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ON THE RESIDUAL TRANSCEDENTAL EXTENSIONS OF A VALUATION. KEY POLYNOMIALS AND AUGMENTED VALUATION

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Let K be a field and v a valuation on K. The problem of extending v to $K(X)$ (the field of rational functions of one inderminate) has been previously considered in some works as [7] and [10]. Particularly in [7], MacLane studied the case when v is discrete and rank one. In solving the problem in this case, MacLane used some notions as key polynomial and augmented valuation.

An extension w of v to $K(X)$ is called residual transcendental (briefly, an r.t. extension) if the residue field of w is a transcendental extension of the residue field of v (MacLane called these extensions "inductive value"). Some aspects of r.t. extensions have been considered in [5, Ch. VI], [9], [1], [2], [3] and [11]. Particularly in [2] and [11] all r.t. extensions of v to $K(X)$ were described using the notion of "minimal pair" (see definition in Section 1). Although in [3] some results on minimal pairs were given, the problem of finding minimal pairs in the general setting seems to be difficult.

In this work we follow, for arbitrary r.t. extensions, MacLane's ideas of key polynomial and augmented valuation and show that these give a powerful tool in the study of all extensions of v to $K(X)$. In particular, the key polynomials over an r.t. extension give us the possibility of defining some new minimal pairs (Theorem 5.1).

Now we briefly describe the content of the paper. Section 1 contains notation, definitions and the main results from [2] and [11], Theorem 1.2 and some consequences of this theorem will be used in this paper.

In section 2, we give some technical results related to the domination of valuations on $K(X)$, which was also introduced by MacLane in [7]. This notion has been used in [4] to describe all valuations on $K(X)$). In Section 3 (after MacLane [7]) key polynomial and augmented valuation are defined.

The key polynomials over an r.t. extension are studied in Section 4. The main results are given in Theorems 4.4 and 4.6. We remark that Theorem 4.6

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and the proof are inspired by MacLane's work [7].

In Section 5 it is proved that a ("comensurable") augmented valuation over a given r.t. extension is also an r.t. extension (Theorem 5.1). By this theorem we can define new minimal pairs, starting from a given one. Theorem 5.5 shows how the augmented valuations are closely related to the domination. Finally in Section 6, using the results of previous sections, we give another proof of a result which asserts that there exist r.t. extensions with given residue field and value group (Theorem 6.4, see [3, Theorem 4.4]).

In a forthcoming paper we will use the results developed here to study all valuations on $K(X)$ and related topics.

1. Notations and definitions

In this section we recall notations, definitions and the main results of [2] and [11] (see Theorem 1.2), which will be used in the rest of this paper. Also some new consequences of Theorem 1.2 are given.

1. Let K be a field and v a valuation on K. We sometimes emphasize this situation saying that (K, v) is a valuation pair. G_v, O_v, k_v and $\rho_v: O_v \to k_v$ represent the value group of v, the valuation ring of v, its residue field and the residue homomorphism, respectively. If $x \in O_v$, we usually write by x^* the image $\rho_v(x)$ of x in k_v. We refer the reader to [5], [6], [12], [13] for general notions and definitions.

Let K'/K be an extension of fields. A valuation v' on K' will be called an *extension* of v if $v'(x)=v(x)$ for all $x \in K$. When v' is an extension of v, we shall identify canonically k_v with a subfield of $k_{v'}$ and G_v with a subgroup of $G_{v'}$.

Throughout this paper, we fix a valuation pair (K, v), an algebraic closure $\bar{K}$ of K and an extension $\bar{v}$ of v to $\bar{K}$. Then $k_{\bar{v}}=\bar{k}_v$ an algebraic closure of k_v and $G_{\bar{v}}=\bar{G}_v=QG_v$, i.e. G_v is the smallest divisible group which contains G_v.

As usual we denote by $K[X]$ and $K(X)$ the polynomial ring and the field of rational functions of an indeterminate X over K, respectively. If $r=f/g$, $f, g \in K[X]$ and f, g are relatively prime, we define the *order* of r by the equality: ord $r=\max(\deg f, \deg g)$. It is easy to see that ord$r=[K(X): K(r)]$.

2. Let w be an extension of v to $K(X)$. According to [8] (see also [1] and [2]), w is called a *residual transcendental* (r.t.)-*extension* of v if k_w/k_v is a transcendental extension. An element (a, δ) of $\bar{K} \times \bar{G}_v$ is usually called a *pair*. If (a, δ) is a pair, we define the valuation $w_{(a,\delta)}$ of $\bar{K}(X)$ (see [5, Ch. VI, par.

10]:

$$w_{(a,\delta)}(f)=\inf(\bar{v}(a_i)+i\delta) \text{ when } f(x)=a_0+a_1(x-a)+\cdots+a_n(x-a)^n\in\bar{K}[X].$$

Usually one says that $w_{(a,\delta)}$ is defined by inf, $\bar{v}$, $a\in\bar{K}$ and $\delta\in\bar{G}_v$. It is easy to see that $w_{(a,\delta)}$ is an r.t. extension of $\bar{v}$ to $\bar{K}(X)$([5, Ch. VI, par. 10], or [1]).

PROPOSITION 1.1. ([1]). *Every r.t. extension w of $\bar{v}$ to $\bar{K}(X)$ is of the form $w=w_{(a,\delta)}$ for a suitable pair (a,δ). Moreover two pairs (a,δ) and (a',δ') define the same valuation on $\bar{K}(X)$, i.e. $w_{(a,\delta)}=w_{(a',\delta')}$, if and only if $\delta=\delta'$ and $\bar{v}(a-a')\geqq\delta$.*

A pair $(a,\delta)\in\bar{K}\times\bar{G}_v$ will be called *minimal with respect to K* if, for every $b\in\bar{K}$ such that $[K(b):K]<[K(a):K]$, one has $\bar{v}(a-b)<\delta$. In [2], [3], [4] and [11] it is shown that the minimal pairs play a prominent part in the definition and in the study of r.t. extensions.

THEOREM 1.2. ([2], [11]). *Let w be an r.t. extensions of v to $K(X)$. Then there exists a pair (a,δ), minimal with respect to K such that w coincides with the restriction of $w_{(a,\delta)}$ to $K(X)$. Moreover one has:*

a) *Denote by f the monic minimal polynomial of a with respect to K and put $\gamma=w(f)$. Then*

$$w(F)=\inf(\bar{v}(F_i(a))+i\gamma), \quad \textit{where} \quad F=F_0+F_1f+\cdots F_sf^s\in K[X],$$

$$\deg F_i<n=\deg f.$$

b) *Let $\tilde{v}$ be the restriction of $\bar{v}$ to $K(a)$. If e is the smallest natural number such that $e\gamma\in G_{\tilde{v}}$ one has:*

$$G_w=G_{\tilde{v}}+Z\gamma \quad \textit{and} \quad [G_w:G_v]=e[G_{\tilde{v}}:G_v].$$

c) *Let $h\in K[X]$. If $\deg h<n=\deg f$ and $w(h)=\bar{v}(h(a))=e\gamma=ew(f)$, then $r=f^e/h$ is the element of O_w of the smallest order such that $r^*\in k_w$ is transcendental over k_v.*

d) *The field $k_{\tilde{v}}$ can be canonically identified with the algebraic closure of k_v in k Moreover, on has $k_w=k_{\tilde{v}}(r^*)$.*

NOTATION 1.3. If w is an r.t. extension of v to $K(X)$, a minimal pair (a,δ) in the previous theorem is called a *minimal pair* of definition of w. In what follows, for every r.t. extension w of v to $K(X)$, we fix a minimal pair of definition (a,δ). Also, the symbols f, γ, $\tilde{v}$, e, h, r and r^* are used as in

Theorem 1.2.

Now we give some consequences of Theorem 1.2, under the same notations and the hypothesis.

COROLLARY 1.4. (cf. [3, Proposition 1.1]) *Let* $A_1, \cdots, A_s, B_1, \cdots, B_t$ *be elements of* $K[X]$ *such that* $\deg A_i<n$, *and* $\deg B_j<n$ *for all* $1\leqq i\leqq s$, $1\leqq j\leqq t$. *If* $w(A_1\cdot\cdots\cdot A_s)=w(B_1\cdot\cdots\cdot B_t)$, *then* $y=\left(\frac{A_1\cdot\cdots\cdot A_s}{B_1\cdot\cdots\cdot B_t}\right)^*=\left(\frac{A_1(a)\cdots A_s(a)}{B_1(a)\cdots B_t(a)}\right)^*\in k_{\bar{v}}$

COROLLARY 1.5. *If* $g\in K[X]$ *satisfies* $w(g)\in G_{\bar{v}}$ *then for* $q\in K[X]$ *such that* $\deg q<n$ *and that* $w(q)=\bar{v}(q(a))=w(g)$, $(g/q)^*\in k_{\bar{v}}[r^*]$. *In particular, if* $w(g)=0$ *and* $g^*\in k_{\bar{v}}$, $\bar{v}(g(a))=0$ *and* $g^*=(g(a))^*$.

