TSUKUBA J. MATH.
Vol. 40 No. 1 (2016), 81–100

ANALYTIC PROPERTIES OF GENERALIZED MORDELL-TORNHEIM TYPE OF MULTIPLE ZETA-FUNCTIONS AND L-FUNCTIONS

By

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Abstract. Analytic properties of three types of multiple zeta functions, that is, the Euler-Zagier type, the Mordell-Tornheim type and the Apostol-Vu type have been studied by a lot of authors. In particular, in the study of multiple zeta functions of the Apostol-Vu type, a generalized multiple zeta function, including both the Euler-Zagier type and the Apostol-Vu type, was introduced. In this paper, similarly we consider generalized multiple zeta-functions and L-functions, which include both the Euler-Zagier type and the Mordell-Tornheim type as special cases. We prove the meromorphic continuation to the multi-dimensional complex space, and give the results on possible singularities.

1. Introduction

The Euler-Zagier type of multiple zeta-function $\zeta_{EZ,r}$ is defined by

$$\zeta_{EZ,r}(s_1,\dots,s_r) = \sum_{1\le m_1<\cdots<m_r} \frac{1}{m_1^{s_1} m_2^{s_2}\cdots m_r^{s_r}}$$

$$= \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty} \frac{1}{m_1^{s_1}(m_1+m_2)^{s_2}\cdots(m_1+\cdots+m_r)^{s_r}}, \tag{1}$$

where $s_1, s_2, \dots, s_r$ are complex variables, and the series (1) is absolutely convergent in the region

$$\{(s_1,\dots,s_r)\in \mathbf{C}^r \mid \mathrm{Re}(s_{r-k+1}+s_{r-k+2}+\cdots+s_r) > k \ \ (k=1,2,\dots,r)\}.$$

2000 *Mathematics Subject Classification*: Primary 11M06, Secondary 40B05.
Key words and phrases: Multiple zeta-function, Multiple L-function, Analytic continuation.
Received December 11, 2015.
Revised January 14, 2016.

The Mordell-Tornheim type and Apostol-Vu type of multiple zeta-functions are defined by

$$\zeta_{MT,r}(s_1,\dots,s_r;s_{r+1}) = \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{1}{m_1^{s_1}\cdots m_r^{s_r}(m_1+\cdots+m_r)^{s_{r+1}}} \tag{2}$$

and

$$\zeta_{AV,r}(s_1,\dots,s_r;s_{r+1}) = \sum_{1\le m_1<\cdots<m_r}\frac{1}{m_1^{s_1}\cdots m_r^{s_r}(m_1+\cdots+m_r)^{s_{r+1}}} \tag{3}$$

where $s_1,\dots,s_r,s_{r+1}$ are complex variables. The series (2) and (3) are absolutely convergent in

$$\{(s_1,\dots,s_r,s_{r+1})\in\mathbf{C}^{r+1}\mid \mathrm{Re}(s_j)>1\ (1\le j\le r),\ \mathrm{Re}(s_{r+1})>0\}. \tag{4}$$

For the meromorphic continuation to the whole space $\mathbf{C}^r$ of (1), Akiyama Egami and Tanigawa [1] and Zhao [11], proved independently of each other. Matsumoto [5] gave an alternative proof of the analytic continuation using the Mellin-Barnes integral formula

$$(1+\lambda)^{-s} = \frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)}\lambda^z\,dz, \tag{5}$$

where $s,\lambda\in\mathbf{C}$, $|\arg\lambda|<\pi$, $\lambda\neq 0$, and $c\in\mathbf{R}$, $-\mathrm{Re}(s)<c<0$ and the path of integration is the vertical line from $c-i\infty$ to $c+i\infty$. Also, Matsumoto [4] proved the meromorphic continuation in the same way for (2) and (3). In particular, Matsumoto introduced the following function in the process of proving the meromorphic continuation of (3). Let $1\le j\le r$, and define

$$\begin{aligned}&\hat{\zeta}_{AV,j,r}(s_1,\dots,s_j;s_{j+1},\dots,s_r;s_{r+1})\\ &\quad= \sum_{1\le m_1<\cdots<m_r}\frac{1}{m_1^{s_1}\cdots m_r^{s_r}(m_1+\cdots+m_j)^{s_{r+1}}}\end{aligned} \tag{6}$$

where $s_1,\dots,s_r,s_{r+1}$ are complex variables. Since $\hat{\zeta}_{AV,r,r}=\zeta_{AV,r}$ and

$$\hat{\zeta}_{AV,1,r}(s_1;s_2,\dots,s_{r+1}) = \zeta_{EZ,r}(s_1+s_{r+1},s_3,\dots,s_r),$$

