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EQUIVARIANT CW COMPLEXES AND SHAPE THEORY

Dedicated to Professor Masahiro Sugawara on his 60th birthday

By

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The aim of this note is to study a discrete group equivariant shape theory by associating an inverse system in the homotopy category of equivariant CW complexes.

1. Introduction

Let G be a discrete group and X a G-space. For a subgroup H of G we denote $X^{H}=\{x\in X; gx=x$ for every $g\in H\}$. For a G-map $f: X\rightarrow Y$ of X to another G-space Y, we denote $f^{H}=f|X^{H}: X^{H}\rightarrow Y^{H}$. Let $\mathcal{H}_{G}$ denote the category of G-spaces and G-homotopy classes of G-maps and $\mathcal{W}_{G}$ the full subcategory of $\mathcal{H}_{G}$ consisting of G-spaces which have the G-homotopy types of G-CW complexes.

THEOREM 1. *There is a functor* $\check{C}_{G}$ *from* $\mathcal{H}_{G}$ *into the pro-category* pro-$\mathcal{W}_{G}$ *of* $\mathcal{W}_{G}$ *so that* $\check{C}_{G}(X)=(X_{\lambda}, [p_{\lambda\lambda'}^{X}]_{G}, \Lambda)$ *has the universal property for the equivariant shape theory with a system G-map* $p^{X}=([p_{\lambda}^{X}]_{G}): X\rightarrow\check{G}_{G}(X)$, *that is,* $p^{X}: X\rightarrow\check{C}_{G}(X)$ *is a G*-CW *expansion of X.*

When G is a finite group, we know that a G-ANR has the G-homotopy type of a G-CW complex and vice versa. Also any numerable covering has a refinement of numerable G-equivariant covering. So, we have

TNEOREM 2. *Let G be a finite group and X a G-space.*

(1) *Any G*-ANR *expansion of X is equivalent to* $p^{X}: X\rightarrow\check{C}_{G}(X)$.

(2) *The expansion* $p^{X}: X\rightarrow\check{C}_{G}(X)$ *is a* (*non-equivariant*) CW *expansion of X. Moreover, if X is a normal G-space, then* $p^{X,H}=([p_{\lambda}^{X,H}]): X^{H}\rightarrow\check{C}_{G}(X)^{H}=(X_{\lambda}^{H}, [p_{\lambda\lambda'}^{X,H}], \Lambda)$ *is a* CW *expansion for every subgroup H of G.*

(3) *Let* $f: X\rightarrow Y$ *be a G-map between normal G-spaces. Then,* $\check{C}_{G}(f): \check{G}_{G}(X)\rightarrow\check{C}_{G}(Y)$ *is an isomorphism in* pro-$\mathcal{W}_{G}$ *if and only if* $f^{H}: X^{H}\rightarrow Y^{H}$ *is a shape*

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equivalence for every subgroup H of G.

The case when G is a finite group is also treated by Pop [10]. But he did not mention on (2) and (3) of Theorem 2. We note also that Antonian-Mardešić [1] defined the equivariant ANR shape for compact groups. Our treatment in the case when G is not a discrete group will be discussed elsewhere.

2. A quick review of shape theory

The general references are [3], [4] and [8]. Borsuk (1968) defined the shape for compact metric spaces, Mardešić-Segal (1971) for compact Hausdorff spaces, Fox (1972) for metric spaces, and Mardešić (1973) and K. Morita (1975) for topological spaces.