PROOF. Let $g=g_0+g_1f+\cdots+g_tf^t$ be the f-expansion of g with $\deg g_i<n$, $0\leqq i\leqq t$. Then, by definition $w(g)=\inf(\bar{v}(g_i(a))+i\gamma)$. Since $w(g)\in G_{\bar{v}}$,

$$\bar{v}(g_i(a))+i\gamma>w(g)=w(q),\quad \text{if}\quad i\not\equiv 0(\bmod e).$$

Since $w(g)=w(q)$, one has $w(g_if^i/_q)\geqq 0$, for all i. Hence $w\left(\frac{g_{es}h^s}{q}\cdot\frac{f^{es}}{h^s}\right)\geqq 0$ and $w\left(\frac{g_{es}h^s}{q}\right)\geqq 0$ because $w\left(\frac{f^{es}}{h^s}\right)=w(r^s)=0$. Therefore,

$$(g/q)^*=(g_0/q)^*+(g_eh/q)^*r^*+\cdots\in k_{\bar{v}}[r^*].$$

Finally, if $w(g)=0$ and $g^*\in k_{\bar{v}}$, then we may take $q=1$, and one has necessarily that $(g_eh)^*=(g_{2e}h^2)^*=\cdots=0$. Hence $g^*=g_0^*=(g(a))^*$, as claimed.

COROLLARY 1.6. *The assignement*: $F\to F^*=\rho_w(F)$ *defines an onto ring-homomorphism* ρ_w: $O_w\cap K[X]\to k_{\bar{v}}[r^*]$.

PROOF. According to Corollary 1.5, it is enough to show that there exists $F\in K[X]\cap O_w$ such that $F^*=r^*$. Indeed, take $t\in K[X]$ such that $\deg t<n$, $w(t)=\bar{v}(t(a))=-w(f^e)=-w(h)=-\bar{v}h(a))$, and that $(th)^*=1$. Then $(tf^e)^*=\left(th\cdot\frac{f^e}{h}\right)^*=(th)^*r^*=r^*$, as claimed.

3. NOTATION 1.7. Let $G=u_0+u_1r^*+\cdots+r^{*s}$ be a monic polynomial of $k_{\bar{v}}[r^*]$. For every $i(0\leqq i<s)$, choose a polynomial $g_i\in K[X]$ such that $\deg g_i<n$, $w(g_i)\geqq 0$ and that $g_i^*=u_i$. Let

$$A=g_0+g_1r+\cdots+r^s=\frac{g_0h^s+g_1h^{s-1}f^e+\cdots+f^{se}}{h^s}.$$

Then $A\in K(X)$, $w(A)=0$, and $A^*=G$. We shall say that the polynomial $g=g_0h^s+g_1h^{s-1}f^e+\cdots+f^{se}$ is a *lifting in* $K[X]$ *of the polynomial* G *in* $k_{\bar{v}}[r^*]$.

Note that G has many liftings in $K[X]$.

2. Domination of r.t. extensions of v to $K(X)$

Let w_1, w_2 be two r.t. extensions of v to $K(X)$. Let (a_i, δ_i) be a minimal pair of definition of w_i, $i=1, 2$. As in Notation 1.3, let f_i be the monic minimal polynomial of a_i, with $\gamma_i = w_i(f_i)$, $\bar{v}_i$ the restriction of $\bar{v}$ to $K(a_i)$, e_i the smallest positive integer such that $e_i\gamma_i \in G_{\bar{v}_i}$, $h_i \in K[X]$ the polynomial such that $\deg h_i < n_i = \deg f_i$ and that $w(h_i) = e_i\gamma_i$, $r_i = f_i^{e_i}/h_i$. Let $r_i^* = \rho_{w_i}(r_i)$, $i=1, 2$.

According to [7] (see also [4]), one says that w_2 *dominants* w_1 (and written by $w_1 < w_2$) if $w_1(g) \leqq w_2(g)$ for all $g \in K[X]$, and $w_1(G) < w_2(G)$ for at least one $G \in K[X]$. This inequality should be understood in $G_{\bar{v}}$ because G_{w_1} and G_{w_2} are of finite index over G_v (see [1] or [2]).

If $w_1 < w_2$, then $O_{w_1} \cap K[X] \subseteqq O_{w_2} \cap K[X]$ and there exists a unique ring homonorphism φ: $k_{\bar{v}_1}[r_1^*] \to k_{\bar{v}_2}[r_2^*]$ such that the following diagram is commutative:

$$\begin{array}{ccc} O_{w_1}\cap K[X] & \longrightarrow & O_{w_2}\cap K[X] \\ \Big\downarrow \rho_1 & & \Big\downarrow \rho_2 \\ k_{\bar{v}_1}[r_1^*] & \xrightarrow{\ \varphi\ } & k_{\bar{v}_2}[r_2^*] \end{array} \tag{1}$$

For the sake of simplicity we write $\rho_i = \rho_{w_i}$, $i=1, 2$ (cf. Corollary 1.6).

PROPOSITION 2.1. *Let w_1, w_2 be two r.t. extensions of v such that $w_1 < w_2$. Consider the diagram* (1), *then*

a) $\varphi(y) \neq 0$, *whenever* $y \in k_{\bar{v}_1}$, $y \neq 0$,

b) $\mathrm{Ker}\,\varphi \neq 0$,

c) $\varphi(\rho_1(F)) = \rho_2(F(a_2))$ *for any* $F \in O_{w_1} \cap K[X]$.

PROOF. a) Clear because $k_{\bar{v}_1}$ is a field.

b) Indeed, since $w_1 < w_2$, there exists $g \in K[X]$ such that $w_1(g) < w_2(g)$. Let m be a positive integer such that $w_1(g^m) = -v(b)$ for some $b \in K$. Then $\rho_1(bg^m) \neq 0$. On the other hand, $\rho_2(bg^m) = \varphi(\rho_1(bg^m)) = 0$, i.e. $\rho_1(bg^m) \in \mathrm{Ker}\,\varphi$.

c) Let $F \in O_{w_1} \cap K[X]$. Then $w_2(F) \geqq w_1(F) \geqq 0$. According to b) it follows that $\varphi(\rho_1(F))$ is algebraic over k_v, so it belongs to $k_{\bar{v}_2}$. If $w_2(F) = 0$, $\varphi(\rho_1(F)) = \rho_2(F) = \rho_2(F(a_2))$ by Corollary 1.5. Assume that $w_2(F) > 0$. Write $F = F_0 + F_1 f_2 + \cdots + F_r f_2^r$, $\deg F_i < \deg f_2$, $0 \leqq i \leqq r$. Then $w_2(F) \leqq w_2(F_0) = \bar{v}(F_0(a_2)) = \bar{v}(F(a_2))$. Hence $\varphi(\rho_1(F)) = \rho_2(F) = 0 = \rho_2(F(a_2))$.

COROLLARY 2.2. *With the notation and hypothesis in Proposition* 2.1, *we have*:
a) $w_1(f_2)<w_2(f_2)$ *and* $n_1 \leqq n_2$,
b) *if* $h \in K[X]$ *and* $\deg h<n_1$, $\bar{v}(h(a_1))=\bar{v}(h(a_2))$.

PROOF. a) Let t be a natural number such that $w_1(f_2^t)=-v(b)$, $b \in K$. Then $\rho_1(bf_2^t)\neq 0$. According to Proposition 2.1 c), $\varphi(\rho_1(bf_2^t))=\rho_2(bf_2^t)=\rho_2(bf_2^t(a_2))=0$. This means that $w_2(f_2)>w_1(f_2)$. Furthermore, since $\varphi(\rho_1(bf_2^t))=0$, $\rho_1(bf_2^t)$ is transcendental over $k_{\bar{v}_1}$. Then, according to [3, Proposition 1.1], there exists a root a_2' of f_2 such that (a_2', δ_1) is a pair of definition of w_1. Now since (a_1, δ_1) is a minimal pair of definition of w_1, $n_1=[K(a_1): K]\leqq n_2=[K(a_2): K]$.

b) If $h \in K[X]$ and $\deg h<n_1$ then, by definition of w_1, $w_1(h)=\bar{v}(h(a_1))$. Let s be a positive integer such that $w_1(h^s)=-v(b)$, $b \in K$. Then $w_1(bh^s)=0$, and $0\neq(bh^s)^*=\rho_1(bh^s)\in k_{\bar{v}_1}$. Thus, according to Proposition 2.1 a), $0\neq\varphi(\rho_1(bh^s))=\rho_2(bh^s)$. Hence $w_2(bh^s)=0$ and $w_2(h)=\bar{v}(h(a_2))$, because $\deg h<n_1\leqq n_2$. It is easy to check that $w_1(h)=\bar{v}(h(a_1))=w_2(h)=\bar{v}(h(a_2))$.

