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ON CLOSED IMAGES OF PERFECT PREIMAGES OF ORTHOCOMPACT DEVELOPABLE SPACES

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1. Introduction.

We consider the following property of the closed images of topological spaces: *For spaces X, Y and a closed mapping $f: X\rightarrow Y$, the following* (*) *holds*:
(*) $Y=Y_0\cup\cup\{Y_n: n\in\omega\}$, *where* $f^{-1}(y)$ *is compact for each* $y\in Y_0$ *and* Y_n *is closed and discrete in* Y *for each* $n\in\omega$.

Originally, Lašnev showed in [7] that (*) holds for a metric space X, and the other cases are listed in [2, pp. 13 and 14]. A few years ago, Chaber proved that (*) holds for a regular σ-space X [3, Theorem 1.1], and he proposed there the problem whether (*) holds or not for the cases when X is a perfect preimage of a regular σ-space or of a Moore space [3, Problems 1.1 and 3.1]. In this paper, we give a characterization of orthocompact developable spaces and give a partial answer to the latter case. We denote the set of all natural numbers by ω. All spaces are assumed to be T_1. All mappings are assumed to be continuous and onto.

2. The main results.

In the sequel, we denote by $[X, Y, Z, f, g]$ the situation that X, Y, Z are spaces, $f: X\rightarrow Y$ is a closed mapping and $g: X\rightarrow Z$ is a perfect mapping. Moreover, we denote by $[X, Y, f]$ the situation that X, Y are spaces and $f: X\rightarrow Y$ is a closed mapping.

Before stating a positive result for some subclass of perfect preimages of Moore spaces, we give the definition of $\mathcal{F}$-preserving families in both sides, which is used to characterize the class of stratifiable μ-spaces by Junnila and the author [6].

DEFINITION 2.1. Let $\mathcal{U}$, $\mathcal{F}$ be families of a space X. We call that $\mathcal{U}$ is *$\mathcal{F}$-preserving in both sides in* X if for each point p of X and for each subfamily $\mathcal{U}_0$ of $\mathcal{U}$, the following two conditions are satisfied:
(1) If $p\in\cap\mathcal{U}_0$, then $p\in F\subset\cap\mathcal{U}_0$ for some $F\in\mathcal{F}$.
(2) If $p\in X-\cup\mathcal{U}_0$, then $p\in F\subset X-\cup\mathcal{U}_0$ for some $F\in\mathcal{F}$.

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$\mathcal{U}$ is called *σ-$\mathcal{F}$-preserving* if $\mathcal{U}=\cup\{\mathcal{U}_n: n\in\omega\}$, where each $\mathcal{U}_n$ is $\mathcal{F}$-preserving in both sides in X.

According to Brandenburg [1], a developable space can be characterized as a space which has a σ-dissectable base, where a family $\mathcal{U}=\{U_\alpha: \alpha\in A\}$ of subsets of a space X is called *dissectable* if for each $\alpha\in A$ there exists a sequence $\{D_{\alpha n}: n\in\omega\}$ of closed subsets of X satisfying the following:

(1) $U_\alpha=\cup\{D_{\alpha n}: n\in\omega\}$ for each $\alpha\in A$.

(2) For each n, $\{D_{\alpha n}: \alpha\in A\}$ is closure-preserving in X.

(3) For each n and each point $p\in\cup\{D_{\alpha n}: \alpha\in A\}$, $\cap\{U_\alpha: \alpha\in A$ and $p\in D_{\alpha n}\}$ is a neighborhood of p in X.

We give here a similar characterization of orthocompact developable spaces. To do so, we introduce the notion of *$\mathcal{O}$-dissectable* families as modified one.

DEFINITION 2.2. Let X be a space and $\mathcal{U}$ a family of subsets of X. We call $\mathcal{U}$ *$\mathcal{O}$-dissectable* if there exists a σ-discrete family $\mathcal{F}$ of closed subsets of X satisfying the following:

(1) $\mathcal{U}$ is $\mathcal{F}$-preserving in both sides in X.

(2) For each $F\in\mathcal{F}$, $\cap\{U\in\mathcal{U}: F\subset U\}$ is a neighborhood of F in X, if it is not empty.

LEMMA 2.3. *If $\mathcal{U}$ is an $\mathcal{O}$-dissectable family of subsets of a space X, then $\mathcal{U}$ is dissectable.*

PROOF. Let $\mathcal{U}=\{U_\alpha: \alpha\in A\}$ and $\mathcal{F}=\cup\{\mathcal{F}_n: n\in\omega\}$ with each $\mathcal{F}_n$ discrete be the same families of the above definition. For each $\alpha\in A$, set

$$D_{\alpha n}=\cup\{F\in\mathcal{F}_n: F\subset U_\alpha\},\quad n\in\omega.$$

Then $\{D_{\alpha n}: n\in\omega\}$, $\alpha\in A$, satisfy the required conditions.

