^{m} u_iA$.

DEFINITION. $(u_i=u_ie_i\in M_A \mid i=1, \cdots, m)$ is called right A-linearly independent if the following condition is satisfied:

$$(*)\qquad (u_1, \cdots, u_m)\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}=0 \ \text{ with } \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}\in \bigoplus_{i=1}^{m} e_iA \Rightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}\in \bigoplus_{i=1}^{m} e_i(\operatorname{rad} A).$$

Then we have the next lemmas.

LEMMA 1.1. *Let $M_A(\neq 0)$ be finitely generated, and $u_i=u_ie_i (i=1, \cdots, m)$ elements of M. Then, $(u_1, \cdots, u_m)$ is an m.s.s. of M if and only if $(u_1, \cdots, u_m)$ is a spanning system of M and is right A-linearly independent.*

PROOF. Define a map $p: \bigoplus_{i=1}^{m} e_iA\to M_A$ by putting $p(e_i)=u_i (i=1, \cdots, m)$. Then p is a projective cover of M if and only if p is an epimorphism and $\operatorname{Ker} p\subset \bigoplus_{i=1}^{m} e_i(\operatorname{rad} A)$, which proves the lemma.

LEMMA 1.2. *Let $M_A(\neq 0)$ be finitely generated, and $(u_i=u_ie_i \mid i=1, \cdots, n)$ a spanning system of M. Then we can choose its subsequence $(u_{i_1}, \cdots, u_{i_m})$ $(m\leqq n)$ as an m.s.s. of M.*

PROOF. Casting out, in turn, redundant elements (as a spanning system of M) from $(u_1, \cdots, u_n)$, we get at last an irredundant spanning system $(u_{i_1}, \cdots, u_{i_m})$ of M. The irredundance of $(u_{i_1}, \cdots, u_{i_m})$ as a spanning system of M implies the right A-linear independence of $(u_{i_1}, \cdots, u_{i_m})$. Because, assume that $\sum_{k=1}^{m} u_{i_k}a_{i_k}=0$ with $a_{i_k}\in e_{i_k}A(k=1, \cdots, m)$, and further that $a_{i_1}\notin e_{i_1}(\mathrm{rad}\, A)$. Then, since $e_{i_1}(\mathrm{rad}\, A)$ is the unique maximal (proper) submodule of $e_{i_1}A$, we have $a_{i_1}A=e_{i_1}A$, and so there is an element b in Ae_{i_1} such that $a_{i_1}b=e_{i_1}$. Then we see $u_{i_1}=-\sum_{k=2}^{m} u_{i_k}a_{i_k}b$, which contradicts the irredundance of $(u_{i_1}, \cdots, u_{i_m})$. Therefore $(u_{i_1}, \cdots, u_{i_m})$ must be right A-linearly independent. Thus the proof is completed by Lemma 1.1.

Accordingly, for a spanning system $(u_1, \cdots, u_n)$ of M, it becomes an m.s.s. of M if and only if it is "minimal" as a spanning system of M, in a sense that any proper subsequence of it is no spanning system of M.

Lemmas 1.1 and 1.2 show also that an m. s. s. of M_A has the properties analogous to a basis of a finite-dimensional vector space over a field. However it is invalid that, to a given right A-linearly independent system $(u_i=u_ie_i\in M_A | i=1, \cdots, l)$, we may always get an m. s. s. of M by adding some elements in M.

EXAMPLE 1. Let A be the trivial extension of $\boldsymbol{R}$ by $\boldsymbol{C}$; $A=\boldsymbol{R}\ltimes\boldsymbol{C}$, where $\boldsymbol{R}$ and $\boldsymbol{C}$ denote respectively the field of real numbers and of complex numbers. Then A is a commutative local artinian ring, and $((0, 1), (0, i))$ is right A-linearly independent in A, but rank $A_A=1$.

More strongly than $(*)$, we may also define as follows:

DEFINITION. $(u_i=u_ie_i\in M_A | i=1, \cdots, m)$ is called right A-independent if the condition below is satisfied:

$$(u_1, \cdots, u_m)\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}=0 \quad \text{with} \quad \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}\in \bigoplus_{i=1}^{m} e_iA \Rightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}=\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

In this case, $\sum_{i=1}^{m} u_iA \approx \bigoplus_{i=1}^{m} e_iA$.

LEMMA 1.3. *Let $M_A(\neq 0)$ be finitely generated. Then M is projective if and only if an (and every) m.s.s. of M is right A-independent.*

PROOF. Trivial.

As for left A-modules, we need later similar definitions; e. g.

DEFINITION. Let ${}_AM(\neq 0)$ be finitely generated, and $\bigoplus_{i=1}^{m} Ae_i \xrightarrow{p} {}_AM$ a projective cover of M. Then, keeping the order of indices, $\begin{pmatrix} p(e_1) \\ \vdots \\ p(e_m) \end{pmatrix}$ is called an m.s.s. of ${}_AM$. In this case we further define $m=\operatorname{rank}_A M$.

DEFINITION. ${}^t(u_i=e_iu_i\in {}_AM \mid i=1, \cdots, m)$ is called left A-linearly independent if the following condition is satisfied:

$$(*')\qquad (a_1, \cdots, a_m)\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}=0 \quad \text{with} \quad (a_1, \cdots, a_m)\in \bigoplus_{i=1}^{m} Ae_i \Rightarrow$$

$$(a_1, \cdots, a_m)\in \bigoplus_{i=1}^{m} (\operatorname{rad} A)e_i .$$

2. Fit matrix theory.

In Sections 2 and 3, we shall treat only matrices of the restricted form, which is as follows:

(I) $m\times n$ matrices $P=(p_{ij})_{i,j}$ with $p_{ij}\in e_iAf_j$ for every (i, j), where m, n, $e_i (1\leqq i\leqq m)$ and $f_j (1\leqq j\leqq n)$ are arbitrarily variable.

(II) Matrix addition is defined only between matrices of the same type in the sense of (I); that is, $m\times n$ matrices $P=(p_{ij})_{i,j}$ with $p_{ij}\in e_iAf_j$ and $Q=(q_{ij})_{i,j}$ with $q_{ij}\in e_iAf_j$.