REMARK 2.3. Now we make some remarks on the relation of domination between r.t. extensions.

a) Let $w_i=w_{(a_i,\delta_i)}$, $i=1, 2$, be two r.t. extensions of $\bar{v}$ to $\bar{K}(X)$. In [4, Proposition 2.1], it is proved that $w_1<w_2$ if and only if $\bar{v}(a_1-a_2)\geqq\delta_1$ and $\delta_1<\delta_2$. When K is not necessarily algebraically closed and w_1, w_2 are two r.t. extensions of v to $K(X)$ such that $w_1<w_2$, we say that w_2 *well dominates* w_1 if there exist minimal pairs of definition (a_i, δ_i) of w_i such that $w_{(a_1,\delta_1)}<w_{(a_2,\delta_2)}$. Ii is clear that if w_2 well dominates w_1, then w_2 dominates w_1. Actually, we do not know if in general the domination implies the well domination. However, this is the case when v is Henselian or of rank one.

b) The relation of domination may be defined also between (not necessarily r.t.) extensions of v to $K(X)$ in the same manner. It is easy to see that the diagram (1) may be defined for any extensions w_1 and w_2 of v to $K(X)$. However, the results in Proposition 2.1 and Corollary 2.2 are valid only when w_1 and w_2 are r.t. extensions.

3. Definitions of key polynomials

1. Let (K, v) be a valuation pair. According to MacLane [7], one says that two elements $a, b \in K$ are of the same order of magnitude or *equivalent in* v and writes $a \sim b$ (in v), when:

$$v(a-b)>v(a)=v(b).$$

It is clear that $\sim$ is an equivalence relation on K. Moreover, if $a\sim b$ and $a'\sim b'$ then $aa'\sim bb'$.

Let A be a suitable subring of K. An element $b\in A$ is said to be *equivalence divisible* in A by $a\in A$ relative to v when there exists $c\in A$ such that $b\sim ca$ (in v). It is easy to see if $a\sim a'$, $b\sim b'$, $c\sim c'$ and $b\sim ca$ then $b'\sim c'a'$.

Let w be a valuation on $K(X)$. According to [7], a *key polynomial* over w is a non-constant polynomial $g(X)\in K[X]$ which satisfies the following:

(i) Irreducibility: If $F, G\in K[X]$ and FG is equivalence divisible in $K[X]$ by g relative to w, then one of the factors is equivalence divisible in $K[X]$ by g.

(ii) Minimal degree: Any non-zero polynomial equivalence divisible in $K[X]$ by g has the degree in X not less than $\deg g(X)$.

(iii) The leading coefficient of g is 1, i.e. g is monic.

A polynomial g with condition (i) is said to be *equivalence irreducible in* w.

PROPOSITION 3.1. *Let $f\in K[X]$ be equivalence irreducible in w. Assume that a product FG of polynomials in $K[X]$ is equivalence divisible by f^i, $i\geqq 1$, and F is not equivalence divisible by f. Then G is equivalence divisible by f^i.*

The proof follows by induction over i and is left to the reader.

2. Let w be a valuation on $K(X)$ and let g be a polynomial in $K[X]$. Suppose an ordered group G contains G_w as an ordered subgroup and take $\gamma\in G$. Then a new valuation $w_1(F)$ may be defined as follows:

$$w_1(F)=\inf(w(F_i)+i\gamma)$$

where $F=F_0+F_1g+\cdots+F_2g^s$, $\deg F_i<\deg g$, $0\leqq i<s$ is the g-expansion of $F\in K[X]$.

For the proof of the following result, we send the reader to [7; Theorems 4.2 and 5.1].

THEOREM 3.2. (MacLane [7]) *If g is a key polynomial over w and $\gamma>w(g)$, then the function w_1 defined above is also a valuation on $K[X]$ (and on $K(X)$), which dominates w.*

According to MacLane's terminology, w_1 will be called the *augmented valuation over w, associated with g and γ*. If $\gamma\in QG_w$, i.e. there exists a positive integer $e\neq 0$ such that $e\gamma\in G_w$, we snall say that w_1 is a *commensurable augmented valuation.*

4. Key polynomials over r. t. extension

In this section we study key polynomials over an r. t. extension. The main results are Theorems 4.4 and 4.6. We remark that Theorem 4.6 and its proof are inspired by MacLane's work [7, Theorem 9.4].

Throughout this section w is an r. t. extension of v to $K(X)$ and (a, δ) is a minimal pair of definition of w. We use the notation in Notation 1.3.

PROPSITION 4.1. *If F is a key polynomial over w, then* $\deg F \geqq n$.

PROOF. It is enough to show that if $g \in K[X]$ is of $\deg g < n$ then g cannot be a key polynomial over w. Indeed, take $q \in K[X]$ such that $\deg q < n$ and that $w(g)+w(q)=0$. Then by Corollary 1.4, $(gq)^* \in k_{\bar{v}}$. Hence there exists $t \in K[X]$ with $\deg t < n$, such that $w(t)=0$, and that $t^*=(qg)^{*-1}$. Therefore $(tqg)^* =1$ and so $w(tqg-1)>0$. Hence the condition (ii) is not satisfied by g.

PROPOSITION 4.2. *For $g \in K[X]$, let $g=qf+g_0$, with* $\deg g_0 < n$. *The following are equivalent*:

a) *g is equivalence divisible in $K[X]$* by f (relative to w).

b) $w(g-qf)=w(g_0)>w(g)$.

PROOF. The implication b) $\Rightarrow$ a) is obvious.

a) $\Rightarrow$ b) Suppose there exists $q_1 \in K[X]$ such that $w(g-q_1 f)>w(g)$. Then $w(g_0+(q-q_1)f)>w(g)$. By definition of w, it follows that $w(g) \leqq w(g_0)$.

Assume that $w(g)=w(g_0)$. Then $w((q-q_1)f)=w(g_0) \in G_{\bar{v}}$. Let

$$(q-q_1)f=h_1 f+\cdots+h_t f^t \ \deg h_i<n, \quad 1 \leqq i \leqq t.$$

Then $w((q-q_1)f)=\inf(\bar{v}(h_i(a))+i\gamma)$. Since $w((q-q_1)f) \in G_{\bar{v}}$,

$$\bar{v}(h_i(a))+i\gamma>w(g_0)=w((q-q_1)f), \quad \text{if} \quad i \not\equiv 0 (\operatorname{mod} e).$$

Hence $w\left(1-\left(\frac{h_1}{g_0} f+\cdots+\frac{h_t}{g_0} f^t\right)\right)>0$, or $w\left(1-\left(\frac{h_1}{g_0} f+\cdots+\frac{h_e h}{g_0} \frac{f^e}{h}+\cdots\right)\right)>0$.

Thus $1=\left(\frac{h_e h}{g_0}\right)^* h^*+\cdots$. But this equallity is impossible, because $\left(\frac{h_e h}{g_0}\right)^*, \cdots$ belongs to $k_{\bar{v}}$ and r^* is transcendental over $k_{\bar{v}}$. Therefore $w(g)<w(g_0)$ as claimed.

COROLLARY 4.3. *The polynomial f (used in the definition of w) is a key polynomial over w.*

PROOF. We show that the conditions (i)-(iii) in the definition of a key polynomial are fulfilled.

(i) Let $A, B\in K[X]$ be such that AB is equivalence divisible relative to w by f. Let $A=A'f+A_0$, $\deg A_0<n$ and $B=B'f+B_0$, $\deg B_0<n$. According to Proposition 4.2, we must prove that $w(A)<w(A_0)$ or $w(B)<w(B_0)$. Assume that $w(A)\geqq w(A_0)$ and $w(B)\geqq w(B_0)$. If we write $A_0B_0=Cf+C_0$ with $\deg C_0<n$, then $w(AB)\geqq w(A_0B_0)=\bar{v}(A_0(a)B_0(a))=\bar{v}(C_0(a))=w(C_0)$. But this is a contradiction. The condition (ii) results by Proposition 4.2 and (iii) is obvious.

Now we try to give a characterization of key polynomials over w. According to Proposition 4.1, we shall treat key polynomials of degree just $n=\deg f$ and key polynomials whose degrees are greater than n separately.

THEOREM 4.4. *Let $g\in K[X]$ be a monic polynomial. Consider the following*:

1) *g is a key polynomial over w and equivalence divisible by f.*

2) *g is a key polynomial over w and of* $\deg g=n=\deg f$.

3) *g is irreducible and there exists a root b of g such that (b, δ) is also a minimal pair of definition of w.*

Then we always have 1) $\Rightarrow$ 2) $\Leftrightarrow$ 3). Moreover, 2) $\Rightarrow$ 1) when $\gamma=w(f)$ does not belong to $G_{\bar{v}}$.

PROOF. 1) $\Rightarrow$ 2) Let $g=qf+g_0$, $\deg g_0<n$. According to Proposition 4.2 b), one has $w(g-qf)=w(g_0)>w(g)$. Now since g is also a key polynomial, q or f is equivalence divisible by g. Being $\deg q<\deg g$, f is equivalence divisible by g. So $\deg f\geqq\deg g$. Hence $\deg f=\deg g$ by Proposition 4.1.