(6) forms a generalized class including as special cases both the Euler-Zagier type (1) and the Apostol-Vu type (3). He, through the recursive structure

$$\zeta_{AV,r}=\hat{\zeta}_{AV,r,r}\to\hat{\zeta}_{AV,r-1,r}\to\hat{\zeta}_{AV,r-2,r}\to\cdots\to\hat{\zeta}_{AV,1,r}=\zeta_{EZ,r} \tag{7}$$

(here $A\to B$ means that A can be expressed as an integral involving B; see (12), (17) and (18) below), discussed analytic properties of those functions.

As an analogue of (6), in this paper we define the following function, and prove the results on meromorphic continuation and singularities. The results will be stated in Section 2.

Definition 1. Let $1 \leq j \leq r$, and define

$$\hat{\zeta}_{MT,j,r}(s_1, \ldots, s_j; s_{j+1}, \ldots, s_{r+1}) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{1}{m_1^{s_1} \cdots m_j^{s_j} (m_1 + \cdots + m_j)^{s_{j+1}} \cdots (m_1 + \cdots + m_r)^{s_{r+1}}}, \tag{8}$$

where $s_1, \ldots, s_r, s_{r+1}$ are complex variables.

Since $\hat{\zeta}_{MT,r,r} = \zeta_{MT,r}$ and

$$\hat{\zeta}_{MT,1,r}(s_1, \ldots, s_j; s_{j+1}, \ldots, s_{r+1}) = \zeta_{EZ,r}(s_1 + s_2, s_3, \ldots, s_{r+1}),$$

we see that (8) forms a generalized class including as specal cases both the Euler-Zagier type (1) and the Mordell-Tornheim type (2), which can be illustrated in the following figure.

$$\begin{tikzcd}
& \hat{\zeta}_{AV,j,r} \arrow[dl, dash] \arrow[dr, dash] & & \hat{\zeta}_{MT,j,r} \arrow[dl, dash] \arrow[dr, dash] & \\
\hat{\zeta}_{AV,r,r} \arrow[d, equal] & & \hat{\zeta}_{AV,1,r}, \hat{\zeta}_{MT,1,r} \arrow[d, equal] & & \hat{\zeta}_{MT,r,r} \arrow[d, equal] \\
\zeta_{AV,r} & & \zeta_{EZ,r} & & \zeta_{MT,r}
\end{tikzcd}$$

The series (8) is absolutely convergent in the region

$$R_{j,r} = \left\{ (s_1, \ldots, s_r, s_{r+1}) \in \mathbf{C}^{r+1} \;\middle|\; \begin{array}{c} \mathrm{Re}(s_{r+2-k} + s_{r+3-k} + \cdots + s_{r+1}) > k \\ (k = 1, 2, \ldots, r-j) \\ \mathrm{Re}(s_{j+1} + s_{j+2} + \cdots + s_{r+1}) > r - j \\ \mathrm{Re}(s_\ell) > 1 \;\; (\ell = 1, 2, \ldots, j) \end{array} \right\},$$

therefore $\hat{\zeta}_{MT,j,r}$ is a regular function in $R_{j,r}$. This fact can be proved by the evaluation

$$\sum_{m=1}^{\infty} \frac{1}{(m+N)^\sigma} < \int_0^\infty \frac{dx}{(x+N)^\sigma} = \frac{1}{\sigma - 1} \frac{1}{N^{\sigma-1}} \quad (\sigma > 1)$$

and the result on the absolutely convergent region (4).

Furthermore, we introduce the following L-function which is a χ-analogue of (8), and we obtain the results on meromorphic continuation and singularities. The results will be stated in Section 2.