Let $\boldsymbol{X}=(X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y}=(Y_\mu, q_{\mu\mu'}, M)$ be inverse systems in a category $\mathcal{C}$. A system map of $\boldsymbol{X}$ to $\boldsymbol{Y}$ consists of $\theta: M\to\Lambda$ and morphisms $f_\mu: X_{\theta(\mu)}\to Y_\mu$ in $\mathcal{C}$ satisfying $q_{\mu\mu'}f_{\mu'}p_{\theta(\mu')\lambda}=f_\mu p_{\theta(\mu)\lambda}$ for $\mu\leqq\mu'$, $\theta(\mu')\leqq\lambda$ and $\theta(\mu)\leqq\lambda$. Two system maps (f_μ, θ) and (f'_μ, θ') are said to be equivalent if each $\mu\in M$ admits a $\lambda\in\Lambda$, $\lambda\geqq\theta(\mu)$ and $\lambda\geqq\theta'(\mu)$, such that $f_\mu p_{\theta(\mu)\lambda}=f'_\mu p_{\theta'(\mu)\lambda}$. The pro-category pro-$\mathcal{C}$ of the category $\mathcal{C}$ is defined by Obj (pro-$\mathcal{C}$)=all inverse systems in $\mathcal{C}$ and Mor$(\boldsymbol{X}, \boldsymbol{Y})$=equivalence classes of system maps of $\boldsymbol{X}$ to $\boldsymbol{Y}$. Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. A $\mathcal{D}$-expansion $p=(p_\lambda): X\to\boldsymbol{X}$ of X is a system map which is characterized by the following universal properties due to Mardešić [4, Ch. I, Th. I]:

(0) $X_\lambda\in\mathcal{D}$ for each $\lambda\in\Lambda$.

(1) For any map $f: X\to K$ with $K\in\mathcal{D}$ there exists a morphism $h_\lambda: X_\lambda\to K$ such that $f=h_\lambda p_\lambda$.

(2) If $f=g_\lambda p_\lambda$ then there is a $\lambda'\geqq\lambda$ such that $h_\lambda p_{\lambda\lambda'}=g_\lambda p_{\lambda\lambda'}$.

We give an exact definition of Čech expansion and Čech system due to Morita. Let $\mathcal{W}$ be the homotopy category of spaces which have homotopy type of CW complexes.

For a space X we associate an inverse system $\check{C}(X)=(X_\lambda, [p^X_{\lambda\lambda'}], \Lambda)$ in $\mathcal{W}$ by

$$\{\mathcal{U}_\lambda\}_{\lambda\in\Lambda}=\text{all numerable coverings of } X,\ \lambda'\geqq\lambda \text{ iff } \mathcal{U}_{\lambda'}\prec\mathcal{U}_\lambda\,;$$
$$X_\lambda=N(\mathcal{U}_\lambda) \text{ and } p^X_{\lambda\lambda'}: N(\mathcal{U}_{\lambda'})\to N(\mathcal{U}_\lambda)\,,$$

where $N(\mathcal{U}_\lambda)$ is the nerve of $\mathcal{U}_\lambda=\{U^\lambda_\alpha\}$ and $p^X_{\lambda\lambda'}$ is a simplicial map defined by choosing $\tilde{p}=p^X_{\lambda\lambda'}$ so that $U^{\lambda'}_\alpha\subset U^\lambda_{\tilde{p}(\alpha)}$. The homotopy class $[p^X_{\lambda\lambda'}]$ is independent of the choice of $\tilde{p}$. Then the inverse system $\check{C}(X)$ in pro-$\mathcal{W}$ well-defined and is called the Čech system of X. Here a pointwise finite covering $\mathcal{U}=\{U_\alpha\}$ of X is called numerable if it admits a locally finite partition of unity $\{\rho_\alpha\}$ i.e., a family of continuous functions $\rho_\alpha: X\to[0, 1]$ with $\sum\rho_\alpha=1$ and $\rho_\alpha^{-1}(0, 1]\subset U_\alpha$ such that $\{\rho_\alpha^{-1}(0, 1]\}$ is a locally finite covering of X. By the locally finite partition of unity $\{\rho_\alpha\}$ subordinate to $\mathcal{U}_\lambda$ we have a map $p_\lambda^X: X\to X_\lambda$ defined by $p_\lambda^X(x)=\sum\rho_\alpha(x)\langle U_\alpha\rangle$ where $\langle U_\alpha\rangle\in X_\lambda$ is the vertex corresponding to U_α. A different choice of the locally finite partition of unity gives another map contiguous to p_λ^X. So, the homotopy class of p_λ^X depends only on $\mathcal{U}_\lambda$ and $p_{\lambda\lambda'}^X p_{\lambda'}^X \simeq p_\lambda^X$. Then $p^X=([p_\lambda^X]): X\to\check{C}(X)$ is a $\mathcal{W}$-expansion and called the Čech expansion of X.