2) $\Rightarrow$ 3) By 2) one has $g=f+g_0$, $\deg g_0<n$. So $w(g)=\inf(w(f), w(g_0))=\inf(\gamma, w(g_0))$. Thus $w(g)\leqq\gamma$. Now we remark that $w(g)=\gamma$. Assume $w(g)<\gamma$. Then $w(f)=w(g-g_0)=\gamma>w(g)$. But this is impossible, because g is a key polynomial over w and $\deg g_0<n=\deg g$.

Let $b_1, \cdots, b_n$ be all roots of g in $\bar{K}$ and $g=\prod_{i=1}^{n}(X-b_i)$. We assert that $w(X-b_i)\geqq\delta$ for at least one index i (here $w=w_{(a,\delta)}$). Indeed, assume that $\bar{v}(a-b_i)<\delta$, $1\leqq i\leqq n$. Then

$$\bar{w}(g)=\sum_i w(X-b_i)=\sum_i\inf(\delta, \bar{v}(a-b_i))=\sum_i\bar{v}(a-b_i)=\bar{v}(g(a)),$$

$$w(f)=w(g)=\bar{w}(g)=\bar{v}(g(a))=\bar{v}(g_0(a))=w(g_0).$$

Then $e=1$, and we may choose $h=g_0$ (see Theorem 1.2, c)). Therefore if we put $r=f/g_0$ then $w(r)=0$ and r^* is transcendental over $k_{\bar{v}_2}$. Consequently, $(g/g_0)^*=r^*+1$ is also transcendental over $k_{\bar{v}_2}$. Hence by [3, Proposition 2.1], there exists a root b of $gg_0=0$ such that (b, δ) is a pair of definition of $\bar{w}$. Now since (a, δ) is a minimal pair and $\deg g_0<n$, it follows that $g_0(b)\neq 0$. In conclusion, one has necessarily $g(b)=0$ and $\bar{v}(b-a)\geqq\delta$. This is a contradiction.

3) $\Rightarrow$ 2) Since (b, δ) is also a minimal pair of definition of w, g is a key polynomial over w and $\deg g=\text{ceg} f=n$.

Now let us assume that $\gamma\notin G_{\bar{v}}$. Then one has the implication 2) $\Rightarrow$ 1). Indeed, we have remarked that $w(g-f)=w(g_0)\geqq w(f)$. Then, since $w(g_0)\in G_{\bar{v}}$, $w(g-f)>w(f)$, i.e. g is equivalence divisible by f.

REMARK 4.5. Now we give an example which shows that the implication 2) $\Rightarrow$ 1) in Proposition 4.4 is not necessarily valid if $\gamma\in G_{\bar{v}}$. For that take an algebraically closed field K and $a, b\in K$ such $v(a-b)=\delta$. Let $w=w_{(a,\delta)}$. Then (a, δ) and (b, δ) are both minimal pairs of definition of w.

Hence, $X-a$ and $X-b$ are both key polynomials over w. But since $w(X-b)=v(a-b)=\delta$, $X-b$ is not equivalence divisible in $K[X]$ (with respect to w) by $X-a$.

For key polynomials over w whose degrees are greater than $n=\deg f$, one has:

THEOREM 4.6. *Take $g\in K[X]$ such that $\deg g>n=\deg f$ and consider the f-expansion of g:*

$$g=g_0+g_1f+\cdots+g_tf^t, \qquad \deg g_i<n,\ 0\leqq i\leqq t.$$

Then the following are equivalent:

1) *g is a key polynomial over w.*

2) *g satisfies the following:*

α) $w(g)=w(g_0)$,

β) $t\equiv 0 (\text{mod } e)$, $g_t=1$, *and* $w(g)=w(f^t)=se\gamma$,

γ) *g is equivalence irreducible in w.*

3) *$t=se$, $w(g)=se\gamma$, g is monic of degree tn and $(g/h^s)^*$ is a monic and irreducible polynomial of degree s in $k_{\bar{v}}[r^*]$ whose constant term is not zero.*

PROOF. 1) $\Rightarrow$ 2). By definition of w, $w(g)\leqq w(g_0)$. If $w(g)<w(g_0)$, $w(g-qf)>w(g)$ for a suitable q of $\deg q<\deg g$. Hence q or f is equivalence divisible by g. But this is impossible. Thus has $w(g)=w(g_0)$.

Further, $w(g)=w(g_0)$ shows that $w(g)\in G_{\bar{v}}$. Now we note that $w(g)=w(g_tf^t)$. Assume that $w(g)<w(g_tf^t)$. Then $w(g-(g_0+\cdots+g_{t-1}f^{t-1}))>w(g)$. But this is also impossible, because g is a key polynomial over w. Hence two remarks $w(g)=w(g_tf^t)$ and $w(g)\in G_{\bar{v}}$ imply that $t\equiv 0(\text{mod } e)$.

Let us show that $g_t=1$. If $g_t\neq 1$ then, since g is monic, $\deg g_t>0$. Take $u\in K[X]$ such that $\deg u<n$, $w(ug_t)=0$ and $(ug_t)^*=1$. This means $w(ug_t-1)>0$. So $w(ug_tf^t-f^t)>w(f^t)=w(ug)$. Now since f is a key polynomial and $\deg g_{t-1}<n$, $ug_tf^t=u'g-d$, where $\deg d<\deg q$ and $w(u')=w(u)$. Thus $w(ug_tf^t-f^t)=w(u'g-d-f^t)>w(f^t)=w(u'g)$. But this is impossible, because g is also a key polynomial over w and $\deg(d+f^t)<\deg g_t=1$.

2) $\Rightarrow$ 1) By β) and γ) conditions (i) and (iii) of a key polynomial are fulfilled. Now let $d\in K[X]$ be equivalence divisible by g. Hence there exists $q\in K[X]$ with $w(qg-d)>w(d)$. We must show $\deg d\geqq\deg g$. Writing $q=\sum_{i=0} q_if_i$ with $\deg q_i<n$, let j be the greatest index t such that $w(q)=\bar{v}(q_j(a))+j\gamma$. Thus in qg one has the term $A=(q_j+q_{j+1}g_{t-1}+\cdots)f^{t+j}$. Then $w(A)=w(q_j)+w(f^{j+t})=w(qg)=w(d)$. Hence if $\deg d<\deg g$ the term A whose degree is at least $\deg g$ must appear in $qg-d$. So the inequality $w(qg-d)>w(d)=w(qg)$ is impossible. Therefore $\deg d\geqq\deg g$, as claimed.

2) $\Rightarrow$ 3) By β) it results

$$t=se\equiv 0(\operatorname{mod} e),\ w(g)=se\gamma=w(h^s)\quad\text{and } g \text{ is monic}$$

(remind that $w(h)=e\gamma$ and $\deg h<n$). Hence one has $w(g/h^s)=0$, and

$$g/h^s=g_0/h^s+g_1f/h^s+\cdots+f^{se}/h^s,$$

$$w(g_if^i/h^s)\geqq 0\quad\text{and}\quad w(g_jf^j/h^s)>0\quad\text{if}\quad j\not\equiv 0(\operatorname{mod} e).$$

If $j=ie$, $w(g_{ie}f^{ie}/h^s)=w\left(\frac{g_{ie}f^{ie}}{h^{s-i}h^i}\right)\geqq 0$. Since $w(f^e/h)=0$, $w(g_{ie}/h^{s-i})\geqq 0$. So $(g_{ie}/h^{s-i})^*\in k_{\bar{v}}$(see Corollary 1.4). Therefore

$$(g/h^s)^*=A_0+A_1r^*+\cdots+r^{*s},\ A_i\in k_{\bar{v}},\ 0\leqq i<s.$$

Now we show that this is an irreducible polynomial of $k_{\bar{v}}[r^*]$. Indeed, assume that A', B', C' are polynomials of $k_{\bar{v}}[r^*]$ such that $(g/h^s)^*C'=A'B'$. Let A, B and C be the liftings of A', B' and C', respectively (see Notation 1.7). Then $(g/h^s)^*(C/h^u)^*=(A/h^q)^*(B/h^t)^*$. Hence

$$w\left(\frac{gC}{h^{s+u}}-\frac{AB}{h^{q+t}}\right)>0.$$

Let $i=q+t-s-u$. If $i\geqq 0$, then $w(gCh^i-AB)>w(h^{q+t})=w(AE)$(see Notation 1.7). Then by condition γ) it follows that, say, A is equivalence divisible by g. Thus for a suitable polynomial $D\in K[X]$, one has

$$w(gD-A)>w(A)=w(gD),$$

$$w\left(g/h^2\cdot\frac{D}{h^{q-s}}-A/h^q\right)>0,\quad\text{or}\quad (g/h^s)^*(D/h^{q-s})^*=(A/h^q)^*=A'.$$

Therefore, since $(D/h^{q-s})^*\in k_{\bar{v}}[r^*]$, A' is divisible by $(g/h^s)^*$. If $i<0$, $w(gC-h^{-i}AB)>w(ABh^{-i})=w(gC)$. Now because g is a key polynomial and $\deg h<n<\deg g$, it results that, say, A is equivalence divisible by g. Thus as above $A'=(A'/h^q)^*$ is divisible by $(g/h^s)^*$ in $k_{\bar{v}}[r^*]$. In conclusion $(g/h^s)^*$ is irreducible in $k_{\bar{v}}[r^*]$ and since $w(g_0)=w(g)=w(h^s)$, its constant terms is not zero.