DEFINITION 2. Let $\chi_1, \chi_2 \dots, \chi_r$ be Diriclet characters of the same modulus q (≥ 2). We define

$$\hat{L}_{MT,j,r}(s_1, \dots, s_j; s_{j+1}, \dots, s_{r+1}; \chi_1, \dots, \chi_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{\chi_1(m_1) \cdots \chi_r(m_r)}{m_1^{s_1} \cdots m_j^{s_j} (m_1 + \cdots + m_j)^{s_{j+1}} \cdots (m_1 + \cdots + m_r)^{s_{r+1}}} \tag{9}$$

where $1 \leq j \leq r$ and $s_1, \dots s_r, s_{r+1}$ are complex variables. The series (9) is absolutely convergent in $R_{j,r}$, and so $\hat{L}_{MT,j,r}$ is a regular function on $R_{j,r}$.

Definition 2 gives a generalized class which includes both

$$\mathscr{L}_{EZ,r}(s_1, \dots, s_r; \chi_1, \dots, \chi_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{\chi_1(m_1) \cdots \chi_r(m_r)}{m_1^{s_1} (m_1 + m_2)^{s_2} \cdots (m_1 + \cdots + m_r)^{s_r}}, \tag{10}$$

and

$$L_{MT,r}(s_1, \dots, s_r; s_{r+1}; \chi_1, \dots, \chi_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{\chi_1(m_1) \cdots \chi_r(m_r)}{m_1^{s_1} \cdots m_r^{s_r} (m_1 + \cdots + m_r)^{s_{r+1}}} \tag{11}$$

as special cases. The series (10) is introduced by Kamano [2], and he proved the meromorphic continuation to $\mathbf{C}^r$. Also (11) is introduced by Wu [10] and he proved some analytic properties (see Theorem 3 in Matsumoto [7]).

REMARK 1. Analytic properties of Apostol-Vu type (3) was also proved by Okamoto [9], whose method is different from the method of Matsumoto [4] through the function (6). Okamoto's method is based on the observation that (3) has the recursive structure

$$\zeta_{AV,r} \to \zeta_{AV,r-1} \to \zeta_{AV,r-2} \to \cdots \to \zeta_{AV,2} \to \zeta, \tag{12}$$

where the right-most ζ denotes the Riemann zeta-function. Thus, analytic properties of (3) can be proved without using the function (6) and the recursive structure (7).

Remark 2. Matsumoto and Tanigawa [8] defined the multiple Dirichlet series

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{a_1(m_1)a_2(m_2)\cdots a_r(m_r)}{m_1^{s_1}(m_1+m_2)^{s_2}\cdots(m_1+\cdots+m_r)^{s_r}}$$

which is a further generalization of (10). They proved its several analytic properties.

2. Statement of Results

Theorem 1. *For $1 \le j \le r$, we have*

(i) *the function $\hat{\zeta}_{MT,j,r}(s_1,\dots,s_j;s_{j+1},\dots,s_{r+1})$ can be continued meromorphically to the whole $\mathbf{C}^{r+1}$-space,*

(ii) *in the case of $j = r-1$, the possible singularities of $\hat{\zeta}_{MT,r-1,r}$ are located only on the subsets of $\mathbf{C}^{r+1}$ defined by one of the following equations;*

$$s_{r+1} = 1,$$

$$s_j + s_r + s_{r+1} = 1 - \ell \quad (1 \le j \le r-1, \ell \ge -1),$$

$$s_{j_1} + s_{j_2} + s_r + s_{r+1} = 2 - \ell \quad (1 \le j_1 < j_2 \le r-1, \ell \ge -1),$$

$$\vdots$$

$$s_{j_1} + \cdots + s_{j_{r-2}} + s_r + s_{r+1} = r - 2 - \ell$$

$$(1 \le j_1 < \cdots < j_{r-2} \le r-1, \ell \ge -1),$$

$$s_1 + \cdots + s_{r-1} + s_r + s_{r+1} = r - 1 - d \quad (d = -1, 0, 1, 3, 5, 7, 9, \dots).$$

Also, in the cases of $1 \le j \le r-2$, possible singularities of $\hat{\zeta}_{MT,j,r}$ are located only on the subsets of $\mathbf{C}^{r+1}$ defined by one of the following equations;

$$s_{r+1} = 1,$$

$$s_r + s_{r+1} = 1 - d \quad (d = -1, 0, 1, 3, 5, 7, 9, \dots),$$

$$s_{r-1} + s_r + s_{r+1} = 3 - \ell \quad (\ell \in \mathbf{N}_0),$$

$$s_{r-2} + s_{r-1} + s_r + s_{r+1} = 4 - \ell \quad (\ell \in \mathbf{N}_0),$$

$$\vdots$$

$$s_{j+2} + s_{j+3} + \cdots + s_r + s_{r+1} = r - j - \ell \quad (\ell \in \mathbf{N}_0),$$