(III) Matrix multiplication is defined only between an $l\times m$ matrix $P=(p_{ij})_{i,j}$ with $p_{ij}\in e_iAf_j$ and an $m\times n$ matrix $Q=(q_{jk})_{j,k}$ with $q_{jk}\in f_jAg_k$. It should be noted that between P and Q common f_j's $(j=1, \cdots, m)$ appear in the same order. These products are sometimes called the fit products.

(IV) Scalar multiplication is not defined. However, an $m\times n$ matrix $P=(p_{ij})_{i,j}$ with $p_{ij}\in e_iAf_j$ is regarded either as $P\in \operatorname{Hom}_A\left(\bigoplus_{j=1}^{n} f_jA, \bigoplus_{i=1}^{m} e_iA\right)$ or as $P\in \operatorname{Hom}_A\left(\bigoplus_{i=1}^{m} Ae_i, \bigoplus_{j=1}^{n} Af_j\right)$. So, in spite of (I) and (III), we shall allow the (fit) products in the forms:

$$P\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}\in \bigoplus_{j=1}^{n} f_jA \quad \text{and} \quad (a_1, \cdots, a_m)P \quad \text{for} \quad (a_1, \cdots, a_m)\in \bigoplus_{i=1}^{m} Ae_i .$$

The matrix theory composed under the restrictions (I)-(IV) is called the fit matrix theory over A, which appeared partly in the literatures. (e.g. [**9**])

First of all we begin with the definition below.

DEFINITION. An $n\times n$ matrix $P=(p_{ij})_{i,j}$ with $p_{ij}\in e_iAf_j$ is called invertible if there exists an $n\times n$ matrix $Q=(q_{ji})_{j,i}$ with $q_{ji}\in f_jAe_i$ such that

$$PQ=\begin{pmatrix} e_1 & & \\ & \ddots & \\ & & e_n \end{pmatrix} \quad \text{and} \quad QP=\begin{pmatrix} f_1 & & \\ & \ddots & \\ & & f_n \end{pmatrix}.$$

In this case, Q is uniquely determined by P, and so we define $Q=P^{-1}$. Also, the diagonal matrices above appeared are called $n\times n$ identity matrices.

REMARK. Let an $n\times n$ matrix $P=(p_{ij})_{i,j}$ with $p_{ij}\in e_iAf_j$ be invertible, and assume that $e_iAf_j=g_iAh_j$ for every i and $j\,(i, j=1, \cdots, n)$. Then we have readily $GE=E$, $EG=G$, $FH=F$ and $HF=H$, where by E, F, G and H we denote respectively $n\times n$ identity matrices $(\delta_{ij}e_i)_{i,j}$, $(\delta_{ij}f_j)_{i,j}$, $(\delta_{ij}g_i)_{i,j}$ and $(\delta_{ij}h_j)_{i,j}$. Therefore, if Q is the inverse of $P=(p_{ij})_{i,j}$ with $p_{ij}\in e_iAf_j$, we see that HQG just becomes the inverse of $P=(p_{ij})_{i,j}$ with $p_{ij}\in g_iAh_j$.

Then we have readily the next.

LEMMA 2.1. *Assume that P and Q are invertible matrices and the product PQ is defined. Then PQ is invertible and* $(PQ)^{-1}=Q^{-1}P^{-1}$.

DEFINITION. An element $a\in eAf$ is called invertible if the 1×1 matrix (a) is invertible.

As is readily seen, $a\in eAf$ is invertible if and only if $a\in eAf\setminus e(\mathrm{rad}\,A)f$.

Now by the analogy of matrices over a field, we want to define elementary matrices.

DEFINITION. The three kinds of $n\times n$ matrices below are called elementary matrices, where $\varepsilon_{ij}(1\leqq i, j\leqq n)$ denote the ordinary matrix units and $f_i(1\leqq i\leqq n)$ are arbitrarily variable.

(EM 1) $\rho_n(j, k)=\sum_{i\neq j,k} f_i\varepsilon_{ii}+f_j\varepsilon_{jk}+f_k\varepsilon_{kj}\,(j\neq k)$ and its transpose ${}^t\rho_n(j, k)$.

(EM 2) $\delta_n(j\,;\,a)=\sum_{i\neq j} f_i\varepsilon_{ii}+a\varepsilon_{jj}$ with $a\in f_jAg_j\setminus f_j(\mathrm{rad}\,A)g_j$.

(EM 3) $\tau_n(j, k\,;\,a)=\sum_{i=1}^{n} f_i\varepsilon_{ii}+a\varepsilon_{jk}$ with $a\in f_jAf_k\,(j\neq k)$.

Obviously every elementary matrix is invertible and its inverse also becomes an elementary matrix.

For an $m\times n$ matrix $P=(p_{ij})_{i,j}$ with $p_{ij}\in e_iAf_j$, we can define elementary column transformations on P, which are induced by multiplications of elementary matrices (except ${}^t\rho_n(j, k)$) from the right. In particular, applying in turn elementary column transformations to P such that $P\not\equiv O \bmod (e_i(\mathrm{rad}\,A)f_j)_{i,j}$, we get at last the reduced column echelon form $\tilde{P}=(\tilde{p}_{ij})_{i,j}$; that is, there is an increasing sequence $1\leqq i_1<i_2<\cdots<i_r\leqq m(1\leqq r\leqq n)$ such that $\tilde{p}_{i_k,k}=e_{i_k}$, $\tilde{p}_{i_k,j}=0(j\neq k)$ for each $k(k=1,\cdots,r)$ and that $\tilde{p}_{ij}\in\mathrm{rad}\,A$ whenever (i,j) belongs to one of the following:

$$\{i<i_1,\ j\geqq 1\},\ \{i_{k-1}<i<i_k,\ j\geqq k\}(k=2,\cdots,r)\quad\text{and}\quad\{i>i_r,\ j\geqq r+1\}.$$

By using this fact we have the next.