3) $\Rightarrow$ 2) By 3) it results that $t=se$, $w(g)=se\gamma=t\gamma$, and $g_t=1$. Hence β) is accomplished. The condition α) is also satisfied because the constant term of $(g/h^s)^*$ is not zero.

Now we are only to show that γ) is also true. For this take $A, B\in K[X]$ such that AB is equivalence divisible by g. Then there exists $D\in K[X]$ such that

$$w(gD-AB)>w(AB)=w(gD).$$

Let i and j be the smallest non-negative integers such that

$$w(A)+i\gamma=\bar{v}(\omega(a)), \quad \text{and} \quad w(B)+j\gamma=\bar{v}(\sigma(a)),$$

where ω, $\sigma\in K[X]$, $\deg\omega<n$ and $\deg\sigma<n$. Then

$$w\Big(\frac{g}{h^s}\cdot\frac{Dh^sf^{i+j}}{\omega\sigma}-\frac{Af^i}{\omega}\cdot\frac{Bf^j}{\sigma}\Big)>0, \quad \text{i.e.,}$$

$$(g/h^s)^*\Big(\frac{Dh^sf^{i+j}}{\omega\sigma}\Big)^*=\Big(\frac{Af^i}{\omega}\Big)^*\Big(\frac{Bf^j}{\sigma}\Big)^*.$$

Here according to Corollary 1.6, all factors are polynomials of $k_{\bar{v}}[r^*]$. So, since $(g/h^s)^*$ is irreducible by hypothesis, it results that it divides, say, $\Big(\frac{Af^i}{\omega}\Big)^*$. Hence one has the equality

$$(g/h^s)^*\cdot G'=\Big(\frac{Af^i}{\omega}\Big)^*, \qquad G'\in k_{\bar{v}}[r^*].$$

According to Notation 1.7, one may write $G'=(G/h^p)^*$ with $G\in K[X]$ and a suitable non-negative integer p. Then

$$w\Big(\frac{gG}{h^{s+p}}-\frac{Af^i}{\omega}\Big)>0, \quad \text{or} \quad w(gG\omega-Af^ih^{s+p})>w(h^{s+p}\omega)=w(gG\omega).$$

Furthermore, by 3) and Proposition 4.2, g is not equivalence divisible by f. Then by Proposition 3.1, it results that $G\omega$ is equivalence divisible by f^i, i.e. $w(G\omega-f^iH)>w(G\omega)=w(f^iH)$, $H\in K[X]$. Hence

$$w(gHf^i-Af^ih^{s+p})>w(gHf^i), \quad \text{or} \quad w(gH-Ah^{s+p})>w(gH).$$

Now let $d\in K[X]$ be such that $w(dh^{s+p})=0$, $\deg d<n$, and that $w(dh^{s+p}-1)>0$. Thus

$$w(gHd-A(dh^{s+p}-1+1))>w(gHd), \quad \text{or} \quad w(gHd-A)>w(gHd)=w(A).$$

This means that A is equivalence divisible by g. In conclusion g is equivalence irreducible. The proof of Theorem 4.6 is complete.

COROLLARY 4.7. *Let G be a monic and irreducible polynomial of $k_{\bar{v}}[r^*]$ whose constant term is not zero. Let g be a lifting of G (see Notation 1.7). Then g is a key polynomial over w. In particular, g is an irreducible polynomial of $K[X]$.*

PROOF. Let $s=\deg G$. Then $\deg g=sen=tn$, and $g=g_0+g_1f+\cdots+f^{se}$, $g_i\in K[X]$, $\deg g_i<n$. The condition that G has a non-zero constant term shows that $w(g)=w(g_0)=w(f^{se})=se\gamma$. Thus, since $G=(g/h^s)^*$, by condition 3) in Theorem 4.6 it results that g is a key polynomial over w.

5. Valuation defined by a key polynomial

In this section we show that key polynomials over an r.t. extension of v to $K(X)$ give new r.t. extensions of v to $K(X)$. In particular, we show that key polynomials may be used to yield minimal pairs.

THEOREM 5.1. *Let w be an r.t. extension of v to $K(X)$ and let f_1 be a key polynomial over w. Take $\gamma_1>G_{\bar{v}}$ such that $\gamma_1>w(f_1)$. Let w_1 be the augmented valuation over w associated with f_1 and γ_1. Then w_1 is an r.t. extension of v to $K(X)$. Moreover there exists a root a_1 of f_1 and $\delta_1\in\bar{G}_v$ such that (a_1, δ_1) is a minimal pair of w_1 with respect to K and w_1 well dominates w.*

PROOF. As usual we keep the notations stated in Notation 1.3. Let (a, δ) be a minimal pair of definition of w. Then two cases are possible $\deg f_1=n=\deg f$ or $\deg f_1>n$ (see Proposition 4.1). We shall consider each case separately.

A) First assume that $\deg f_1=n$. Then according to condition 3) in Theorem 4.4, there exists a root a_1 of f_1 such that (a_1, δ) is also a minimal pair of definition of w. Hence we may assume that $f_1=f$ and $a_1=a$. Since (a, δ) is a minimal pair of definition of w, one has

$$w(f)=\gamma=\inf(\bar{v}(A_i)+i\delta), \quad \text{where } f=\sum_{i=1}^{n} A_i(X-a)^i,\ A_i\in\bar{K}$$

$$\text{and} \quad \delta=\sup_{1\leq i\leq n}\frac{\gamma-\bar{v}(A_i)}{i}.$$

Now let us define

$$\delta_1 = \sup_{1\leq i\leq n} \frac{\gamma_1 - \bar{v}(A_i)}{i}.$$

Since by hypothesis $\gamma_1 > \gamma$, it results that $\delta_1 > \delta$. Therefore (a, δ_1) is also a minimal pair because (a, δ) is a minimal pair with respect to K. Let w' be the restriciton of $w_{(a,\delta_1)}$ to $K(X)$. Then by Theorem 1.2, for $F \in K[X]$

$$w'(F) = \inf(\bar{v}(F_i(a)) + i\gamma_1) = \inf(w(F_i) + i\gamma_1),$$

where $F = F_0 + F_1 f + \cdots + F_s f^s$, $\deg F_i < n$, $0 \leqq i \leqq s$. Hence $w_1 = w'$ by definition of an augmented valuation. Therefore w_1 well dominates w because $w_{(a,\delta)} < w_{(a,\delta_1)}$ by [4, Proposition 2.1].

B) Next assume that $\deg f_1 = n_1 > n$. Then by assertion 3) in Theorem 4.6, there exists a positive integer s such that $w(f_1/h^s) = 0$ and that $(f_1/h^s)^*$ is an irreducible polynomial of $k_{\bar{v}}[r^*]$. Then, according to [3, Prosition 1.1] there exists an element $a_1 \in \bar{K}$ such that $f_1(a_1)h^s(a_1) = 0$ and that (a, δ) is a pair of definition of $w_{(a,\delta)}$, or equivalently $\bar{v}(a_1 - a) \geqq \delta$. Now since $\deg h < n$ and (a, δ) is a minimal pair with respect to K, one has necessarily $f_1(a_1) = 0$. Writing $f_1 = \sum_{i=1}^{n_1} A'_i (X - a_1)^i$, $A'_i \in \bar{K}$, define

$$\delta_1 = \sup_{1\leq i\leq n_1} \frac{\gamma_1 - \bar{v}(A'_i)}{i}.$$

In what follows we shall show that w_1 is an r.t. extension of v to $K(X)$ and that (a_1, δ_1) is a minimal pair of definition of w_1 with $\delta < \delta_1$, or w_1 well dominates w. We shall divide the proof in several steps.

B1) At this point we introduce an useful notation. Let us denote by P the subring of $K[X]$ whose elements are fractions $p = F/G$ such that $w(p) \geqq 0$, and that every irreducible factor of G has the degree smaller than n. According to Corollaries 1.4 and 1.6 it results that for every $p \in P$ the mapping $p \rightsquigarrow p^*$ gives a surjective ring homomorphism $\rho: P \to k_{\bar{v}}[r^*]$.