$$s_{k_1} + s_{j+1} + \cdots + s_r + s_{r+1} = 1 - \ell' \quad (1 \leq k_1 \leq j, \ell' \geq -(r-j)),$$

$$s_{k_1} + s_{k_2} + s_{j+1} + \cdots + s_r + s_{r+1} = 2 - \ell'$$
$$(1 \leq k_1 < k_2 \leq j, \ell' \geq -(r-j)),$$
$$\vdots$$
$$s_{k_1} + \cdots + s_{k_{j-1}} + s_{j+1} + \cdots + s_r + s_{r+1} = j - 1 - \ell'$$
$$(1 \leq k_1 < \cdots < k_{j-1} \leq j, \ell' \geq -(r-j)),$$

$$s_1 + \cdots + s_j + s_{j+1} + \cdots + s_r + s_{r+1} = j - \ell' \quad (\ell' \geq -(r-j)).$$

(iii) *each of these singularities can be canceled by the corresponding linear factor, and*

(iv) $\hat{\zeta}_{MT,j,r}$ *is of polynomial order with respect to* $|\mathrm{Im}(s_{r+1})|$.

THEOREM 2. *For* $1 \leq j \leq r$, *we have*

(i) *the function* $\hat{L}_{MT,j,r}(s_1, \ldots, s_j; s_{j+1}, \ldots, s_{r+1}; \chi_1, \ldots \chi_r)$ *can be continued meromorphically to the* $\mathbf{C}^{r+1}$*-space.*

(ii) *If none of the characters* $\chi_1, \ldots, \chi_r$ *are principal, then* $\hat{L}_{MT,j,r}$ *is entire. If* $\chi_{t_1}, \ldots, \chi_{t_k}$ $(1 \leq t_1 < \cdots < t_k \leq j)$ *and* $\chi_{r-d_1}, \ldots, \chi_{r-d_h}$ $(1 \leq d_1 < \cdots < d_h \leq r-j)$ *are principal character and other characters are non-principal, in the case of* $j = r-1$, *then possible singularities are located only on the subsets of* $\mathbf{C}^{r+1}$ *defined by one of the following equation;*

$$\begin{aligned}
&s_{t_{u(1)}} + s_r + s_{r+1} = 1 - \ell \quad (1 \leq u(1) \leq k, \ell \geq -\delta_r),\\
&s_{t_{u(1)}} + s_{t_{u(2)}} + s_r + s_{r+1} = 2 - \ell \quad (1 \leq u(1) < u(2) \leq k, \ell \geq -\delta_r),\\
&\quad\vdots\\
&s_{t_{u(1)}} + \cdots + s_{t_{u(k-1)}} + s_r + s_{r+1} = k - 1 - \ell\\
&\qquad (1 \leq u(1) < \cdots < u(k-1) \leq k, \ell \geq -\delta_r),\\
&s_{t_1} + \cdots + s_{t_k} + s_r + s_{r+1} = k - \ell \quad (\ell \geq -\delta_r),
\end{aligned} \tag{13}$$

where

$$\delta_r = \begin{cases} 1 & (\chi_r \text{ is principal}) \\ 0 & (\chi_r \text{ is non principal}) \end{cases},$$

also in the cases of $1 \leq j \leq r-2$, *then possible singularities are located only on the subsets of* $\mathbf{C}^{r+1}$ *defined by one of the following equation*;

$$
\begin{aligned}
&s_{r-d_1+1} + s_{r-d_1+2} + \cdots + s_{r+1} = d_1 + 1 - \ell_0 \quad (\ell_0 \in \mathbf{N}_0), \\
&\quad \vdots \\
&s_{r-d_h+1} + s_{r-d_h+2} + \cdots + s_{r+1} = d_h + 1 - \ell_0 \quad (\ell_0 \in \mathbf{N}_0), \\
&s_{t_{u(1)}} + s_{j+1} + \cdots + s_r + s_{r+1} = 1 - \ell' \quad (1 \leq u(1) \leq k, \ell' \geq -\Delta_j), \\
&s_{t_{u(1)}} + s_{t_{u(2)}} + s_{j+1} + \cdots + s_r + s_{r+1} = 2 - \ell' \\
&\qquad (1 \leq u(1) < u(2) \leq k, \ell' \geq -\Delta_j), \\
&\quad \vdots \\
&s_{u(1)} + \cdots + s_{u(j-1)} + s_{j+1} + \cdots + s_r + s_{r+1} = j - 1 - \ell' \\
&\qquad (1 \leq u(1) < \cdots < u(j-1) \leq k, \ell' \geq -\Delta_j), \\
&s_1 + \cdots + s_j + s_{j+1} + \cdots + s_r + s_{r+1} = j - \ell' \quad (\ell' \geq -\Delta_j),
\end{aligned}
\tag{14}
$$