PROPOSITION 2.2. *An invertible matrix is expressed as a product of a finite number of elementary matrices.*

PROOF. Under the same notations as above (together with $m=n$), let P be an invertible $n\times n$ matrix, and $\tilde{P}$ its reduced column echelon form. Then $\tilde{P}$ is expressed in the form:

$$\tilde{P}=PE_1\cdots E_t,$$

where $E_k(1\leqq k\leqq t)$ denote elementary matrices, and hence by Lemma 2.1 $\tilde{P}$ is an invertible matrix. On the other hand, $\tilde{P}$ must be the identity matrix $\begin{pmatrix} e_1 & & \\ & \ddots & \\ & & e_n\end{pmatrix}$; otherwise, $r<n$ and so $\tilde{p}_{ij}\in\mathrm{rad}\,A$ for every $(i,j)\in\{i\geqq 1,\ j\geqq r+1\}$, and consequently $\tilde{P}^{-1}\tilde{P}$ is no identity matrix, a contradiction. Thus $P=E_t^{-1}\cdots E_1^{-1}$, which proves the lemma.

Turn next our attention to finitely generated right A-modules.

DEFINITION. Let $M_A(\neq 0)$ be finitely generated, and let $(u_1,\cdots,u_m)$ with $u_i=u_ie_i(i=1,\cdots,m)$ be an m. s. s. of M. Then the three kinds of substitutions below are called elementary substitutions in M_A.

(ES 1) transposition: interchanging u_j and $u_k(j\neq k)$.

(ES 2) dilatation: replacing u_j by u_ja with $a\in e_iAg_j\setminus e_i(\mathrm{rad}\,A)g_j$.

(ES 3) transvection: replacing u_j by u_j+u_ka with $a\in e_kAe_j(j\neq k)$.

These are realized by multiplications of elementary matrices $\rho_m(j,k)$, $\delta_m(j;a)$ and $\tau_m(k,j;a)$ respectively from the right. Hence we obtain the following.

THEOREM 2.3. *Let $(u_1,\cdots,u_m)$ and $(v_1,\cdots,v_m)$ be given two m.s.s.'s of a finitely generated module M_A. Then the one is obtained from the other by applying finitely many times of elementary substitutions in M_A.*

PROOF. Assume that $(u_1,\cdots,u_m)$ and $(v_1,\cdots,v_m)$ are determined respectively by the projective covers $\overset{m}{\underset{i=1}{\oplus}} e_iA \xrightarrow{p} M_A$ and $\overset{m}{\underset{j=1}{\oplus}} f_jA \xrightarrow{p'} M_A$. Then by the uniqueness of projective covers of M there is a commutative diagram below:

$$\begin{array}{ccc} \overset{m}{\underset{i=1}{\oplus}} e_iA & \xrightarrow{\ \ p\ \ } & M_A \\ {\scriptstyle P}\Big\uparrow\wr & & \Big\| \\ \overset{m}{\underset{j=1}{\oplus}} f_jA & \xrightarrow{\ \ p'\ \ } & M_A, \end{array}$$

where $P=(p_{ij})_{i,j}$ with $p_{ij}\in e_iAf_j$ must be invertible. Since $p(e_i)=u_i$ and $p'(f_j)=v_j$, we have readily

$$(v_1, \cdots, v_m)=(u_1, \cdots, u_m)P.$$

Hence the theorem follows from Proposition 2.2.

Finally we shall come back to the fit matrix theory over A.

DEFININION. Let $R=(r_{ij})_{i,j}$ with $r_{ij}\in e_iAf_j$ be an $m\times n$ matrix $(\neq O)$ and set further $R=(\boldsymbol{r}_1, \cdots, \boldsymbol{r}_n)=\begin{pmatrix} \boldsymbol{s}_1 \\ \vdots \\ \boldsymbol{s}_m \end{pmatrix}$. Then we define respectively

$$\text{column rank}\, R=\text{rank} \sum_{j=1}^{n} \boldsymbol{r}_jA\Big(\subset \overset{m}{\underset{i=1}{\oplus}} e_iA\Big), \text{ and}$$

$$\text{row rank}\, R=\text{rank} \sum_{i=1}^{m} A\boldsymbol{s}_i \Big(\subset \overset{n}{\underset{j=1}{\oplus}} Af_j\Big).$$

Obviously column rank $R\neq$ row rank R in general.

Now the following is a direct consequence of Lemma 1.2.

LEMMA 2.4. *Let* $R=(\boldsymbol{r}_1, \cdots, \boldsymbol{r}_n)=\begin{pmatrix} \boldsymbol{s}_1 \\ \vdots \\ \boldsymbol{s}_m \end{pmatrix}$ *be an* $m\times n$ *matrix* $(\neq O)$.

Then column rank $R=t$ *if and only if there exists a subsequence* $(\boldsymbol{r}_{j_1}, \cdots, \boldsymbol{r}_{j_t})$ *of* $(\boldsymbol{r}_1, \cdots, \boldsymbol{r}_n)$ *such that it becomes an m.s.s. of* $\sum_{j=1}^{n} \boldsymbol{r}_jA$. *Similarly,*

row rank $R=r$ *if and only if there exists a subsequence* ${}^t(\boldsymbol{s}_{i_1}, \cdots, \boldsymbol{s}_{i_r})$ *of* ${}^t(\boldsymbol{s}_1, \cdots, \boldsymbol{s}_m)$ *such that it becomes an m.s.s. of* $\sum_{i=1}^{m} A\boldsymbol{s}_i$.

Moreover, these ranks remain invariable whenever we multiply R by invertible matrices. Namely,

PROPOSITION 2.5. *Let* $R=(r_{ij})_{i,j}$ *with* $r_{ij}\in e_iAf_j$ *be an* $m\times n$ *matrix* $(\neq O)$ *and let P and Q be invertible matrices. If the product PRQ is defined, then we have* *column rank PRQ=column rank R, and row rank PRQ= row rank R.*

PROOF. We have only to show the first equality. Set now $R=(\boldsymbol{r}_1, \cdots, \boldsymbol{r}_n)$ and assume column rank $R=t$. Then by Lemma 2.4 there is an m. s. s. $(\boldsymbol{r}_{j_1}, \cdots, \boldsymbol{r}_{j_t})$ of $\sum_{j=1}^{n} \boldsymbol{r}_j A$. Since the other $\boldsymbol{r}_{j_k} (t+1 \leqq k \leqq n)$ becomes a right A-linear combination of $(\boldsymbol{r}_{j_1}, \cdots, \boldsymbol{r}_{j_t})$ by Lemma 1.1, there is a $t \times (n-t)$ matrix B such that

$$(\boldsymbol{r}_{j_{t+1}}, \cdots, \boldsymbol{r}_{j_n}) = (\boldsymbol{r}_{j_1}, \cdots, \boldsymbol{r}_{j_t}) B .$$