Any $\mathcal{W}$-expansion $X\to\boldsymbol{X}$ is equivalent to the Čech expansion $p^X: X\to\check{C}(X)$. The equivalence class of $\mathcal{W}$-expansion of X is called the shape of X.

3. Equivariant Čech system $\check{C}_G(X)$ (Proof of Theorem 1)

Let G be a discrete group and X a G-space. An open covering $\mathcal{U}=\{U_\alpha\}$ of X is called a numerable G-equivariant covering if $gU_\alpha=U_{g\alpha}\in\mathcal{U}$ for each $U_\alpha\in\mathcal{U}$ and $g\in G$ and if $\mathcal{U}$ has a locally finite partition of unity $\{\rho_\alpha\}$ such that $\rho_{g\alpha}(x)=\rho_\alpha(g^{-1}x)$ for any $g\in G$ and the following three sets have finite differences:

$$\{g\in G\,;\, g\alpha=\alpha \text{ i.e., } \rho_{g\alpha}=\rho_\alpha\}\subset\{g\in G\,;\, gU_\alpha=U_\alpha\}\subset\{g\in G\,;\, gU_\alpha\cap U_\alpha\neq\emptyset\}.$$

The nerves $X_\lambda=N(\mathcal{U}_\lambda)$ of the numerable G-equivariant coverings $\mathcal{U}_\lambda$ of X induce an inverse system $\check{C}_G(X)=(X_\lambda, [p_{\lambda\lambda'}^X]_G, \Lambda)$ in $\mathcal{W}_G$ with a system G-map $p^X=([p_\lambda^X]_G: X\to X_\lambda)$ such that $p_\lambda^X\simeq_G p_{\lambda\lambda'}^X p_{\lambda'}^X$. The G-homotopy classes $[p_\lambda^X]_G$ and $[p_{\lambda\lambda'}^X]_G$ are also well-defined by the argument using contiguity as in the non-equivariant case.

For a G-map $f: X\to Y$ a system G-map $\check{C}_G(f)=([f_\mu]_G, \theta): \check{C}_G(X)=(X_\lambda, [p_{\lambda\lambda'}^X]_G, \Lambda)\to\check{C}_G(Y)=(Y_\mu, [p_{\mu\mu'}^X]_G, M)$ is defined so that $p_\mu^Y f\simeq_G f_\mu p_{\theta(\mu)}^X$. In fact, a numerable G-equivariant covering $\mathcal{V}_\mu=\{V_\beta^\mu, \rho_\beta\}$ of Y induces a covering $f^{-1}\mathcal{V}_\mu=\{f^{-1}(V_\beta^\mu), \rho_\beta f\}$ of X, which is numerable G-equivariant and may be denoted by $\mathcal{U}_{\theta(\mu)}$, and $f_\mu: N(f^{-1}\mathcal{V}_\mu)\to N(\mathcal{V}_\mu)$ defined by the natural inclusion satisfies the required G-homotopy equality.

Hereafter we will omit $[\ ]_G$ to avoid complexity of notation.

LEMMA 3.1. *Let K be a G-CW complex. Then, the system G-map $p^K: K\to\check{C}_G(K)$ is an isomorphism in pro-$\mathcal{W}_G$.*

LEMMA 3.2. *For a G-space X we take a G-map $p_\lambda^X: X\to X_\lambda$ in the system*

G-map $p^X=(p_\lambda^X)\colon X\to\check{C}_G(X)$ *and consider a system G-map* $\check{C}_G(p_\lambda)=((p_\lambda^X)_\mu, \varphi_\lambda)\colon \check{C}_G(X)\to\check{C}_G(X_\lambda)$. *Then, there is a* ν *with* $\nu\geqq\lambda$ *and* $\nu\geqq\varphi_\lambda(\mu)$ *such that* $p_\mu^\lambda p_{\lambda\nu}^X\simeq_G (p_\lambda^X)_\mu p_{\varphi_\lambda(\mu)\nu}^X$, *where* p_μ^λ *denotes* $p_\mu^{X_\lambda}$.