where $\Delta_j = \delta_r + \delta_{r-1} + \cdots + \delta_{r-j}$. *Moreover, if* χ_r *is principal character, then*

$$s_{r+1} = 1$$

is a possible singularity in addition to the above possible singularities (13) *and* (14).

(iii) *each of these singularities can be canceled by the corresponding linear factor, and*

(iv) $\hat{L}_{MT,j,r}$ *is of polynomial order with respect to* $|\mathrm{Im}(s_{r+1})|$.

Remark 3. In both Theorem 1 and Theorem 2, the case $j = r$ is known (see Theorem 4 and Theorem 5 below). It is interesting that the feature of possible singularities in the case $j = r-1$ is different from that in the cases $1 \leq j \leq r-2$.

3. Proof of Theorem 1

The proof of Theorem 1 and Theorem 2 is similar to the argument of Matsumoto [3], [4], [5], [6], [7]. The basic point is the use of the following integral representation.

LEMMA 3. *We have*

$$\hat{\zeta}_{MT,j,r}(s_1,\dots s_j;s_{j+1},\dots,s_r,s_{r+1})$$
$$=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}$$
$$\times\hat{\zeta}_{MT,j,r-1}(s_1,\dots,s_j;s_{j+1},\dots,s_{r-1},s_r+s_{r+1}+z)\zeta(-z)\,dz \tag{15}$$

and

$$\hat{L}_{MT,j,r}(s_1,\dots,s_j;s_{j+1},\dots,s_{r+1};\chi_1,\dots,\chi_r)$$
$$=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}$$
$$\times\hat{L}_{MT,j,r-1}(s_1,\dots,s_j;s_{j+1},\dots,s_{r-1},s_r+s_{r+1}+z;\chi_1,\dots,\chi_{r-1})$$
$$\times L(-z,\chi_r)\,dz, \tag{16}$$

where $L(-z,\chi_r)$ *is the Dirichlet L-function attached to* χ_r, $1\le j\le r-1$ *and* $-\mathrm{Re}(s_{r+1})<c<-1$.

PROOF OF LEMMA 3. We prove only for $\hat{L}_{MT,j,r}$. Using the Mellin-Barnes integral formula (5) for the multiple sum (9) with $\lambda=m_r/(m_1+\cdots+m_{r-1})$, we can formally obtain

$$\hat{L}_{MT,j,r}(s_1,\dots,s_j;s_{j+1},\dots,s_{r+1};\chi_1,\dots,\chi_r)$$
$$=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{\chi_1(m_1)\cdots\chi_r(m_r)}{m_1^{s_1}\cdots m_j^{s_j}(m_1+\cdots+m_j)^{s_{j+1}}\cdots(m_1+\cdots+m_{r-1})^{s_r+s_{r+1}}}$$
$$\times\left(1+\frac{m_r}{m_1+\cdots+m_{r-1}}\right)^{-s_{r+1}}$$
$$=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{\chi_1(m_1)\cdots\chi_r(m_r)}{m_1^{s_1}\cdots m_j^{s_j}(m_1+\cdots+m_j)^{s_{j+1}}\cdots(m_1+\cdots+m_{r-1})^{s_r+s_{r+1}}}$$
$$\times\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}\left(\frac{m_r}{m_1+\cdots+m_{r-1}}\right)^{z}dz$$
$$=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}$$

$$\times \sum_{m_1=1}^{\infty} \cdots \sum_{m_{r-1}=1}^{\infty} \frac{\chi_1(m_1)\cdots\chi_{r-1}(m_{r-1})}{m_1^{s_1}\cdots m_j^{s_j}(m_1+\cdots+m_j)^{s_{j+1}}\cdots(m_1+\cdots+m_{r-1})^{s_r+s_{r+1}+z}}$$

$$\times \sum_{m_r=1}^{\infty} \frac{\chi_r(m_r)}{m_r^{-z}}\, dz$$

$$= \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}$$

$$\times \hat{L}_{MT,j,r-1}(s_1,\ldots,s_j;s_{j+1},\ldots,s_{r-1},s_r+s_{r+1}+z;\chi_1,\ldots,\chi_{r-1})L(-z,\chi_r)\, dz.$$