At first noting that $PR=(P\boldsymbol{r}_1, \cdots, P\boldsymbol{r}_n)$, $(P\boldsymbol{r}_{j_{t+1}}, \cdots, P\boldsymbol{r}_{j_n})=(P\boldsymbol{r}_{j_1}, \cdots, P\boldsymbol{r}_{t_t})$ B and that, since $(\boldsymbol{r}_{j_1}, \cdots, \boldsymbol{r}_{j_t})$ is right A-linearly independent, $(P\boldsymbol{r}_{j_1}, \cdots, P\boldsymbol{r}_{j_t})$ is so, we get at once an m. s. s. $(P\boldsymbol{r}_{j_1}, \cdots, P\boldsymbol{r}_{j_t})$ of $\sum_{j=1}^{n} (P\boldsymbol{r}_j)A$. This shows column rank $PR=t$.

To show next column rank $RQ=t$, by Proposition 2.2 we may assume that Q is an elementary matrix. However the right A-module $\sum_{j=1}^{n} \boldsymbol{r}_j A$ remains invariable, as a whole, by applying an elementary column transformation to R. Hence from the definition of column ranks it follows that column rank $RQ=t$. Thus the lemma is proved.

EXAMPLE 2. Let $\Delta \supset \Gamma$ be a division ring extension such that $[\Delta_\Gamma : \Gamma]=2$; that is, $\Delta_\Gamma = \Gamma \oplus d_0 \Gamma$. Set now respectively

$$A=\begin{pmatrix} \Delta & \Delta \\ 0 & \Gamma \end{pmatrix}, \quad e_1=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad u=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } v=\begin{pmatrix} 0 & d_0 \\ 0 & 0 \end{pmatrix}.$$

Then A is an artinian ring with rad $A=\begin{pmatrix} 0 & \Delta \\ 0 & 0 \end{pmatrix}$, and we see $u=e_1 u e_2$ and $v=e_1 v e_2$. For the next 3×4 matrix:

$$R=\begin{pmatrix} e_1 & 0 & 0 & 0 \\ 0 & e_2 & 0 & 0 \\ 0 & 0 & u & v \end{pmatrix},$$

we get column rank $R=4$ but row rank $R=3$. Further remark that the sequence consisting of its column vectors is right A-independent and also that the sequence consisting of its row vectors is left A-independent.

3. Relation matrices.

We begin with the definition of relation matrices.

DEFINITION. Let M_A be a finitely presented non-projective module, and let $(**)$ below denote a minimal projective presentation of M:

$$(**)\qquad \bigoplus_{j=1}^{n} f_jA \xrightarrow{p_1} \bigoplus_{i=1}^{m} e_iA \xrightarrow{p_0} M_A \longrightarrow 0 \quad \text{(exact)},$$

where p_0 denotes a projective cover of M and p_1 induces a projective cover of $\operatorname{Ker} p_0$. Set now $p_1(f_j)=\boldsymbol{r}_j=\begin{pmatrix} r_{1j} \\ \vdots \\ r_{mj} \end{pmatrix}(j=1, \cdots, n)$. Then the $m\times n$ matrix $R=(r_{ij})_{i,j}$ with $r_{ij}\in e_i(\mathrm{rad}\, A)f_j$ may be regarded as an m. s. s. of $\operatorname{Ker} p_0$, which is called the relation matrix of M_A associated with $(**)$.

In this case, of course $(\boldsymbol{r}_1, \cdots, \boldsymbol{r}_n)$ is right A-linearly independent, and further

$$(1)\qquad p_1\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}=R\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \text{for every} \quad \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}\in \bigoplus_{j=1}^{n} f_jA.$$

For another minimal projective presentation $(**')$ of M_A, we get the following commutative diagram:

$$(**')\qquad \begin{array}{ccccccc} \bigoplus_{j=1}^{n} f_jA & \xrightarrow{p_1} & \bigoplus_{i=1}^{m} e_iA & \xrightarrow{p_0} & M_A & \longrightarrow & 0 \text{ (exact)} \\ {\scriptstyle Q^{-1}}\downarrow\wr & & {\scriptstyle P}\downarrow\wr & & \| & & \\ \bigoplus_{j=1}^{n} h_jA & \xrightarrow{p_1'} & \bigoplus_{i=1}^{m} g_iA & \xrightarrow{p_0'} & M_A & \longrightarrow & 0 \text{ (exact)}, \end{array}$$

whence we have the next.

LEMMA 3.1. *Let M_A be a finitely presented non-projective module, and R a relaton matrix of M. Then every relation matrix of M is expressed in the form: PRQ, where P and Q denote invertible matrices.*

Taking now A-duals of $(**)$, we get

$$(***)\qquad \bigoplus_{i=1}^{m} Ae_i \xrightarrow{p_1^*} \bigoplus_{j=1}^{n} Af_j \xrightarrow{q} \operatorname{Coker} p_1^* \to 0 \text{ (exact)},$$

where $p_1^*=\mathrm{Hom}_A(p_1, A)$ and q denotes the canonical epimorphism, and it follows readily that

$$(2)\qquad p_1^*(a_1, \cdots, a_m)=(a_1, \cdots, a_m)R \text{ for every } (a_1, \cdots, a_m)\in \bigoplus_{i=1}^{m} Ae_i.$$

Hence $\operatorname{Im} p_1^*$ is a finitely generated module $(\neq 0)$ contained in $\bigoplus_{j=1}^{n}(\mathrm{rad}\, A)f_j$, and so q is a projective cover of $\operatorname{Coker} p_1^*$ and $\operatorname{Coker} p_1^*$ is a finitely presented non-projective left A-module.

After Auslander and Reiten [3], we shall adopt the next:

DEFINITION. $M_A \in \mathrm{mod}_P A$ implies that M_A is finitely presented and that M_A has no projective module ($\neq 0$) as its direct summand. Similarly ${}_A M \in \mathrm{mod}_P A^{op}$ is defined.

Such a property can be characterized by the fit matrix theory over A.

