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COALGEBRA ACTIONS ON AZUMAYA ALGEBRAS

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Introduction.

The notion of measuring actions of coalgebras on an algebra unifies the notions of algebra automorphisms, of derivations and of higher derivations. In this paper we examine such actions of a k-coalgebra C on an Azumaya k-algebra A, where k is a commutative ring. In (2.4) we show a 1-1 correspondence between the set of measurings $C\to\operatorname{End} A$ and the set of certain right C^*-submodules of $C^*\otimes A$. Using this result, we show a Noether-Skolem type theorem (3.1): For example, *if k is a field, then any measuring $C\to\operatorname{End} A$ is inner for arbitrary C and A.*

Throughout the paper we fix a commutative ring k with 1. A linear map, an algebra, a coalgebra, $\otimes$, Hom and End mean a k-linear map, a k-algebra, a k-algebra, a k-coalgebra, $\otimes_k$, Hom_k and End_k, respectively. We fix an algebra A and a coalgebra C. C^* denotes $\operatorname{Hom}(C, k)$, the dual algebra of C [**9**, Prop. 1.1.1, p. 9].

1. Preliminaries.

Let Δ, ε be the structure maps of C and write

$$\Delta(c)=\sum_{(c)} c_{(1)}\otimes c_{(2)} \qquad \text{for } c\in C.$$

The k-module $\operatorname{Hom}(C, A)$ is an algebra with the $*$-product [**9**, p. 69]. $\operatorname{Hom}(C, A)^{\times}$ denotes the group of units in $\operatorname{Hom}(C, A)$.

1.1. DEFINITION. A linear map $f: C\to\operatorname{End} A$ is called a *measuring*, if $a\mapsto (c\mapsto f(c)(a))$, $A\to\operatorname{Hom}(C, A)$ is an algebra map, or equivalently if

$$f(c)(1)=\varepsilon(c)1,$$
$$f(c)(ab)=\sum_{(c)} f(c_{(1)})(a)f(c_{(2)})(b)$$

for $c\in C$, $a, b\in A$ [**9**, Def. p. 138]. We denote by

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$$\mathrm{Meas}(C, \mathrm{End}\, A)$$

the set of measurings $C \to \mathrm{End}\, A$.

For any $u \in \mathrm{Hom}(C, A)^{\times}$, the linear map $inn\, u : C \to \mathrm{End}\, A$ determined by

$$inn\, u(c)(a) = \sum_{(c)} u(c_{(1)}) a u^{-1}(c_{(2)}) \qquad c \in C,\ a \in A \tag{1.2}$$

is a measuring. Thus we have a map

$$inn : \mathrm{Hom}(C, A)^{\times} \longrightarrow \mathrm{Meas}(C, \mathrm{End}\, A). \tag{1.3}$$

1.4. DEFINITION (cf. [**2**, Def. 1.2, p. 674]). We write

$$\mathrm{Inn}(C, \mathrm{End}\, A) = \text{the image of } inn$$

and call an element of this set an *inner measuring*.

2. A 1-1 correspondence.

Throughout this section, let A be an Azumaya algebra [**6**, p. 95]. Thus A is a progenerator k-module and

$$A \otimes A \simeq \mathrm{End}\, A \qquad \text{via } a \otimes b \mapsto (x \mapsto axb). \tag{2.1}$$

Let D be an arbitrary algebra. $\mathrm{Alg}(A, D \otimes A)$ denotes the set of algebra maps $A \to D \otimes A$.

2.2. DEFINITION. $\mathbf{I}(D \otimes A)$ denotes the set of right D-submodules I of $D \otimes A$ such that

$$\kappa : I \otimes A \longrightarrow D \otimes A\,, \qquad \kappa(x \otimes a) = x(1 \otimes a)$$

is an isomorphism.

2.3. PROPOSITION. *Let A, D be as above.*

(1) *Let $f \in \mathrm{Alg}(A, D \otimes A)$ and define*

$$I_f = \{x \in D \otimes A \mid f(a)x = x(1 \otimes a) \quad \text{for all } a \in A\}.$$

Then $I_f \in \mathbf{I}(D \otimes A)$.

(2) *Let $I \in \mathbf{I}(D \otimes A)$ and suppose $\kappa^{-1}(1 \otimes 1) = \Sigma_i x_i \otimes a_i$. Define $f_I \in \mathrm{Hom}(A, D \otimes A)$ by*

$$f_I(a) = \Sigma_i x_i (1 \otimes a a_i), \qquad a \in A.$$

Then f_I is an algebra map.