Now, we prove that $\sum_{m=1}^{\infty}$ and $\int_{(c)}$ can be exchanged. Put $z=c+iw$ $(-\infty<w<\infty)$. It is enough to prove that

$$I_{j,r} = \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\chi_1(m_1)\cdots\chi_r(m_r)}{m_1^{s_1}\cdots m_j^{s_j}(m_1+\cdots+m_j)^{s_{j+1}}\cdots(m_1+\cdots+m_{r-1})^{s_r+s_{r+1}}}\right.$$

$$\left.\times\left(\frac{m_r}{m_1+\cdots+m_{r-1}}\right)^{c+iw}\frac{\Gamma(s_{r+1}+c+iw)\Gamma(-c-iw)}{\Gamma(s_{r+1})}\right| dw$$

$$= \hat{\zeta}_{MT,j,r-1}(\sigma_1,\ldots,\sigma_j;\sigma_{j+1},\ldots,\sigma_{r-1},\sigma_r+\sigma_{r+1}+c)\zeta(-c)$$

$$\times \frac{1}{|\Gamma(s_{r+1})|}\int_{-\infty}^{\infty}|\Gamma(s_{r+1}+c+iw)\Gamma(-c-iw)|\, dw$$

is bounded for each $(s_1,s_2,\ldots,s_{r+1})\in R_{j,r}$. By using the Stirling's formula we have

$$\begin{aligned}
&|\Gamma(s_{r+1}+c+iw)\Gamma(-c-iw)|\\
&\quad=\sqrt{2\pi}\left|\exp\left\{\left(s_{r+1}+c+iw-\frac{1}{2}\right)\log(s_{r+1}+c+iw)\right\}\right|\\
&\qquad\times|\exp(-s_{r+1}-c-iw)|(1+O(|w|^{-1}))\quad(|w|\to\infty)\\
&\quad=\sqrt{2\pi}\exp\{-w\arg(s_{r+1}+c+iw)\}O(|w|^{\sigma_{r+1}+c+1/2})\\
&\quad=O\left(\exp\left(-\frac{\pi}{2}|w|\right)\right),
\end{aligned}$$

hence

$$\int_{-\infty}^{\infty}|\Gamma(s_{r+1}+c+iw)\Gamma(-c-iw)|\, dw = O(1).$$

This implies the assertion. $\square$

These integral representations (15), (16), give the following inductive structure;

$$\hat{\zeta}_{MT,j,r} \to \hat{\zeta}_{MT,j,r-1} \to \hat{\zeta}_{MT,j,r-2} \to \cdots \to \hat{\zeta}_{MT,j,j+1} \to \hat{\zeta}_{MT,j,j} = \zeta_{MT,j}, \quad (17)$$

$$\hat{L}_{MT,j,r} \to \hat{L}_{MT,j,r-1} \to \hat{L}_{MT,j,r-2} \to \cdots \to \hat{L}_{MT,j,j+1} \to \hat{L}_{MT,j,j} = L_{MT,j}. \quad (18)$$

THEOREM 4 (K. Matsumoto [4]). (i) *The function* $\zeta_{MT,r}(s_1,\ldots,s_r;s_{r+1})$ *can be meromorphically continued to the whole* $\mathbf{C}^{r+1}$*-space.*

(ii) *The possible singularities of* $\zeta_{MT,r}$ *are located only on the subsets of* $\mathbf{C}^{r+1}$ *defined by one of the following equations*;

$$s_j + s_{r+1} = 1 - \ell \quad (1 \le j \le r, \ell \in \mathbf{N}_0),$$

$$s_{j_1} + s_{j_2} + s_{r+1} = 2 - \ell \quad (1 \le j_1 < j_2 \le r, \ell \in \mathbf{N}_0),$$

$$\cdots$$

$$s_{j_1} + \cdots + s_{j_{r-1}} + s_{r+1} = r - 1 - \ell \quad (1 \le j_1 < \cdots < j_{r-1} \le r, \ell \in \mathbf{N}_0),$$

$$s_1 + s_2 + \cdots + s_r + s_{r+1} = r,$$

where $\mathbf{N}_0$ *denotes the set of non-negative integer.*

(iii) *Each of these singularities can be cancelled by the corresponding linear factor.*

(iv) $\zeta_{MT,r}$ *is of polynomial order with respect to* $|\mathrm{Im}(s_{r+1})|$.