If $p \in O_{w_1}$, let us denote by p^{**} the image of p into the residue field k_{w_1}. According to MacLane's Theorem (see Theorem 3.2), one has $w < w_1$. So if $p \in P$, then $p \in O_{w_1}$. Hence the mapping $p \rightsquigarrow p^{**}$ gives a ring homomorphism $\rho_1: P \to k_{w_1}$. Finally, it is easy to see that the mapping $p^* \rightsquigarrow p^{**}$ gives a k_v-algebras homomorphism $\varphi: k_{\bar{v}}[r^*] \to k_{w_1}$, which makes the following diagram commutative

$$(2) \qquad \begin{array}{ccc} & p & \\ {}^{\rho}\swarrow & & \searrow^{\rho_1} \\ k_{\bar{v}}[r^*] & \xrightarrow{\ \varphi\ } & k_{w_1}. \end{array}$$

B2) Since $\gamma_1 > w(f_1)$ and $w(h)=w_1(h)$ we note that the kernel of φ is generated by $(f_1/h^s)^*$. This implies that for every $z \in k_{\bar{v}}[r^*]$, $\varphi(z)$ is algebraic over k_v.

B3) Let e_1 be the smallest positive integer such that $e_1\gamma_1 \in G_w$. We claim that there exists a polynomial $h_1 \in K[X]$ such that $\deg h_1 < \deg f_1 = n_1$ and that $w(h_1)=w_1(h_1)=e_1\gamma_1$.

According to Theorem 1.2, since $G_w = G_{\bar{v}} + Z\gamma$ and $e\gamma \in G_{\bar{v}}$, one has $e_1\gamma_1 = w(gf^i)$ for suitable g and i with $\deg g < n$, $0 \leqq i < e$. Then, if $gf^i = qf_1 + h_1$, $\deg h_1 < n_1$, we have $w(gf^i) = e_1\gamma_1 = w(h_1)$. Indeed, assume that $e_1\gamma_1 > w(h_1)$. Then $w(qf_1 + h_1) > w(h_1)$. But this is impossible, because f_1 is a key polynomial over w and $\deg h_1 < n_1$. Further, if $e_1\gamma_1 < w(h_1)$ then $w(gf^i - qf_1) > w(gf^i) = e_1\gamma_1$. Since f_1 is a key polynomial over w, it results that one of the polynomials g or f is equivalence divisible by f_1. But this is also impossible since $\deg n < n_1$. Hence $w(gf^i) = e_1\gamma_1 = w(h_1)$.

B4) Now we shall prove that, if we put $r_1 = f_1^{e_1}/h_1$ with h_1 as above (see B3)), $w_1(r_1)=0$ and $r_1^{**} \in k_{w_1}$ is transcendental over k_v. Moreover, r_1 is the element of $K(X)$ of the smallest degree with these properties.

For the sake of simplicity, in the rest of this proof we shall express r_1^{**} by y. Assume that $y \in k_{w_1}$ is algebraic over k_v. Then there exists $b_0, \cdots, b_{t-1} \in K$ such that $v(b_i) \geqq 0$, $0 \leqq i < t$, and that

$$b_0^* + \cdots + b_{t-1}^* y^{t-1} + y^t = 0.$$

Let us consider the polynomial $G = b_0 h_1^t + b_1 h_1^{t-1} f_1^{e_1} + \cdots + b_{t-1} h_1 f_1^{(t-1)e_1} + f_1^{te_1}$. Then $w_1(G) > w(h_1^t) = t_1 e_1$.

On the other hand, since $\deg h_1 < n_1$, $\deg b_i h_1^{t-i} f_1^{e_1 i} < te_1 n_1$. So, in the f_1-expansion of G, the term $f_1^{te_1}$ must appear. But, then, according to the definition of w_1, one has $w_1(G) \leqq t_1 e_1$. This is a contradiction. Therefore y is transcendental over k_v.

Furthermore, suppose $p = F/H \in K(X)$ satisfies $w_1(p)=0$ and $\deg p = [K(X) : K(p)] = \max(\deg F, \deg H) < \deg r_1 = e_1 n_1$. Let

$$F = F_0 + F_1 f_1 + \cdots + F_m f_1^m, \quad \text{and} \quad H = H_0 + H_1 f_1 + \cdots + H_q f_1^q$$

be the f_1-expansions of F and H respectively. Since $w_1(p)=0$, one has

$$w_1(F) = \inf_i (w(F_i) + i\gamma_1) = w_1(H) = \inf_j (w(H_j) + j\gamma_1).$$

Now since $\deg p < e_1 n_1$, it follows that $m < e_1$, $q < e_1$. So there exists only one index, say i, such that

$$w_1(F) = w(F_i) + i\gamma_1 = w_1(H) = w(H_i) + i\gamma_1.$$

But then $p^{**}=\left(\frac{F_i}{H_i}\right)^{**}$, because

$$p=\frac{F}{H}=\frac{F_i}{H_i}\cdot\frac{F_0/F_if_1^i+\cdots+1+\cdots}{H_0/H_1f_1^i+\cdots+1+\cdots}.$$

To end the proof of B4) it is enough to show that p^{**} is algebraic over k_v. Indeed, since $\deg F_i<n_1$, $\deg C_i<n_1$, $w_1(F_i)=w(F_i)=w_1(H_i)=w(H_i)$. Let d be a positive integer such that $dw(F_i)=v(c)$ for a suitable $c\in K$. Then we have (see (2)):

$$p^{*d}=\left(\frac{F_i}{H_i}\right)^{*d}=((F_i^d/c)/(H_i^d/c))^*=(F_i^d/c)^*/(H_i^d/c)^*.$$

$$\varphi(p^{*d})=\varphi((F_i^d/c)^*)/\varphi((H_i^d/c)^*)=p^{**d}.$$

Hence p^{**} is also algebraic over k_v(see B1)).

B5) Finally we shall prove that the pair (a_1, δ_1) defined above (see B)) is a minimal pair of definition of w_1 (with respect to K). For this we show that $[K(b): K]\geqq n_1$, whenever (b, δ_1) is a pair of definition of w_1 with respect to K. We shall prove that $\deg g\geqq n_1$ if g is the minimal polynomial of b over K.

Indeed, let us assume that $\deg g<n_1$. According to the definition of an augmented valuation, one has $w_1(g)=w(g)$. Take a suitable positive integer t such that $tw(g)=v(c)$, $c\in K$. Then $w(g^t/c)=w_1(g^t/c)=0$, and by diagram (2) one has $0\neq(g^t/c)^{**}=\varphi((g^t/c)^*)$. Hence $(g^t/c)^{**}$ is algebraic over k_v. But this contradicts the assumption that (b, δ_1) is a pair of definition of w_1 (see [1]).

Furthermore, since $(f_1^{e_1}/h_1)^{**}$ is transcendental over k_v, according to [3, Proposition 1.1] it follows that there exists a root a_1' of $f_1h_1=0$ such that (a_1', δ_1) is a pair of definition of w_1. Since $\deg h_1<n_1=\deg f_1$ it follows that a_1' is necessarily a root of f_1 and that (a_1', δ_1) is a minimal pair of definition of w_1.

And we have the inequality $\delta<\delta_1$ because $w(f_1)<\gamma_1$(see B)). The proof of Theorem 5.1 is complete.

REMARK 5.2. Let w_1, w_2 be two r.t. extensions of v to $K(X)$ such that $w_1<w_2$. In general, we do not know if w_2 well dominates w_1. However, according to Theorem 5.1, if w_2 is an augmented valuation over w_1, then w_2 well domintes w_1. Now we shall give an example which shows that the "well domination" is not a special property of an augmented valuation, i.e. it is possible that w_2 well dominates w_1 even if w_2 is not an augmented valuation over w_1.

Let K be the field of 3-adic numbers and v the 3-adic valuation on K. Let $a_2=\sqrt[3]{3}$. The minimal polynomial of a_2 is $f_2=X^3-3$. Let ω be a primitive cube-root of 1. Then, $a_2, \omega a_2, \omega^2 a_2$ are all roots of f_2. One has $\sup(\bar{v}(a_2-\omega a_2)$,

$\bar{v}(a_2-\omega^2 a_2))=4/3$. Hence according to [3, Proposition 3.2, b)], $(a_2, 2)$ is a minimal pair. Let w_2 be the restriction of $w_{(a_2,2)}$ to $K(X)$ and w_1 the restriction of $w_{(0,0)}$ to $K(X)$. It is easy to see that w_2 well domintes w_1. However, w_2 is not an augmented valuation over w_1. Indeed, if w_2 would be an augmented valuation over w_1, then for every polynomial g of $\deg g<3$ we have $w_1(g)=w_2(g)$. But it is easy to see that $w_1(X^2-3)<w_2(X^2-3)$.