THEOREM 3.2. *Let M_A be a finitely presented non-projective module, and $R=\begin{pmatrix} s_1 \\ \vdots \\ s_m \end{pmatrix}$ the relation matrix associated with* (**). *Then the conditions below are equivalent to each other.*

(i) $M_A \in \mathrm{mod}_P A$.

(ii) ${}^t(s_1, \cdots, s_m)$ *is left A-linearly independent.*

(iii) (***) *expressed above becomes a minimal projective presentation of Coker $p_1{}^*$.*

In these cases, further R also becomes the relation matrix of Coker $p_1{}^*$ associated with (***); *that is, Coker $p_1{}^* = TrM_A \in \mathrm{mod}_P A^{op}$ after Auslander and Reiten* [3].

PROOF. (i) $\Leftrightarrow$ (iii) is well known (Cf. [2] and [11]).

(i) $\Leftrightarrow$ (ii): To prove this, we have only to show that $M_A \notin \mathrm{mod}_P A$ if and only if row rank $R<m$. Assume first $M_A = N_A \oplus L_A$ with a projective module $L(\neq 0)$. Let $\bigoplus_{i=1}^{s} g_i A \xrightarrow{\sigma} N_A\,(1 \leqq s < m)$ and $\bigoplus_{i=s+1}^{m} g_i A \xrightarrow{\rho} L_A$ be the projective covers of N_A and L_A respectively. Then we have a projective cover of $M_A: \bigoplus_{i=1}^{m} g_i A \xrightarrow{\sigma\oplus\rho} M_A$, and since Ker $\sigma\oplus\rho=$ Ker $\sigma \subset \bigoplus_{i=1}^{s} g_i A$ the relation matrix R' of Ker $\sigma\oplus\rho$ is of the form: $\begin{array}{r} s\{ \\ \\ \end{array}\!\!\begin{pmatrix} * \\ O \end{pmatrix}$, i.e. row rank $R'<m$. On the other hand, by Lemma 3.1 we see $R'=PRQ$ with invertible matrices P and Q. Therefore row rank $R=$ row rank $R'<m$ by Proposition 2.5.

Conversely assume row rank $R<m$. In view of Lemma 2.4, by applying elementary row transformations to R we may reach to an matrix $R' = \begin{array}{r} s\{ \\ \\ \end{array}\!\!\begin{pmatrix} * \\ O \end{pmatrix}$ $(1 \leqq s < m)$. By Proposition 2.2 there is an invertible matrix $P=(p_{ij})_{i,j}$ with $p_{ij} \in g_i A e_j$ such that $R'=PR$. Considering the commutative diagram below:

$$\begin{array}{ccccccc}
\bigoplus_{j=1}^{n} f_jA & \xrightarrow{p_1} & \bigoplus_{i=1}^{m} e_iA & \xrightarrow{p_0} & M_A & \longrightarrow & 0 \text{ (exact)} \\
\| & & P\downarrow\wr & & \| & & \\
\bigoplus_{j=1}^{n} f_jA & \xrightarrow{p_1'} & \bigoplus_{i=1}^{m} g_iA & \xrightarrow{p_0'} & M_A & \longrightarrow & 0 \text{ (exact)},
\end{array}$$

R' is an m. s. s. of Ker p_0'. So set respectively

$$N_A = p_0'\Big(\bigoplus_{i=1}^{s} g_iA\Big) \quad \text{and} \quad L_A = p_0'\Big(\bigoplus_{i=s+1}^{m} g_iA\Big).$$

Then, since there follows Ker $p_0' \subset \bigoplus_{i=1}^{s} g_iA$ from the form of R', we have readily

$$M_A = N_A \oplus L_A \text{ and } \bigoplus_{i=s+1}^{m} g_iA \approx L_A\,; \text{ that is, } M_A \notin \mathrm{mod}_P A.$$

(ii) $\Leftrightarrow$ (iii): To prove this, we have only to show that row rank $R=m$ if and only if Ker $p_1^* \subset \bigoplus_{j=1}^{n} f_j(\mathrm{rad}\, A)$. However this is obvious by (2). Thus the proof is completed.

It should be noted that, in case $M_A \in \mathrm{mod}_P A$ with a relation matrix $R = (\boldsymbol{r}_1, \cdots, \boldsymbol{r}_n) = \begin{pmatrix} \boldsymbol{s}_1 \\ \vdots \\ \boldsymbol{s}_m \end{pmatrix}$, we may regard as:

$$M_A = \bigoplus_{i=1}^{m} e_iA \Big/ \sum_{j=1}^{n} \boldsymbol{r}_jA \quad \text{and} \quad TrM_A = \bigoplus_{j=1}^{n} Af_j \Big/ \sum_{i=1}^{m} A\boldsymbol{s}_i. \quad \text{(Cf. [11])}$$

REMARK. If M_A is a finitely presented non-projective module with a $1\times n$ relation matrix R, then obviously M_A is indecomposable and so $M_A \in \mathrm{mod}_P A$.

The following is useful to construct indecomposables, which is also observed by H. Asashiba.

COROLLARY 3.3. *Let M_A be a finitely presented non-projective module with an $m\times 1$ relation matrix $R = \begin{pmatrix} r_{11} \\ \vdots \\ r_{m1} \end{pmatrix}$. Then the conditions below are equivalent to each other.*

(i) *M_A is an indecomposable module.*

(ii) $M_A \in \mathrm{mod}_P A$

(iii) *${}^t(r_{11}, \cdots, r_{m1})$ is left A-linearly independent.*

PROOF. We have only to prove the equivalence (i)$\Leftrightarrow$(ii). To do so, we first assume that M_A is decomposed such as:

$$M_A = N_A \oplus L_A \quad (N \neq 0 \text{ and } L \neq 0).$$

Denote by $\bigoplus_{i=1}^{s} g_i A \xrightarrow{\sigma} N_A$ $(1 \leqq s < m)$ and $\bigoplus_{i=s+1}^{m} g_i A \xrightarrow{\rho} L_A$ respectively the projective covers of N and L. Then $\bigoplus_{i=1}^{m} g_i A \xrightarrow{\sigma \oplus \rho} M_A$ becomes a projective cover of M. Since top $(\operatorname{Ker} \sigma \oplus \rho)$ is simple and since $\operatorname{Ker} \sigma \oplus \rho = \operatorname{Ker} \sigma \oplus \operatorname{Ker} \rho$, we have either $\operatorname{Ker} \sigma = 0$ or $\operatorname{Ker} \rho = 0$; that is, either of N and L must be projective; i.e. $M_A \notin \operatorname{mod}_P A$.