LEMMA 3.3 (Universal property for equivariant shape). *Let* $p^X=(p_\lambda^X)\colon X\to \check{C}_G(X)=(X_\lambda, p_{\lambda\lambda'}^X, \Lambda)$ *be the system G-map defined above. Let* K *be a G*-CW *complex and* $f\colon X\to K$ *a G-map.*

(1) *There exist a* λ *and a G-map* $h\colon X_\lambda\to K$ *such that* $f\simeq_G hp_\lambda^X$.

(2) *If* $f\simeq_G gp_\lambda^X$ *for any other G-map* $g\colon X_\lambda\to K$, *then there is a* ν *with* $\nu\geqq\lambda$ *such that* $hp_{\lambda\nu}^X\simeq_G gp_{\lambda\nu}^X$.

PROOF OF LEMMA 3.3 AND THEOREM 1 FROM LEMMAS 3.1 AND 3.2. Lemma 3.3 is a detailed restatement of Theorem 1. Lemmas 3.1 and 3.2 imply Lemma 3.3 in a standard way. In fact, the system G-map $\check{C}_G(f)\colon \check{C}_G(X)\to\check{C}_G(K)$ consists of $\theta\colon M\to\Lambda$ and G-maps $f_\mu\colon X_{\theta(\mu)}\to K_\mu$. By Lemma 3.1 we have a μ and a G-map $q\colon K_\mu\to K$ such that $qp_\mu^K\simeq_G id_K$. Now it suffices to define $\lambda=\theta(\mu)$ and $h=qf_\mu$ to prove (1), because $qf_\mu p_{\theta(\mu)}^X\simeq_G qp_\mu^K f\simeq_G f$. To prove (2) we note that $qg_\mu p_{\theta_\lambda(\mu)}^\lambda\simeq_G g$ replacing X, f and θ with X_λ, g and θ_λ respectively. By Lemma 3.1 there is a G-map $q'\colon (X_\lambda)_\nu\to X_\lambda$ with $\nu\geqq\theta_\lambda(\mu)$ such that $q'p_\nu^\lambda\simeq_G id$ and $p_\nu^\lambda q' p_{\nu\tilde{\nu}}^\lambda\simeq_G p_{\nu\tilde{\nu}}^\lambda$ for some $\tilde{\nu}\geqq\nu$, where $p_{\nu\nu'}^\lambda$ denotes $p_{\nu\nu'}^{X_\lambda}$. So, $gq'p_{\nu\tilde{\nu}}^\lambda\simeq_G qg_\mu p_{\nu\tilde{\nu}}^\lambda$. Here we retake $\theta_\lambda(\mu)=\nu$. Take the G-map $(p_\lambda^X)_{\tilde{\nu}}\colon X_{\nu'}\to(X_\lambda)_{\tilde{\nu}}$ by putting $\nu'=\varphi_\lambda(\tilde{\nu})$. Then, since $gp_\lambda^X\simeq_G f$, we have a $\tilde{\nu}'$ with $\tilde{\nu}'\geqq\nu'$ and $\tilde{\nu}'\geqq\theta(\mu)$ such that $g_\mu p_{\nu\tilde{\nu}}^\lambda(p_\lambda^X)_{\tilde{\nu}} p_{\nu'\tilde{\nu}'}^X\simeq_G f_\mu p_{\theta(\mu)\tilde{\nu}'}^X$. So, $gq'p_{\nu\tilde{\nu}}^\lambda(p_\lambda^X)_{\tilde{\nu}} p_{\nu'\tilde{\nu}'}^X\simeq_G qf_\mu p_{\theta(\mu)\tilde{\nu}'}^X$. On the other hand by Lemma 3.2 we have $p_{\nu\tilde{\nu}}^\lambda(p_\lambda^X)_{\tilde{\nu}} p_{\nu'\tilde{\nu}'}^X\simeq_G p_\nu^\lambda p_{\lambda\tilde{\nu}'}^X$, if necessary retaking a larger $\tilde{\nu}'$. Hence, $gp_{\lambda\tilde{\nu}'}^X\simeq_G qf_\mu p_{\theta(\mu)\tilde{\nu}'}^X\simeq_G hp_{\lambda\tilde{\nu}'}^X$. q.e.d.