On the other hand, Theorem 5.5 gives a characterization of r.t. valuations which are augmented valuations over another r.t. extensions of v to $K(X)$. First we prove the following:

LEMMA 5.3. *Let w_1, w_2 be r.t. extensions of v to $K(X)$ such that $w_1<w_2$. Let $g\in K[X]$ be a monic polynomial of the smallest degree such that $w_1(g)<w_2(g)$. Then g is a key polynomial over w_1.*

PROOF. Let (a_i, δ_i) be a minimal pair of definition of w_i and let f_i be the monic minimal polynomial of a_i, $i=1, 2$. According to Corollary 2.2 b) it follows that, if $h\in K[X]$ is of $\deg h<\deg f_1$, $w_1(h)=\bar{v}(h(a_1))=\bar{v}(h(a_2))=w_2(h)$. Hence $\deg g\geqq\deg f_1$. Let t be a positive integer such that $w_1(g^t)=-v(c)$, $c\in K$. Then $w_1(cg^t)=0$, and $(cg^t)^*$ is a non-zero element of $k_{\bar{v}}[r_1^*]$. The hypothesis $w_1(g)< w_2(g)$ yields $\varphi((cg^t)^*)=0$. But then, according to Proposition 2.1 a), it results that $(cg^t)^*$ is transcendental over k_v. Therefore, according to [3, Proposition 1.1] there exists a root b of g such that (b, δ_1) is a pair of definition of w_1.

Now we consider the cases $\deg g=\deg f_1$ and $\deg g>\deg f_1$ separately.

Suppose $\deg g=\deg f_1$. Then (b, δ) is also a minimal pair of definition for w_1. Hence according to Corollary 4.3, g is a key polynomial over w_1.

Now let us assume that $\deg g>\deg f_1$ and

$$g=A_0+A_1f_1+\cdots+A_tf_1^t,\quad \deg A_i<\deg f,\quad 0\leqq i\leqq t.$$

To show that g is a key polynomial over w_1 we shall prove that g satisfies the condition 3) in Theorem 4.6.

a) First we claim that $w_1(g)=w_1(A_0)$. Since $w_1(g)\leqq w_1(A_0)$ we show that $w_1(g)<w_1(A_0)$ implies a contradiction. Indeed, assume that $w_1(g_2)<w_1(A_0)$. Let $g=A_0+f_1q$, $\deg q<\deg g$ be the f_1-expansion of g. Then $w_1(g-f_1q)=w_1(A_0)> w_1(g)$. Hence $w_1(g)=w_1(f_1q)<w_1(A_0)$. Since $\deg q<\deg g$ then, by hypothesis on g, one has

$$w_2(f_1q)=w_1(f_1q)=w_1(g)<w_2(g).$$

Thus $w_1(g)=w_1(f_1q)=w_2(f_1q)=w_2(g-f_1q)=w_2(A_0)=w_1(A_0)$. So we get a desired contradiction.

b) Next we show that $t=se_1$ and $w_1(g)=se_1\gamma_1$. Note that by a) it follows that $w_1(g)=w_1(A_0)=\bar{v}(A_0(a_1))\in G_{\tilde{v}}$, where $\tilde{v}$ is the restriction of $\bar{v}$ to $K(a_1)$. Now we remark that $w_1(g)=w_1(A_1f_1^t)$. Indeed, if $w_1(g)<w_1(A_tf_1^t)$, then $w_1(B)=w_1(g)$, where $B=g-A_tf_1^t$. Hence $w_1(A_tf_1^t)>w_1(B)$. Further, since $\deg B<\deg g$, one has $w_1(B)=w_2(B)$. And $w_1(A_tf_1^t)=w_2(A_tf_1^t)$. On the other hand, since $w_2(g)>w_1(g)=w_1(B)=w_2(B)$, one has $w_1(A_tf_1^t)=w_2(g-B)=w_2(B)=w_1(B)$ a contradiction. Therefore $w_1(g)=w_1(A_tf_1^t)$. Since $w_1(g)\in G_{\tilde{v}}$, it follows that $w_1(f_1^t)\in G_{\tilde{v}}$, or $t=se_1$.

Now we shall prove that $A_t=1$, or A_t is of degree 0. Since $w_1(g)\in G_{\tilde{v}}$, there exists $h\in K[X]$, $\deg h<\deg f_1$ such that $w_1(g)=w_1(h)=\bar{v}(h(a_1))$. Hence $0\neq(g/h)^*\in k_{\tilde{v}}[r_1^*]$. We show that $(g/h)^*$ is in fact an irreducible polynomial of $k_{\tilde{v}}[r_1^*]$. Note that by hypothesis $\varphi((g/h)^*)=0$. Hence to prove that $(g/h)^*$ is irreducible it is enough to show that $(g/h)^*$ is the kernel of φ.

Let $m\in k_{\tilde{v}}[r^*]$ be the monic generator of the kernel of φ, i.e.

$$m=u_0+u_1r_1^*+\cdots+u_{p-1}r_1^{*p-1}+r_1^{*p}.$$

Since $w_1(g)\in G_{\tilde{v}}$, $w_1(A_if_1^i)>w_1(g)=w_1(h)$, for every $i\not\equiv0(\mathrm{mod}\, e_1)$. Thus

$$\begin{aligned}(g/h)^*&=(A_0/h)^*+(A_{e_1}h_1/h)^*(f_1^{e_1}/h_1)^*+\cdots+(A_th_1^s/h)^*(f_1^{e_1}/h_1)^{*s}\\&=u_0'+u_1'r_1^*+\cdots+u_s'r_1^{*s},\qquad u_i'=\Big(\frac{A_{ie_1}h_1^i}{h}\Big)^*\in k_{\tilde{v}},\ 0\leqq i\leqq s.\end{aligned}$$

Now since m is the kernel of φ, it follows that $p\leqq s$. Let $M=m_0+m_1f_1^{e_1}+\cdots+f_1^{pe_1}$ be a lifting of m in $K[X]$. Since $\varphi(m)=0$, it follows that $w_1(M)<w_2(M)$. Thus $\deg M\geqq\deg g$, or $pe_1\deg f_1\geqq se_1\deg f_1+\deg A_t$. This inequality together with the inequality $p\leqq s$ implies that $s=p$ and $A_s=1$. Therefore it results that $w_1(g)=w_1(f_1^t)=se_1\gamma_1$ as claimed.

c) Finally we shall prove that $(g/h_1^s)^*$ is an irreducible polynomial of $k_{\tilde{v}}[r_1^*]$, with non-zero constant term. Indeed, by a) and b) one has $w_1(g)=w_1(A_0)=se_1\gamma_1=w_1(h_1^s)$. On the other hand, since $w_1(g)\in G_{\tilde{v}}$, $w_1(A_if_1^i)>w_1(g)$ if $i\not\equiv0(\mathrm{mod}\, e_1)$. Hence

$$(g/h_1^s)^*=(A_0/h_1^s)^*+(A_{e_1}/h_1^{s-1})^*r_1^*+\cdots+r_1^{*s},\ (A_0/h_1^s)^*\neq0.$$

In the same way as for $(g/h)^*$, we see that $(g/h_1^s)^*\in\mathrm{Ker}\,\varphi$. So $(g/h_1^s)^*$ is divisible by m. But, since we have already proved that $s=p$, it follows that $(g/h_1^s)^*$ is also an irreducible polynomial of $k_{\tilde{v}}[r_1^*]$ whose constant term $(A_0/h_1^s)^*$ is not-zero.

REMARK 5.4. A) Note that the diagram (1) can be derived only by the hypothesis that w_2 is an extension (but not necessarily an r. t. extension) of v to $K(X)$ which dominates w_1. So it is easy to see that Lemma 5.3 is true without the hypothesis that w_2 is an r. t. extension of v.

B) By the proof of Lemma 5.3 it follows that $(g/h_1^s)^*$ is the kernel of φ.

THEOREM 5.5. *Let w_1, w_2 be r.t. extensions of v to $K(X)$ such that $w_1<w_2$. Let (a_i, δ_i) be a minimal pair of definition of w_i and let f_i be the monic minimal polynomial of a_i (with respect to K), $i=1, 2$. The following assertions are equivalent*:

1) *f_2 is a key polynomial over w_1.*

2) *f_2 is the polynomial in $K[X]$ of the smallest degree such that $w_1(f_2)<w_2(f_2)$.*

In this case w_2 is an augmented valuation over w_1 and w_2 well dominates w_1.

PROOF. 1)$\Rightarrow$2) First, let us assume that $\deg f_1=\deg f_2$. Then, according to Theorem 4.4 3), there exists a root b of f_2 such that (b, δ_1) is also a minimal pair of definition of w_1. Hence we may assume that $f_1=f_2$ and that $b=a_1$. The inequality $w_1(f_2)<w_2(f_2)$ follows by Corollary 2.2.

Next, let us assume that $n_1=\deg f_1<n_2=\deg f_2$. According to Theorem 4.6 3), there exists a suitable positive integer s such that $(f_2/h_1^s)^*=\rho_1(f_2/h_1^s)$ is an irreducible polynomial of $k_{\bar{\delta}}[r_1^*]$ (see Notation 1.3). And by diagram (1) and Corollary 2.2 a), it follows that $\varphi((f_2/h_1^*)^*)=0$. Hence $(f_2/h_1^s)^*$ is an irreducible polynomial which generates $\operatorname{Ker}\varphi$.