LEMMA 2.4. *For a family $\mathcal{U}$ of subsets of a space X, $\mathcal{U}$ is $\mathcal{O}$-dissectable if and only if $\mathcal{U}$ is interior-preserving and $\mathcal{F}$-preserving in both sides in X for some σ-discrete family $\mathcal{F}$ of closed subsets of X.*

PROOF. Only if part: Assume that $\mathcal{U}$ and $\mathcal{F}$ satisfy the conditions (1) and (2) of Definition 2.2. To see that $\mathcal{U}$ is interior-preserving in X, let $p\in\cap\mathcal{U}_0$ for $\mathcal{U}_0\subset\mathcal{U}$. There exists $F\in\mathcal{F}$ such that $p\in F\subset\cap\mathcal{U}_0$. By (2), $\cap\mathcal{U}_0$ is a neighborhood of p in X, implying that $\cap\mathcal{U}_0$ is open in X. If part is trivial.

LEMMA 2.5. *Let X be an orthocompact developable space. Then each open cover of X has an $\mathcal{O}$-dissectable open refinement.*

PROOF. It suffices to show that each interior-preserving open cover of a semistratifiable space is $\mathcal{F}$-preserving in both sides in X for some σ-discrete family $\mathcal{F}$ of closed subsets of X. Then it is $\mathcal{O}$-dissectable by the above lemma. Let $\mathcal{U}=\{U_\alpha : \alpha\in A\}$ be an interior-preserving open cover of X. For each point $p\in X$, let $\delta(p)=\{\alpha\in A : p\in U_\alpha\}$ and let $\Delta=\{\delta(p) : p\in X\}$. For each $\delta\in\Delta$ and $k\in\omega$, set

$$F(k,\ \delta)=(\bigcap\{U_\alpha : \alpha\in\delta\})_k-\bigcup\{U_\alpha : \alpha\in A-\delta\},$$

where $\{(\bigcap\{U_\alpha : \alpha\in\delta\})_k : k\in\omega\}$ is the semi-stratifiability of an open subset $\bigcap\{U_\alpha : \alpha\in\delta\}$. Set

$$\mathcal{F}(k)=\{F(k,\ \delta) : \delta\in\Delta\},\ k\in\omega.$$

Then $\mathcal{F}=\bigcup\{\mathcal{F}(k) : k\in\omega\}$ is a σ-discrete family of closed subsets of X and it is easy to see that $\mathcal{U}$ is $\mathcal{F}$-preserving in both sides in X. This completes the proof.

THEOREM 2.6. *For a space X, the following are equivalent* :

(1) *X is an orthocompact developable space.*

(2) *X has a σ-discrete family $\mathcal{F}$ of closed subsets and has a base $\bigcup\{\mathcal{V}_n : n\in\omega\}$, where each $\mathcal{V}_n$ is interior-preserving and $\mathcal{F}$-preserving in both sides in X.*

(3) *X has a σ-$\mathcal{O}$-dissectable base.*

PROOF. (1)→(2) : Let $\{\mathcal{U}_n : n\in\omega\}$ be a development for X. By the above lemma, for each n there exists a σ-discrete family $\mathcal{F}_n$ of closed subsets of X such that $\mathcal{U}_n$ has an open refinement $\mathcal{V}_n$ such that $\mathcal{V}_n$ is $\mathcal{F}_n$-preserving in both sides and interior-preserving in X. Letting $\mathcal{F}=\bigcup\{\mathcal{F}_n : n\in\omega\}$ we have the required base $\bigcup\{\mathcal{V}_n : n\in\omega\}$.

(2)→(3) follows directly from Lemma 2.4.

(3)→(1) : By Lemma 2.3, X has a σ-dissectable base. Therefore X is developable by [1]. By Lemma 2.4, every open cover of X has a σ-interior-preserving open refinement. The countable metacompactness of X implies that every open cover of X has an interior-preserving open refinement, i.e., X is orthocompact. This completes the proof.

COROLLARY 2.7. *Every orthocompact developable space has a σ-$\mathcal{F}$-preserving base for some σ-discrete family $\mathcal{F}$ of closed subsets of it.*

LEMMA 2.8. [11, Lemma 5.4]. *Let $\mathcal{F}$ be a hereditarily closure-preserving family of closed subsets of a space Y. For each $n\in\omega$, let*

$$Y_n=\bigcup\{F_1\cap\cdots\cap F_n : F_1,\cdots,F_n\in\mathcal{F} \text{ and } F_1\cap\cdots\cap F_n \textit{ is a non-empty finite subset of } Y\}.$$

Then each Y_n is closed and discrete in Y.