PROOF OF LEMMA 3.1. We consider a natural G-map $\sigma\colon |S(K)|\to K$ for the geometric realization of the singular complex of K. Since $|S(K)|^H=|S(K^H)|$, we see that σ is a G-homotopy equivalence. Since a G-homotopy equivalence induces an isomorphism $\check{C}_G(\cdot)$ in pro-$\mathcal{W}_G$, the proof reduces to the following two lemmas.

LEMMA 3.4. *For a G-space* X, $|S(X)|$ *admits a G-equivariant triangulation.*

LEMMA 3.5. *For a G-equivariantly triangulated G-space* K, $p^K\colon K\to\check{G}_G(K)$ *is an isomorphism in* pro-$\mathcal{W}_G$. *Moreover, suppose* μ *is given then there are a* $\tilde{\mu}$ $(\geqq\mu)$ *and a G-map* $q\colon K_{\tilde{\mu}}\to K$ *such that* q *is the G-homotopy inverse to* $p_{\tilde{\mu}}^K$.

PROOF OF LEMMA 3.4. We know that there is a G-homeomorphism between

$|S(X)|$ and $|\mathrm{Sd}\, S(X)|$ where $\mathrm{Sd}\, S(X)$ is a barycentric subdivision of the singular s.s. complex $S(X)$ of X. Note that the natural quotient map $|\mathrm{Sd}\, S(X)| \to |\mathrm{Sd}\, S(X)/G|$ restricts to a homeomorphism on any cell of $|\mathrm{Sd}\, S(X)|$. So, a triangulation of the regular CW complex $|\mathrm{Sd}\, S(X)/G|$ lifts to a G-equivariant triangulation of $|\mathrm{Sd}\, S(X)|$. q.e.d.

PROOF OF LEMMA 3.5. For each vertex v we take an open star neighborhood U_v. Then, $v_1, \cdots, v_n$ are the vertices of the same simplex if and only if $U_{v_1} \cap \cdots \cap U_{v_n}$ is not empty. If necessary by taking a barycentric subdivision, we may assume the following: If gv and v are in the same simplex of K then $gv=v$ and hence $U_{gv} \cap U_v \neq \emptyset$ implies $gv=v$. We put $\bar{\rho}_v(x)$=the coefficient of x with respect to v. Then the G-map $\bar{p}: K \to N(\{U_v\})$ defined by $\{\bar{\rho}_v\}$ is not only a bijection but also a G-homeomorphism. Note here that $\bar{\rho}_v(gx)=\bar{\rho}_v(x)$ if $gv=v$. Now we make the support of $\bar{\rho}_v$ smaller and get a locally finite G-equivariant partition of unity ρ_v so that $\mathcal{V}=\{U_v, \rho_v\}$ is a numerable G-equivariant covering and $p: K \to N(\mathcal{V})$, defined by $\{\rho_v\}$, is G-homotopic to $\bar{p}: K \to N(\mathcal{V})$. If we take a subdivision of K fine enough at first, we may assume that $\mathcal{V} < \mathcal{V}_\mu$. Take this $\mathcal{V}$ as $\mathcal{V}_{\bar{\mu}}$. Then $p_{\bar{\mu}}: K \to K_{\bar{\mu}} = N(\mathcal{V}_{\bar{\mu}})$ is a G-homotopy equivalence. This finishes the proof of Lemma 3.5 and also Lemma 3.1. q.e.d.