Futhermore, let $g\in K[X]$ be of the smallest degree such that $w_1(g)<w_2(g)$. By Lemma 5.3, g is a key polynomial over w_1. And, by Theorem 4.6 3) and Corollary 2.2, we may assume $\deg g>n_1$. By Theorem 4.6 3), it results that for a suitable t, $(g/h_1^t)^*$ is an irreducible polynomial and that $(g/h_1^t)^+\in\operatorname{Ker}\varphi$. So $\deg(g/h_1^t)=\deg(f_1/h_1^s)$. This means that $\deg g=\deg f_2$.

The implication 2)$\Rightarrow$2) is a special case of Lemma 5.3.

Finally, it is clear that w_2 is the augmented valuation over w_1 associated with f_2 and $\gamma_2=w_2(f_2)$. So by Theorem 5.1, w_2 well dominates w_1.

6. Some applications

In this section we use the above results on key polynomials and augmented valuations over an r. t. extension to give a new proof of a result [3, Theorem 4.4]. We begin by a completion of Theorem 5.1.

Let w be an r. t. extension of v to $K(X)$. Let (a, δ) be a minimal pair of w with respect to K. As usual we shall use Notation 1.3. If g is a key polynomial over w such that $\deg g>\deg f$, then according to Theorem 4.6 3), there exists a positive integer s such that $(g/h^s)^*$ is an irreducible polynomial of $k_{\bar{v}}[r^*]$. The polynomial $(g/h^s)^*$ will be called *the residue of the key polynomial g*.

LEMMA 6.1. *Let w be an r.t. extension of v to $K(X)$ and let w_1 be the augmented valuation over w associated with a key polynomial g over w and $\gamma_1 \in G_{\bar{v}}$ with $\gamma_1 > w(g)$. Then*

a) *k_{w_1} is cannonically isomorphic to k_w if* $\deg g = \deg f$.

If $\deg g > \deg f$, *then*

b) *$k_{w_1} \cong (k_{\bar{v}}[r^*]/(g/h^s)^*)(t)$, t transcendental over $k_{\bar{v}}$, and*

c) *$G_{w_1} = G_w + Z\gamma_1$.*

PROOF. a) According to Theorem 4.4 3), there exists a root b of g such that (b, δ) is a minimal pair of definition of w. Thus according to Theorem 1.2 d), the residue field k_w is cannonically isomorphic to $k_{v'}(t)$, where v' is the restriction of $\bar{v}$ to $K(b)$ and t is transcendental over $k_{v'}$. Now according to the step A in Theorem 5.1, the augmented valuation w_1 has a minimal pair (b, δ_1), where $\delta < \delta_1$. Also according to Theorem 1.2 d), it follows that $k_{w_1} \cong k_{v'}(u)$, where u is a variable, i. e. $k_w \cong k_{w_1}$ as claimed.

b) Let us consider the diagram (1). Since g is the polynomial of the smallest degree such that $w(g) < w_1(g)$, according to the proof of Lemma 5.3 (see Remark 5.4 B)) it results that $(g/h^s)^*$ (the residue of the key polynomial g) is the kernel of φ. Since according to Theorem 1.2 d), k_{w_1} is isomorphic to the field of the rational function of one variable over the algebraic closure of k_v in k_{w_1}, we are only to prove that the image of φ in k_{w_1} coincides with the algebraic closure of k_v in k_{w_1}. Indeed, according to Theorem 5.1, w_1 has a minimal pair of definition (a_1, δ_1) where a_1 is a root of g. Hence if $y \in k_{w_1}$ is algebraic over k_v then, according to Theorem 1.2 d), there exists $F \in K[X]$ such that $\deg F < \deg g$, $w_1(F) = \bar{v}(F(a_1)) = 0$, and that F^{**} is just y. Now since $\deg F < \deg g$, $w(F) = w_1(F) = 0$ and $\varphi(F^*) = F^{**} = y$, where F^* is the residue of F in k_w. To complete the proof, it suffices to remark that the image of φ is included in the algebraic closure of k_v in k_{w_1} because the kernel of φ is not trivial.

The part c) results from the definition of an augmented valuation.

LEMMA 6.2. *Let w be an r.t. extension of v to $K(X)$. Assume that there exists a subgroup G of G_v such that $G_w < G$ and that the quotient group G/G_w is cyclic. Then there exists a key polynomial g over w and $\gamma_1 \in G_v$, with $\gamma_1 > w(g)$ such that, $G_{w_1} = G$ and k_{w_1} is k_v-isomorphic to k_w, where w_1 is the augumented valuation over w defined by g and γ_1.*

PAOOP. As usual we shall use Notation 1.3. Let (a, δ) be a minimal pair of definition of w. Two cases are possible: $e=1$ or $e>1$.

If $e=1$, then $G_w=G_{\bar{v}}$. Take $\gamma_1\in G$ such that $\gamma_1>\gamma=w(f)$ and the coset $\bar{\gamma}_1$ of γ_1 modulo G_w generates G/G_w. Let w_1 be the augmented valuation over w defined by f and γ_1. Then by Theorem 1.2 and Lemma 6.1 a), $G_{w_1}=G_{\bar{v}}+Z\gamma_1=G_w+Z\gamma_1=G$ and $k_{w_1}\cong k_{\bar{v}}(t)$ is k_v-isomorphic to $k_w\cong k_{\bar{v}}(r^*)$.

Now assume that $e>1$. Let $g=f^e+u$, taking $u\in K[X]$ such that $\deg u<\deg f$ and that $w(u)=\bar{v}(u(a))=w(f^e)=e\gamma$. Then by Theorem 4.6 3), g is a key polynomial over w end $(g/h)^*=r^*+y$ with $0\neq y\in k_{\bar{v}}$. Take $\gamma_1\in G$ such that $\gamma_1>w(g)$ and that the coset $\bar{\gamma}_1$ or γ_1 modulo G_w generates G/G_w. Let w_1 be the augmented valuation over w associated with g and γ_1. Then since, $(g/h)^*$ is of degree 1, according to Lemma 6.1 b) and c) it follows that $k_{w_1}\cong k_{\bar{v}}(t)$ is k_v-isomorphic to k_w and that $G_{w_1}=G$.

LEMMA 6.3. *Let w be an r.t. extension of v to $K(X)$ and let $k_w=k'(t)$ where k' is a finite extension of k_v and t is transcendental over k_v. Let k/k' be a finite simple extension, i.e. $k=k'(\alpha)$. Then there exists a key polynomial g over w and $\gamma_1\in \bar{G}_v$ with $\gamma_1>w(g)$ such that, if w_1 is the augmented valuation over w associated with g and γ_1,*

$$k_{w_1}\cong k(t) \quad \text{and} \quad G_{w_1}=G_w.$$

PROOP. Using Notation 1.3, we may assume that $k'=k_{\bar{v}}$ and that $k=k_{\bar{v}}(\alpha)$. Let $G\in k_{\bar{v}}[r^*]$ be the monic minimal polynomial of α. We may assume that $k\neq k_{\bar{v}}$, or G is of degree greater than 1. Let g be a lifting of G in $K[X]$. According to Corollary 4.7, we know that g is a key polynomial over w.

Take $\gamma_1\in G_w$ such that $\gamma_1>w(g)$ and let w_1 be the augmented valuation over w associated with g and γ_1. The proof of Lemma 6.3 follows from Lemma 6.1 b) and c).

THEOREM 6.4. *Let (K, v) be a valuation pair, k a finite extension field of k_v and G an ordered group such that G/G_v is a finite group. Then there exists an r.t. extension w of v to $K(X)$ such that $G_w\cong G$ and $k_w\cong k(t)$, t transcendental over k.*

PROOP. Since G/G_v is finite we may assume that $G_v\subseteq G\subseteq G_{\bar{v}}$, and that there exists a chain of subgroups $G_v=G_0\subset G_1\subset\cdots\subset G_m=G$ such that G_{i+1}/G_i is a non-trivial cyclic group, $i=0, \cdots, m-1$.

Let w_0 be the r. t. extenion of v to $K(X)$ defined by the minimal pair (0, 0). Then $k_{w_0}=k_v(X^*)$ (as usual X^* is the image of X in the residue field), and

$G_{w_0}=G_v$. By repeated application of Lemma 6.2 we can define, starting from w_0, an r. t. extension w' of v to $K(X)$ such that $G_{w'}=G$ and $k_{w'}=k_v(t')$, where t' is transcendental over k_v.

Furthermore, since k/k_v is a finite extension, we can define a tower of fields $k_v=k_0\subset k_1\subset \cdots \subset k_n=k$ such that k_{i+1}/k_i is a simple extension for all i, $0\leqq i<n$. By repeated application of Lemma 6.3, we can define, starting from w', an r. t. extension w of v to $K(X)$ such that $G_w=G_{w'}=G$, and $k_w\cong k(t)$, where t is transcendental over k. The proof of Theorem 6.4 is complete.

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