We state the main result.

THEOREM 2.9. *If in* $[X, Y, Z, f, g]$ Z *is an orthocompact Moore space, then* (*) *holds.*

PROOF. By virtue of Corollary 2.7, it suffices to show that if in $[X, Y, Z, f, g]$ Z is a regular space which has a σ-$\mathcal{F}$-preserving base for some σ-discrete (more generally, σ-locally finite) family $\mathcal{F}$ of closed subsets of Z, then (*) holds.

Let $\mathcal{U}=\cup\{\mathcal{U}_n : n\in\omega\}$ be a base for Z, where each $\mathcal{U}_n$ is $\mathcal{F}$-preserving in both sides in Z. Let $\mathcal{F}=\cup\{\mathcal{F}_n' : n\in\omega\}$, where each $\mathcal{F}_n'$ is a locally finite closed cover of Z. For each n, let $\mathcal{F}_n$ be the totality of finite intersections of members of $\cup\{\mathcal{F}_i' : i\leqq n\}$. Then $\{\mathcal{F}_n : n\in\omega\}$ is a sequence of locally finite and finitely multiplicative closed covers of Z such that $\mathcal{F}_n\subset\mathcal{F}_{n+1}$ for each n. Obviously, each $\mathcal{U}_n$ is $\cup_n\mathcal{F}_n$-preserving in both sides in Z and $\cup_n \mathcal{F}_n$ is a network for Z. Thus, we can assume $\mathcal{U}_n\subset\mathcal{U}_{n+1}$ for each n. For each n, write

$$\mathcal{E}_n=g^{-1}(\mathcal{F}_n)=\{E_\lambda : \lambda\in\Lambda_n\}.$$

For each n, $k\in\omega$, let $\Delta_n(k)$ be the totality of subsets δ of Λ_n such $|\delta|=k$ and

$$Y(\delta)=\cap\{f(E_\lambda) : \lambda\in\delta\}$$

is a non-empty finite subset of Y. By Lemma 2.8,

$$Y_n(k)=\cup\{Y(\delta) : \delta\in\Delta_n(k)\}$$

is closed and discrete in Y. Set

$$Y_0=Y-\cup\{Y_n(k) : n,\ k\in\omega\}.$$

We shall show that for each $y\in Y_0$, $f^{-1}(y)$ is compact in X. To do it, we establish the following claims:

Claim 1: For each $n\in\omega$,

$$\mathcal{E}_n(y)=\{E\in\mathcal{E}_n : E\cap f^{-1}(y)\neq\phi\}$$

is finite.

To see it, assume the contrary, i.e., that for some m, $\mathcal{E}_m(y)$ is infinite. Choose an infinite sequence $\{E_m, E_{m+1},\cdots\}\subset\mathcal{E}_m(y)$ and $x_0\in f^{-1}(y)$. Observe that for each k

$$E_k'=\cap\{E\in\mathcal{E}_k : x_0\in E\}\in\mathcal{E}_k.$$

Since $y\in Y_0$, $f(E_k')\cap f(E_k)$ is infinite for each $k\geqq m$, we can choose a sequence $\{y_k : k\geqq m\}$ of distinct points of Y such that

$$y_k\in f(E_k')\cap f(E_k),\ k\geqq m.$$

Choose two points p_k, $p_k'\in X$ for each $k\geqq m$ such that

$$p_k\in f^{-1}(y_k)\cap E_k \text{ and } p_k'\in f^{-1}(y_k)\cap E_k'$$

for each k. Recall that $\cup\{\mathcal{E}_n : n\in\omega\}$ is a Σ-network for Y in the sense of Nagami [8]. Therefore, $\{p_k'\}$ has a cluster point in Y. So, $\{y_k : k\geqq m\}$ consequently has a cluster point in Y. But this is a contradiction because $\{p_k : k \geqq m\}$ is discrete in X and f is a closed mapping. Hence $\mathcal{E}_n(y)$ is finite for each n. (The proof of this part have been done referring to [12, Theorem 1.3].)

Claim 2: $g(f^{-1}(y))$ is Lindelöf.

In fact, by Claim 1, for each n

$$\mathcal{F}_n(y) = \{F \in \mathcal{F}_n : g^{-1}(F) \in \mathcal{E}_n(y)\}$$

is finite. It is obvious that

$$\cup \{\mathcal{F}_n(y) : n \in \omega\} / g(f^{-1}(y))$$

is a countable network for the subspace $g(f^{-1}(y))$. This implies that $g(f^{-1}(y))$ is Lindelöf.