PROOF OF LEMMA 3.2. Note that X_λ is equivariantly triangulated. By the proof of Lemma 3.5 we have a $\bar{\mu}$ $(\geqq \mu)$ and a subdivision X'_λ of X_λ such that $\mathcal{V}_{\bar{\mu}}$ is the open star covering of X'_λ and $p^\lambda_{\bar{\mu}}: X'_\lambda \to (X_\lambda)_{\bar{\mu}} = N(\mathcal{V}_{\bar{\mu}})$ is G-homotopic to the natural identification. The G-map p^X_λ induces a numerable G-equivariant covering $\mathcal{V}_\nu = (p^X_\lambda)^{-1}(\mathcal{V}_{\bar{\mu}})$ of X and the natural inclusion $(p^X_\lambda)_{\bar{\mu}}: X_\nu = N(\mathcal{V}_\nu) \to (X_\lambda)_{\bar{\mu}} = N(\mathcal{V}_{\bar{\mu}})$. The G-map $p^X_{\lambda\nu}$ is the composition of the inclusion $X_\nu \to X'_\lambda$ with a simplicial G-map $X'_\lambda \to X_\lambda$ given by choosing a refinement. Hence $p^\lambda_{\bar{\mu}} p^X_{\lambda\nu} \simeq_G (p^X_\lambda)_{\bar{\mu}}$. This implies Lemma 3.2 and completes a proof of Theorem 1. q.e.d.

4. The case when G is a finite group

Let G be a finite group and X a G-space. Then (1) of Theorem 2 is a consequence of Theorem 1 and the fact that a G-ANR has the G-homotopy of a G-CW complex and vice versa (cf. [9] and [4, Appendix] or [10]). Pop [10] also defines the equivariant shape theory for a finite group G. In the case that X is normal, (2) and (3) of Theorem 2 enrich the result.

LEMMA 4.1. *Let $G=\{g_1, \cdots, g_n\}$ be a finite group. For any numerable covering $\mathcal{V}=\{U_\alpha, \rho_\alpha\}$ of a G-space X we have a numerable G-equivariant cover-*

ing $\mathcal{V}$ of X such that $\mathcal{V} \prec \mathcal{U}$.

PROOF. It suffices to take the covering $\mathcal{V}$ consisting of $g_1^{-1}U_{\alpha_1} \cap \cdots \cap g_n^{-1}U_{\alpha_n}$ with $\rho_{\alpha_1}(g_1x) \cdots \rho_{\alpha_n}(g_nx)$. In fact, $g_i(g_1^{-1}U_{\alpha_1} \cap \cdots \cap g_n^{-1}U_{\alpha_n}) \subset U_{\alpha_i}$ and the sum $\sum \rho_{\alpha_1}(g_1x) \cdots \rho_{\alpha_n}(g_nx)$ is equal to $(\sum \rho_{\alpha_1}(g_1x)) \cdots (\sum \rho_{\alpha_n}(g_nx)) = 1$. Note that we do not require $gV_\beta \cap V_\beta \neq \emptyset$ implies $gV_\beta = V_\beta$ for the numerable G-equivariant covering. q.e.d.

PROOF OF (2) OF THEOREM 2. Lemma 4.1 implies that $p^X: X \to \check{C}_G(X)$ is also a (non-equivariant) CW expansion of X [4, Ch. I, §1, Th. 1; §2, Rem. 3]. Assume that X is a normal space. For a subgroup H of G any numerable covering $\mathcal{U}_H$ of the closed subspace X^H extends to a numerable covering $\mathcal{U}$ of X i.e., $\mathcal{U}_H = \{U \cap X^H ; U \in \mathcal{U}\}$. We may assume that if $U \cap X^H = \emptyset$ then U is not H-invariant for $U \in \mathcal{U}$. So, we see that $\check{C}_G(X)^H \simeq \check{C}_{W(H)}(X^H)$ for a normal G-space X where $W(H) = N(H)/H$ and $N(H) = \{g \in G ; gHg^{-1} = H\}$. Now we have proved (2) of Theorem 2 by considering X^H a $W(H)$-space. q.e.d.