(3) *$f \mapsto I_f$ and $I \mapsto f_I$ establish a 1-1 correspondence between $\mathrm{Alg}(A, D \otimes A)$ and $\mathbf{I}(D \otimes A)$.*

PROOF. We modify the proof of [**6**, Prop. 1.2, p. 107].

Let ${}_f(D\otimes A)$ denote the k-module $D\otimes A$ with the twisted A-bimodule structure represented by

$$A\otimes A \xrightarrow{f\otimes 1} D\otimes A\otimes A \xrightarrow{1\otimes(2.1)} D\otimes \mathrm{End}\, A \subset \mathrm{End}\,(D\otimes A).$$

Then I_f is identified with the A-centralizer of ${}_f(D\otimes A)$. This, together with [**6**, Cor. 5.3, p. 95], implies $I_f \in \mathbf{I}(D\otimes A)$.

f_I coincides with the composition of algebra maps

$$A \longrightarrow \mathrm{End}_{-D\otimes A}(I\otimes A) \xrightarrow{\simeq} \mathrm{End}_{-D\otimes A}(D\otimes A) = D\otimes A\,,$$

where the first map is $a \mapsto (x\otimes b \mapsto x\otimes ab)$ and the second is $g \mapsto \kappa\circ g\circ\kappa^{-1}$. This is a unique algebra map making $\kappa: I\otimes A \simeq {}_{f_I}(D\otimes A)$ into an A-bimodule isomorphism, so we have

$$f = f_{I_f}, \qquad I = I_{f_I}. \qquad \text{Q.E.D.}$$

2.4. THEOREM. *Let A be an Azumaya algebra, let C be a coalgebra and let $D=C^*$.*

(1) *There is a 1-1 correspondence between* $\mathrm{Meas}(C, \mathrm{End}\, A)$ *and* $\mathbf{I}(D\otimes A)$, *which is given by $f \mapsto I_f$, $I \mapsto f_I$ in* (2.3) *through the natural identification*

(2.5) $$\mathrm{Meas}(C, \mathrm{End}\, A) = \mathrm{Alg}(A, D\otimes A).$$

(2) *If $f \mapsto I$ in* (1), *then f is inner if and only if $I \simeq D$ as right D-modules.*

PROOF. (1) By definition (1.1) we have $\mathrm{Meas}(C, \mathrm{End}\, A) = \mathrm{Alg}(A, \mathrm{Hom}(C, A))$ by adjointness. Since A is a finitely generated projective k-module, we have $D\otimes A = \mathrm{Hom}(C, A)$. Thus we have (2.5). Then part (1) follows from (2.3) immediately.

(2) We have the correspondences

$$\begin{aligned} inn\, u &\longleftrightarrow (a \mapsto u(1\otimes a)u^{-1}) && \text{in (2.5)} \\ &\longleftrightarrow uD && \text{in (2.3)(3)} \end{aligned}$$

for $h \in (D\otimes A)^\times$. If $h: D \to I$, $I \in \mathbf{I}(D\otimes A)$, is a right D-module isomorphism with $u = h(1)$ (so $I = uD$), then $u \in (D\otimes A)^\times$, since we have the right $D\otimes A$-module isomorphism

$$D\otimes A = D\otimes_D (D\otimes A) \xrightarrow[h\otimes 1]{\simeq} I\otimes_D (D\otimes A) \xrightarrow[\kappa]{\simeq} D\otimes A$$

sending $1\otimes 1$ to u. Thus part (2) follows. Q.E.D.

2.6. FACT. *Let A, C, D be as in* (2.4). *Suppose C is cocommutative. Then*:

(1) $\mathrm{Meas}(C, \mathrm{End}\, A)$ *forms a group with respect to the $*$-product.*

(2) *$f \mapsto I_f$ in* (2.3) *induces an exact sequence of groups*

$$1 \longrightarrow \mathrm{Inn}(C, \mathrm{End}\, A) \longrightarrow \mathrm{Meas}(C, \mathrm{End}\, A) \xrightarrow{\phi} \mathrm{Pic}(D)$$

and

$$\mathrm{Im}\, \phi = \{I \in \mathrm{Pic}(D) \mid I \otimes A \simeq D \otimes A \text{ as right or left } D \otimes A\text{-modules}\},$$

where $\mathrm{Pic}(D)$ *is the Picard group of* D.