Conversely, if $M_A \notin \operatorname{mod}_P A$ then evidently M is decomposable. Thus we obtain the corollary.

Next we want to characterize the indecomposability of modules in $\operatorname{mod}_P A$ by using their relation matrices. For this purpose we need the next.

DEFINITION. An $n \times n$ matrix $T = (t_{ij})_{i,j}$ with $t_{ij} \in e_i A e_j$ is called idempotent if $T^2 = T$.

LEMMA 3.4. *Let $T = (t_{ij})_{i,j}$ with $t_{ij} \in e_i A e_j$ be an $n \times n$ matrix. Then T is idempotent if and only if there exist an invertible $n \times n$ matrix $P = (p_{jk})_{j,k}$ with $p_{jk} \in g_j A e_k$ and a diagonal $n \times n$ matrix*

$$D = \begin{pmatrix} g_1 & & & & & \\ & \ddots & & & & \\ & & g_s & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

$(0 \leqq s \leqq n)$ *such that $T = P^{-1} D P$.*

PROOF. We have only to prove the only if part. Let T be an idempotent matrix which is neither zero matrix nor identity matrix, and set $M_A = \bigoplus_{i=1}^{n} e_i A$. Then, since $T \in \operatorname{End} M_A$ we have

$$M_A = T M_A \oplus (E - T) M_A,$$

where E denotes the identity matrix $\begin{pmatrix} e_1 & & \\ & \ddots & \\ & & e_n \end{pmatrix}$. Hence TM as well as $(E-T)M$ is finitely generated projective, and so we get their projective covers below:

$$\bigoplus_{j=1}^{s} g_j A \xrightarrow[\sim]{\sigma} T M_A \ (1 \leqq s < n) \quad \text{and} \quad \bigoplus_{j=s+1}^{n} g_j A \xrightarrow[\sim]{\rho} (E - T) M_A$$

Therefore there is an invertible matrix P such that $P^{-1}=\sigma\oplus\rho : \bigoplus_{j=1}^{n} g_jA \simeq M_A$.

Set further $\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_n \end{pmatrix}=(\boldsymbol{g}_1, \cdots, \boldsymbol{g}_n)$. Then, for any j $(1\leqq j\leqq s)$ we have $P^{-1}\boldsymbol{g}_j \in TM$ and hence $TP^{-1}\boldsymbol{g}_j=P^{-1}\boldsymbol{g}_j$ because $T^2=T$; i.e.

$$PTP^{-1}\boldsymbol{g}_j=\boldsymbol{g}_j \qquad \text{for every } j\ (j=1, \cdots, s).$$

On the other hand, for any j $(s+1\leqq j\leqq n)$ we have $P^{-1}\boldsymbol{g}_j\in(E-T)M$ and hence $TP^{-1}\boldsymbol{g}_j=\boldsymbol{0}$; i.e.

$$PTP^{-1}\boldsymbol{g}_j=\boldsymbol{0} \qquad \text{for every } j\ (j=s+1, \cdots, n).$$

Consequently it follows that

$$PTP^{-1}=(\boldsymbol{g}_1, \cdots, \boldsymbol{g}_s, \boldsymbol{0}, \cdots, \boldsymbol{0})=\begin{pmatrix} g_1 & & & & \\ & \ddots & & & \\ & & g_s & & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

which proves the lemma.

The following will be applied, in fact, to connected semiperfect rings. (Cf. [12])

THEOREM 3.5. *Let M_A be a module in* $\mathrm{mod}_P A$ *and R an $m\times n$ relation matrix of M_A ($m\geqq 2$ and $n\geqq 2$). Then M_A is indecomposable if and only if the condition below is satisfied:* ($\circ$) *If $TR=RS$ for non-zero idempotent matrices T and S, then both T and S must be identity matrices.*

PROOF. We shall prove that M_A is decomposable if and only if there exist "proper" idempotent matrices T and S such that $TR=RS$, where proper matrices mean that they are neither zero matrices nor identity matrices.

Assume first that M_A is decomposable; i.e. $M=M_1\oplus M_2$, and let $\bigoplus_{i=1}^{s} g_iA\xrightarrow{\sigma}M_1$ $(1\leqq s<m)$ and $\bigoplus_{i=s+1}^{m} g_iA \xrightarrow{\rho} M_2$ be the projective covers of M_1 and M_2 respectively. Since $\mathrm{Ker}\,\sigma\neq 0$ and $\mathrm{Ker}\,\rho\neq 0$ by the assumption that $M_A\in\mathrm{mod}_P A$, we get further the projective covers of $\mathrm{Ker}\,\sigma$ and $\mathrm{Ker}\,\rho$; i.e. $\bigoplus_{j=1}^{t} h_jA \xrightarrow{\alpha} \mathrm{Ker}\,\sigma$ $(1\leqq t<n)$ add $\bigoplus_{j=t+1}^{n} h_jA \xrightarrow{\beta} \mathrm{Ker}\,\rho$.

Then the sequence below:

$$\bigoplus_{j=1}^{n} h_jA \xrightarrow{\alpha\oplus\beta} \bigoplus_{i=1}^{m} g_iA \xrightarrow{\sigma\oplus\rho} M_A \to 0 \quad \text{(exact)}$$

becomes a minimal projective presentation of M_A. Let R' be the relation matrix of M_A associated with the above. Then we see easily

$$R'=\begin{pmatrix} R_1 & O \\ O & R_2 \end{pmatrix} \text{ with } R_1\neq O \text{ and } R_2\neq O\,,$$

where R_1 and R_2 denote respectively an $s\times t$ matrix and an $(m-s)\times(n-t)$ matrix. On the other hand, by Lemma 3.1 there are invertible matrices $P=(p_{ki})_{k,i}$ with $p_{ki}\in g_kAe_i$ and $Q=(q_{jl})_{j,l}$ with $q_{jl}\in f_jAh_l$ such that $R'=PRQ$, and hence

$$PRQ=\begin{pmatrix} R_1 & O \\ O & R_2 \end{pmatrix}.$$

Take now the diagonal $m\times m$ (resp. $n\times n$) matrix below:

$$D_1=\begin{pmatrix} g_1 & & & & & \\ & \ddots & & & & \\ & & g_s & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \quad \text{resp. } D_2=\begin{pmatrix} h_1 & & & & & \\ & \ddots & & & & \\ & & h_t & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

Multiplying PRQ by D_1 (resp. D_2) from the left (resp. the right), we have

$$D_1PRQ=\begin{pmatrix} R_1 & O \\ O & O \end{pmatrix}=PRQD_2\,,$$

whence it follows immediately that $(P^{-1}D_1P)R=R(QD_2Q^{-1})$.