Claim 3: There exists a sequence $\{y_n : n \in \omega\}$ of points of Y satisfying the following:

(1) $E \cap f^{-1}(y_n) \neq \phi$ for each $E \in \mathcal{E}_k(y)$ and $n \geqq k$.

(2) If $N \subset \omega$ is infinite, then $\{y_n : n \in N\}$ has a cluster point in Y.

In fact, by Claim 1, each $\mathcal{E}_n(y)$ is finite. Since $y \in Y_0$ and

$$y \in \cap \{f(E) : E \in \mathcal{E}_n(y)\},$$

$\cap \{f(E) : E \in \mathcal{E}_n(y)\}$ is infinite. Thus, we can choose a sequence $\{y_n : n \in \omega\}$ of points of Y such that for each n

$$y_{n+1} \in \cap \{f(E) : E \in \mathcal{E}_{n+1}(y)\} - \{y_1, \cdots, y_n\}.$$

It is obvious to see that $\{y_n : n \in \omega\}$ satisfies (1). Let N be an infinite subset of ω. Since for a point $x_0 \in f^{-1}(y)$,

$$E_n' = \cap \{E \in \mathcal{E}_n : x_0 \in E\} \in \mathcal{E}_n(y), \ n \in N,$$

there exists by Claim 3(1),

$$p_n \in f^{-1}(y_n) \cap E_n', \ n \in N.$$

By the same reason as in the proof of Claim 1, $\{y_n : n \in N\}$ has a cluster point in Y.

Finally we show that $f^{-1}(y)$ is compact in X. Assume that $f^{-1}(y)$ is not compact in X. Then $g(f^{-1}(y))$ is not so in Z because g is a perfect mapping. Recall that by Claim 2 $g(f^{-1}(y))$ is Lindelöf. By the argument of [3, Theorem 1] there exists an increasing open cover $\{U_i : i \in \omega\}$ of $g(f^{-1}(y))$ such that for each i

$$g(f^{-1}(y)) \cap (U_{i+1} - \bar{U}_i) \neq \phi.$$

Take points $p_1 \in U_1$ and

$$p_{i+1} \in g(f^{-1}(y)) \cap (U_{i+1} - \bar{U}_i)$$

for each i. Set

$$A_i = Z - \cup \{U \in \mathcal{U}_i : U \cap g(f^{-1}(y)) = \phi\}$$

for each i. Then $\{A_i : i \in \omega\}$ is a decreasing sequence of closed subsets of Z such that

$$g(f^{-1}(y)) = \cap \{A_i : i \in \omega\}.$$

Since $\mathcal{U}_i$ is $\cup_k \mathcal{F}_k$-preserving in both sides in Z, there exists $F_i \in \cup_k \mathcal{F}_k$ such that $p_1 \in F_1 \subset U_1$ and

$$p_{i+1} \in F_{i+1} \subset (U_{i+1} - \bar{U}_i) \cap A_i.$$

By Claim 3 (1), we can choose $\{y_{n(i)} : i \in \omega\}$ such that

$$F_i \cap g(f^{-1}(y_{n(i)})) \neq \phi \text{ and } n(i) < n(i+1)$$

for each i. If we take for each i

$$x_i \in g^{-1}(F_i) \cap f^{-1}(y_{n(i)}),$$

then by Claim 3 (2), $\{g(x_i) : i \in \omega\}$ has a cluster point z in Z. Since $g(x_i) \in F_i$, $i \in \omega$, and $\{F_i : i \in \omega\}$ is discrete in the subspace $g(f^{-1}(y))$, z must belong to $Z - g(f^{-1}(y))$. Since $g(f^{-1}(y)) = \cap \{A_i : i \in \omega\}$, there exists $m \in \omega$ such that $z \notin A_n$ for every $n \geqq m$. But this is a contradiction because $g(x_n) \in A_m$ for every $n \geqq m$ and A_m is closed in Z. Hence we have shown that $f^{-1}(y)$ is compact in X. This completes the proof.

From here, we assume that all p-spaces are regular. In [4], Filippov showed that (*) holds if X is a paracompact p-space in $[X, Y, f]$. We generalize it as follows:

COROLLARY 2.10. *If in* $[X, Y, f]$ X *is an orthocompoct, d-paracompact* p*-space, then* (*) *holds.*

PROOF. By [9, Theorem 4.4] there exists a perfect mapping of X onto a Moore space Z. By [5, Theorems 3.2 and 3.3] Z is orthocompact. Thus, by the theorem (*) holds.

REMARK. We know that Veličko showed that (*) holds if X is a metacompact, completely regular p-space [13], as a generalization of Filippov's result. But, Corollary 2.10 is not the corollary of Veličko's, because there exists an orthocompact Moore space X which is not metacompact [13, Theorem 2].

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