PROOF. As is easily verified, if C is cocommutative (so D is commutative), then $\mathrm{Meas}(C, \mathrm{End}\, A)$ is a sub-monoid of $\mathrm{Hom}(C, \mathrm{End}\, A)$ and the natural bijection

$$\mathrm{Meas}(C, \mathrm{End}\, A) = \mathrm{Alg}(A, D \otimes A) \simeq \mathrm{End}_{D\text{-}\mathrm{Alg}}(D \otimes A)$$

is a monoid isomorphism. Moreover since $D \otimes A$ is an Azumaya D-algebra, the assertions follow from [**6**, Cor. 5.4, p. 95 and Prop. 1.2, p. 107]. Q. E. D.

3. A Noether-Skolem theorem.

3.1. THEOREM. *Let C be a coalgebra and let $D = C^*$. Then any measuring* $C \to \mathrm{End}\, A$ *is inner for an arbitrary Azumaya algebra A, if either*

(a) *C is cocommutative and the Picard group* $\mathrm{Pic}(D)$ *of D is trivial,*

(b) *k, the base ring, is artinian and C is a finitely generated k-module, or*

(c) *k is a field* (*and C is arbitrary*).

PROOF in case (a). This follows from (2.6).

PROOF in case (b). By (2.4) we have only to show each $I \in \mathbf{I}(D \otimes A)$ is isomorphic to D as a right D-module. Multiplying a primitive idempotent, we may assume k is local artinian. Then A is a free k-module of finite rank, say n. We have

$$I^n \simeq I \otimes A \simeq D \otimes A \simeq D^n$$

as right D-modules, where $(\ )^n$ means the direct sum of n copies of $(\ )$. Since D is right artinian, we can apply the Krull-Schmidt theorem to have $I \simeq D$.

Q. E. D.

More generally, the conclusion of (3.1) holds true, if k is the direct product $\prod k_i$ of finitely many commutative rings k_i such that all finitely generated projective k_i-modules are free and if each Dk_i is contained in the class $\mathfrak{N}$ defined as follows. Let $\mathfrak{N}$ be the class of rings R with 1 satisfying: *A right R-module M is isomorphic to R, if there exists $n \geqq 1$ such that $M^n \simeq R^n$ as right R-modules.* All right artinian rings are contained in $\mathfrak{N}$.

3.2. LEMMA. (1) *If* $R/\mathrm{Rad}\,R\in\mathfrak{R}$, *then* $R\in\mathfrak{R}$, *where* Rad R *is the Jacobson radical of* R.

(2) $\mathfrak{R}$ *is closed under possibly infinite direct products.*

PROOF. (1) This follows from [**1**, (2.12) Prop., p. 90].

(2) Let $R=\prod R_\lambda$. Suppose $M^n\simeq R^n$. Then $M\simeq\prod MR_\lambda$, since so is $M^n=R^n$. Suppose $R_\lambda\in\mathfrak{R}$ for all λ. Then $MR_\lambda\simeq R_\lambda$, since $M^n\simeq R^n$ implies $(MR_\lambda)^n\simeq R_\lambda^n$. Thus we have

$$M\simeq\prod MR_\lambda\simeq\prod R_\lambda=R$$

as right R-modules. Hence $R\in\mathfrak{R}$. Q. E. D.

PROOF in case (c). By (3.2)(1), it is enough to show $D/\mathrm{Rad}\,D\in\mathfrak{R}$. By [**5**, 2.1.5. Prop. (a), p. 224], $D/\mathrm{Rad}\,D\simeq C_0^*$, where C_0 is the coradical [**9**, Def., p. 181] of C. Since C_0^* is a direct product of finite dimensional (simple) algebras [**5**, p. 223], $D/\mathrm{Rad}\,D=C_0^*\in\mathfrak{R}$ by (3.2)(2). Q. E. D.

3.3. REMARKS. (1) Sweedler [**8**, Thm. 9.5, p. 236] extended the classical results of Noether-Skolem and of Jacobson to Hopf algebra actions. His result cannot be covered by ours, unless $D=B$ in the notation of [**8**].

(2) Blattner and Montgomery [**3**, Thm. 2.15] prove a Noether-Skolem theorem for Hopf-Galois extensions, generalizing [**7**, Thm. 6]. Their result follows immediately from (3.1)(c), since, in their notation, an action of H on B trivial on Z gives rise to a Z-linear measuring $Z\otimes H\to\mathrm{End}_Z B$.

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