Conversely, assume that there exist proper idempotent matrices T and S such that $TR=RS$. Then, by Lemma 3.4 T and S are expressed respectively in the forms:

$$T=P^{-1}D_1P \quad \text{and} \quad S=QD_2Q^{-1}\,,$$

where $P=(p_{ki})_{k,i}$ with $p_{ki}\in g_kAe_i$ and $Q=(q_{jl})_{j,l}$ with $q_{jl}\in f_jAh_l$ are invertible matrices, and where

$$D_1=\begin{pmatrix} g_1 & & & & \\ & \ddots & & & \\ & & g_s & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \quad (1\leqq s<m)$$

and

$$D_2=\begin{pmatrix} h_1 & & & & \\ & \ddots & & & \\ & & h_t & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \quad (1\leqq t<n).$$

Therefore we have $D_1(PRQ)=(PRQ)D_2$, whence it readily follows that

$$PRQ=\begin{pmatrix} R_1 & O \\ O & R_2 \end{pmatrix},$$

where R_1 and R_2 denote respectively an $s\times t$ matrix and an $(m-s)\times(n-t)$ matrix. Considering now the commutative diagram below:

$$(**') \qquad \begin{array}{ccccccc} \bigoplus_{j=1}^{n} f_jA & \xrightarrow{p_1} & \bigoplus_{i=1}^{m} e_iA & \xrightarrow{p_0} & M_A & \longrightarrow & 0 \text{ (exact)} \\ Q^{-1}\downarrow\wr & & P\downarrow\wr & & \Vert & & \\ \bigoplus_{j=1}^{n} h_jA & \xrightarrow{p_1'} & \bigoplus_{i=1}^{m} g_iA & \xrightarrow{p_0'} & M_A & \longrightarrow & 0 \text{ (exact)}, \end{array}$$

PRQ is the relation matrix associated with $(**')$. From the above form of PRQ it readily follows that

$$\operatorname{Ker} p_0'=\operatorname{Ker} p_0'\cap\Big(\bigoplus_{i=1}^{s} g_iA\Big)\oplus\operatorname{Ker} p_0'\cap\Big(\bigoplus_{i=s+1}^{m} g_iA\Big),$$

whence we have

$$M_A=p_0'\Big(\bigoplus_{i=1}^{s} g_iA\Big)\oplus p_0'\Big(\bigoplus_{i=s+1}^{m} g_iA\Big).$$

Hence M_A is decomposable.

Accordingly we have proved that M_A is indecomposable if and only if the condition below is satisfied:

($\circ'$) If $TR=RS$ for non-zero idempotent matrices T and S, then either of T and S must be an identity matrix.

But, by the assumption that $M_A \in \mathrm{mod}_P A$, if S (resp. T) is the identity matrix then T (resp. S) must be the identity matrix. Because, if $R=TR$ with a proper idempotent matrix $T=P^{-1}D_1P$ expressed above, then $PR=D_1(PR)= {}^{s\{}\begin{pmatrix} * \\ O \end{pmatrix}$ and row rank $PR=m$ by Proposition 2.5 and by Theorem 3.2, a contradiction. Similarly, if $R=RS$ with a proper idempotent matrix $S=QD_2Q^{-1}$ expressed above, then we are again led to a contradiction. Thus the proof is completed.

4. Semiperfect rings satisfying (TSF).

As is well known, in a representation-finite artinian ring its two-sided ideals constitute always a distributive lattice (Cf. [**6**]), and so from the representation theory of distributive lattices it follows that the number of its two-sided ideals is finite (Cf. [**5**]).

In this section we want to show that such a property holds good under a weaker condition than the above. We first adopt the next.

DEFINITION. A right A-module M is called top-simple if top M_A is a simple module.

Then, as was stated in the introduction, we consider the following condition:

(TSF) The number of all the isomorphism classes of top-simple right A-modules is finite.

THEOREM 4.1. *Let A be a semiperfect ring satisfying* (TSF). *Then A has only a finite number of two-sided ideals.*

PROOF. First of all, by Morita equivalence we may assume that A itself is a basic ring, and let

$$1=\sum_{k=1}^{t} e_k$$

be the decomposition of 1 into pairwise orthogonal primitive idempotents. It should be noted that $e_iA \approx e_jA$ only if $i=j$. We shall distinguish two steps.

Step 1. ${}_AI_A \subset \mathrm{rad}\, A$.

Let us set $\boldsymbol{F}=\{{}_AI_A \mid I\subset \mathrm{rad}\, A\}$. At first remark that, for each I in $\boldsymbol{F}$,

e_kA/e_kI $(k=1, \cdots, t)$ are not isomorphic to each other; because, top $e_kA/e_kI\approx e_kA/e_k(\mathrm{rad}\,A)$ $(k=1, \cdots, t)$ and they are not isomorphic to each other. Secondly remark that, for any (I, J) $(I\neq J)$ in $\boldsymbol{F}\times\boldsymbol{F}$, there exists at least one, say k $(1\leqq k\leqq t)$, such that $e_kA/e_kI\not\approx e_kA/e_kJ$; because, if $e_kA/e_kI\approx e_kA/e_kJ$ for every k $(k=1, \cdots, t)$ then

$$[A/I]_A=\bigoplus_{k=1}^{t} e_kA/e_kI\approx \bigoplus_{k=1}^{t} e_kA/e_kJ=[A/J]_A \text{ and so } I=J,$$

a contradiction.