LEMMA 4.2. *Let G be a finite group. Let X and Y be G-CW complexes and $h_H: X^H \to Y^H$ maps satisfying $g_* h_H \simeq h_{H'} g_*$ for every pair of subgroups $H' \subset gHg^{-1}$ where $g_*(x) = gx$. Then there is a G-map $f: X \to Y$ such that $f \mid X^H \simeq h_H$ for every subgroup H of G.*

PROOF. Choose a family of representatives $\{H_1, \cdots, H_m\}$ of conjugacy classes of subgroups of G. For G-0-cell $\sigma: \Delta^0 \times G/H_i \to X$ we define $f \mid X^0$ by $f(\sigma(\Delta^0 \times gH_i/H_i)) = g_* h_{H_i}(\sigma(\Delta^0 \times H_i/H_i))$. Assume that a G-map $f \mid X^{n-1}$ is defined and for $H = H_i$ there are given homotopies between $f \mid \sigma(\Delta^k \times H/H)$ and $h_H \mid \sigma(\Delta^k \times H/H)$ in Y^H which extend the homotopies on the boundaries as an induction hypothesis for $k < n$. Then, for a G-n-cell $\sigma: \Delta^n \times G/H \to X$ with $H = H_i$, $h_H \mid \sigma(\partial \Delta^n \times H/H)$ is homotopic to $f \mid \sigma(\partial \Delta^n \times H/H)$. We can now define $f \mid \sigma(\Delta^n \times H/H)$ by the homotopy on the collar and by h_H on the interior. Extending f on $\sigma(\Delta^n \times G/H)$ so that f becomes G-equivariant, $f \mid X^n$ satisfies also the induction hypothesis. So, we get a G-map $f: X \to Y$ such that $f \mid X^H \simeq h_H$. q.e.d.

PROOF OF (3) OF THEOREM 2. If $f: X \to Y$ induces an isomorphism $\check{C}_G(f): \check{C}_G(X) \to \check{C}_G(Y)$ in pro-$\mathcal{W}_G$, then all $\check{C}_G(f)^H: \check{C}_G(X)^H \to \check{C}_G(Y)^H$ are isomorphisms in pro-$\mathcal{W}$. This means that all $f^H: X^H \to Y^H$ are shape equivalences by (2) of Theorem 2. Now suppose that all $f^H: X^H \to Y^H$ are shape equivalences. Then, also by (2) of Theorem 2, $\check{C}_G(f)^H = (f_\mu^H, \lambda): \check{C}_G(X)^H \to \check{C}_G(Y)^H$ are isomorphisms in pro-$\mathcal{W}$. Let $q_H=((q_H)_\lambda, \mu)\colon \check{C}_G(Y)^H \to \check{C}_G(X)^H$ be pro-$\mathcal{W}$ inverses of $\check{C}_G(f)^H$. Then $(q_H)_\lambda f_\mu^H p_{\lambda'\tilde{\lambda}}^{X,H} \simeq p_{\lambda\tilde{\lambda}}^{X,H}$ for some $\tilde{\lambda} \geqq \lambda'$ and $f_\mu^H p_{\lambda'\tilde{\lambda}}^{X,H}(q_H)_{\tilde{\lambda}} p_{\mu'\tilde{\mu}}^{Y,H} \simeq p_{\mu\tilde{\mu}}^{Y,H}$ for some $\tilde{\mu} \geqq \mu'$. Here we abbreviate $\mu=\mu(\lambda)$, $\lambda'=\lambda(\mu)$ and $\mu'=\mu(\tilde{\lambda})$. By taking μ, λ', $\tilde{\lambda}$, μ' and $\tilde{\mu}$ equal to or bigger than the ones for each H, we may assume that they do not depend on H. Note that if $H' \subset gHg^{-1}$ then $g_* f_\mu^H \simeq f_\mu^{H'} g_*$, $g_* p_{\lambda\lambda'}^{X,H} \simeq p_{\lambda\lambda'}^{X,H'} g_*$ and $g_* p_{\mu\mu'}^{Y,H} \simeq p_{\mu\mu'}^{Y,H'} g_*$. We have in this case the following diagram:

$$\begin{tikzcd}
Y_{\tilde{\mu}}^H \arrow[r] \arrow[d, "g_*"] \arrow[dr, phantom, "\circlearrowright"] & Y_{\mu'}^H \arrow[r, "(q_H)_{\tilde{\lambda}}"] \arrow[d, "g_*"] & X_{\tilde{\lambda}}^H \arrow[r] \arrow[d, "g_*"] \arrow[dr, phantom, "\circlearrowright"] & X_{\lambda'}^H \arrow[r, "f_\mu^H"] \arrow[d, "g_*"] \arrow[dr, phantom, "\circlearrowright"] & Y_\mu^H \arrow[r, "(q_H)_\lambda"] \arrow[d, "g_*"] & X_\lambda^H \arrow[d, "g_*"] \\
Y_{\tilde{\mu}}^{H'} \arrow[r] & Y_{\mu'}^{H'} \arrow[r, "(q_{H'})_{\tilde{\lambda}}"] & X_{\tilde{\lambda}}^{H'} \arrow[r] & X_{\lambda'}^{H'} \arrow[r, "f_\mu^{H'}"] & Y_\mu^{H'} \arrow[r, "(q_{H'})_\lambda"] & X_\lambda^{H'}
\end{tikzcd}$$

In the diagram we omit to write $p_{\mu'\tilde{\mu}}^{Y,H}$, $p_{\mu'\tilde{\mu}}^{Y,H'}$, $p_{\lambda'\tilde{\lambda}}^{X,H}$ and $p_{\lambda'\tilde{\lambda}}^{X,H'}$. Not necessarily $g_*(q_H)_\lambda \simeq (q_{H'})_\lambda g_*$ but we have $g_*(q_H)_\lambda p_{\mu\tilde{\mu}}^{Y,H} \simeq (q_{H'})_\lambda p_{\mu\tilde{\mu}}^{Y,H'} g_*$, because $g_*(q_H)_\lambda p_{\mu\tilde{\mu}}^{Y,H}$ $\simeq g_*(q_H)_{\tilde{\lambda}} f_\mu^H p_{\lambda'\tilde{\lambda}}^{X,H}(q_H)_\lambda p_{\mu'\tilde{\mu}}^{Y,H} \simeq g_* p_{\lambda\tilde{\lambda}}^{X,H}(q_H)_{\tilde{\lambda}} p_{\mu'\tilde{\mu}}^{Y,H} \simeq p_{\lambda\tilde{\lambda}}^{X,H'} g_*(q_H)_{\tilde{\lambda}} p_{\mu'\tilde{\mu}}^{Y,H} \simeq (q_{H'})_\lambda f_\mu^{H'} p_{\lambda'\tilde{\lambda}}^{X,H'}$ $\cdot g_*(q_H)_{\tilde{\lambda}} p_{\mu'\tilde{\mu}}^{Y,H} \simeq (q_{H'})_\lambda g_* f_\mu^H p_{\lambda'\tilde{\lambda}}^{X,H}(q_H)_{\tilde{\lambda}} p_{\mu'\tilde{\mu}}^{Y,H} \simeq (q_{H'})_\lambda g_* p_{\mu\tilde{\mu}}^{Y,H} \simeq (q_{H'})_\lambda p_{\mu\tilde{\mu}}^{Y,H'} g_*$. This means that we may assume $g_*(q_H)_\lambda \simeq (q_{H'})_\lambda g_*$ for every H, H' and g by retaking $\tilde{\mu}$ as $\mu(\lambda)$. By Lemma 4.2 we get a new G-map $q_\lambda\colon Y_{\mu(\lambda)} \to X_\lambda$ such that $q_\lambda^H \simeq (q_H)_\lambda$ for every subgroup H of G. Note that $q_\lambda^H f_{\mu(\lambda)}^H p_{\lambda(\mu(\lambda))\tilde{\lambda}}^{X,H} \simeq p_{\lambda\tilde{\lambda}}^{X,H}$ for some $\tilde{\lambda} \geqq \lambda(\mu(\lambda))$ and every H. So, applying the same argument of Lemma 4.2, we can get a G-homotopy between $q_\lambda f_{\mu(\lambda)} p_{\lambda(\mu(\lambda))\tilde{\lambda}}^X$ and $p_{\lambda\tilde{\lambda}}^X$. Also, we have a G-homotopy between $f_\mu q_{\lambda(\mu)} p_{\mu(\lambda(\mu))\tilde{\mu}}^Y$ and $p_{\mu\tilde{\mu}}^Y$ for some $\tilde{\mu} \geqq \mu(\lambda(\mu))$. q. e. d.

Reserences

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