Now, for a right A-module M, denote by $[\![M_A]\!]$ the isomorphism class of M_A, and set respectively

$$\boldsymbol{G}=\{[\![e_kA/e_kI]\!] \mid 1\leqq k\leqq t,\ I\in\boldsymbol{F}\} \text{ and } s=\#\boldsymbol{G}.$$

Then by the assumption (TSF) and by the first remark we have $t\leqq s<\infty$. For each $[\![M_A]\!]\in\boldsymbol{G}$ we shall assign a natural number l $(1\leqq l\leqq s)$; that is, there is a bijection $\varphi: \boldsymbol{G}\simeq\{1, \cdots, s\}$.

To count $\#\boldsymbol{F}$ we shall define the map $\psi: \boldsymbol{F}\to\{1, \cdots, s\}^{(t)}$ as follows:

$$\psi(I)=(\varphi([\![e_1A/e_1I]\!]), \cdots, \varphi([\![e_tA/e_tI]\!])) \qquad \text{for each } I\in\boldsymbol{F}.$$

By the second remark ψ becomes an injective map. From the first remark again it follows that

$$\#\boldsymbol{F}\leqq {}_sP_t<\infty.$$

Step 2. ${}_AI_A\subset A$.

Set now $\boldsymbol{H}=\{{}_AI_A \mid I\subset A\}$. For each I in $\boldsymbol{H}$, since $I=\sum_{k=1}^{t} e_kIe_k+\sum_{i\neq j} e_iIe_j$, we can express it as follows:

$$I=\sum_{k\in\Lambda} e_kAe_k+I\cap\mathrm{rad}\,A,$$

where $\Lambda=\Lambda(I)=\{k \mid e_k\in I, 1\leqq k\leqq t\}$, and this expression is uniquely determined by I. Since $I\cap\mathrm{rad}\,A\in\boldsymbol{F}$ we can conclude that $\#\boldsymbol{H}\leqq 2^t{}_sP_t<\infty$. Thus the theorem is proved.

The next will be required to illustrate examples later.

LEMMA 4.2. *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be right ideals contained in* $e(\mathrm{rad}\,A)$. *Then,* $eA/\mathfrak{m}\approx eA/\mathfrak{n}$ *if and only if there exists an invertible element* a *in* eAe *such that* $\mathfrak{n}=a\mathfrak{m}$.

PROOF. Assume first that $eA/\mathfrak{m}\overset{\alpha}{\simeq}eA/\mathfrak{n}$. By the uniqueness of projective covers, we get the next commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m} & \dashrightarrow & eA & \longrightarrow & eA/\mathfrak{m} & \longrightarrow & 0 \quad \text{(exact)} \\ & & \downarrow\wr & & a\downarrow\wr & & \alpha\downarrow\wr & & \\ 0 & \longrightarrow & \mathfrak{n} & \longrightarrow & eA & \dashrightarrow & eA/\mathfrak{n} & \longrightarrow & 0 \quad \text{(exact)}; \end{array}$$

that is, there is an invertible element a in eAe such that $\mathfrak{n}=a\mathfrak{m}$. Hence the only if part is proved. Whereas the if part is obvious, and thus the lemma is proved.

EXAMPLE 1. (Already appeared.) $A=\boldsymbol{R}\ltimes\boldsymbol{C}$. The number of the two-sided ideals of A is infinite, but that of the isomorphism classes of its two-sided ideals is finite.

EXAMPLE 2. (Already appeared.) $A=\begin{pmatrix}\Delta & \Delta \\ 0 & \Gamma\end{pmatrix}$ with $[\Delta_\Gamma:\Gamma]=2$. As for (TSF), in view of the structure of A the following right ideals $\begin{pmatrix}0 & d\Gamma \\ 0 & 0\end{pmatrix}$ with $0\neq d\in\Delta$ only give rise to discussion. But, for such two right ideals $\begin{pmatrix}0 & d_1\Gamma \\ 0 & 0\end{pmatrix}$ and $\begin{pmatrix}0 & d_2\Gamma \\ 0 & 0\end{pmatrix}$, there is an invertible element $\begin{pmatrix}d_2d_1^{-1} & 0 \\ 0 & 0\end{pmatrix}$ in e_1Ae_1 such that $\begin{pmatrix}d_2d_1^{-1} & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & d_1\Gamma \\ 0 & 0\end{pmatrix}=\begin{pmatrix}0 & d_2\Gamma \\ 0 & 0\end{pmatrix}$. Hence by Lemma 4.2 A satisfies (TSF). Of course, the number of the two-sided ideals of A is five.

EXAMPLE 3. (Rosenberg and Zelinsky [**10**])

Let F be a field and $K=F(x_1, x_2, \cdots)$ the rational function field in countably infinite indeterminates $x_1, x_2, \cdots$ over F. We define the ring monomorphism: $K\xrightarrow{\sigma}K$ by $\sigma(x_i)=x_{i+1}$ $(i=1, 2, \cdots)$ and $\sigma|F=1_F$. Further define the K-K-bimodule N as follows:

${}_KN={}_KK$ and N_K is defined by $n*k=n\sigma(k)$ for $n\in N$ and $k\in K$.

Let then A be the trivial extension of K by N; i.e. $A=K\ltimes N$. A is a local left artinian ring, but is not right artinian, and A has only three two-sided ideals. But A does not satisfy (TSF), which is shown as follows:

Obviously $[\text{rad } A]_A=(0, N_K)$. So if $A/(0, \mathfrak{m})\approx A/(0, \mathfrak{n})$ for $\mathfrak{m}_K$ and $\mathfrak{n}_K\subset N_K$, then by Lemma 4.2 there exists an invertible element (k, n) $(k\neq 0)$ in A such that $(k, n)(0, \mathfrak{m})=(0, \mathfrak{n})$; i.e. $\mathfrak{n}=k\mathfrak{m}$, and hence $\dim \mathfrak{m}_K=\dim \mathfrak{n}_K$. But, since $[N_K: K]=[K: \sigma(K)]\geqq\aleph_0$, there exist right K-submodules $\mathfrak{m}_i$ of N_K such that $\dim [\mathfrak{m}_i]_K=i$ $(i=1, 2, \cdots)$. Accordingly, by Lemma 4.2 $A/(0, \mathfrak{m}_i)$ $(i=1, 2, \cdots)$ are not isomorphic to each other.

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