PROOF OF THEOREM 1. When $j = r$ the assertion is Theorem 4. If $j = r-1$, (15) implies

$$\hat{\zeta}_{MT,r-1,r}(s_1,\ldots,s_{r-1};s_r,s_{r+1}) = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}\zeta_{MT,r-1}(s_1,\ldots,s_{r-1};s_r+s_{r+1}+z)\zeta(-z)\,dz \quad (19)$$

where $-\mathrm{Re}(s_{r+1}) < c < -1$. By Theorem 4, the poles of $\zeta_{MT,r-1}(s_1,\ldots,s_{r-1};s_r+s_{r+1}+z)$ as in a z-plane are

$$z = -s_j - s_r - s_{r+1} + 1 - \ell \quad (1 \le j \le r-1, \ell \in \mathbf{N}_0),$$

$$z = -s_{j_1} - s_{j_2} - s_r - s_{r+1} + 2 - \ell \quad (1 \le j_1 < j_2 \le r-1, \ell \in \mathbf{N}_0),$$

$$\vdots$$

$$z = -s_{j_1} - \cdots - s_{j_{r-2}} - s_r - s_{r+1} + r - 2 - \ell$$
$$(1 \le j_1 < \cdots < j_{r-1} \le r-1, \ell \in \mathbf{N}_0),$$
$$z = -s_1 - \cdots - s_{r-1} - s_r - s_{r+1} + r - 1,$$

all of which are located to the left of $\mathrm{Re}(z) = c$. The other poles of the integrand on the right-hand side of (19) are $z = -s_{r+1} - n$ $(n \in \mathbf{N}_0)$, $z = n$ $(n \in \mathbf{N}_0)$ and $z = -1$. We shift the path of integration to the right to $\mathrm{Re}(z) = N - \varepsilon$, where N is a positive integer. Because $\zeta_{MT,r-1}(s_1, \ldots, s_{r-1}; s_r)$ is of polynomial order with respect to $|\mathrm{Im}(s_r)|$, using Stirling's formula we obtain

$$\left| \int_{c \pm iT}^{N-\varepsilon \pm iT} \frac{\Gamma(s_{r+1} + z)\Gamma(-z)}{\Gamma(s_{r+1})} \zeta_{MT,r-1}(s_1, \ldots, s_{r-1}; s_r + s_{r+1} + z)\zeta(-z)\, dz \right|$$
$$\ll g(T)e^{-\pi T} \quad (T \to \infty),$$

where g is a certain polynomial. Hence, the shift of the path of integration is possible, and we obtain

$$\begin{aligned}
&\hat{\zeta}_{MT,r-1,r}(s_1, \ldots, s_{r-1}; s_r, s_{r+1}) \\
&\quad = \frac{1}{s_{r+1} - 1} \zeta_{MT,r-1}(s_1, \ldots, s_{r-1}; s_r + s_{r+1} - 1) \\
&\qquad - \frac{1}{2} \zeta_{MT,r-1}(s_1, \ldots, s_{r-1}; s_r + s_{r+1}) \\
&\qquad + \sum_{n=1}^{[N/2]} \binom{-s_{r+1}}{2n-1} \zeta_{MT,r-1}(s_1, \ldots, s_{r-1}; s_r + s_{r+1} + 2n - 1)\zeta(1 - 2n) \\
&\qquad + \frac{1}{2\pi i} \int_{(N-\varepsilon)} \frac{\Gamma(s_{r+1} + z)\Gamma(-z)}{\Gamma(s_{r+1})} \\
&\qquad \times \zeta_{MT,r-1}(s_1, \ldots, s_{r-1}; s_r + s_{r+1} + z)\zeta(-z)\, dz. \qquad (20)
\end{aligned}$$

The poles of the integrand of the last integral term is listed above, and hence we see that this integral is holomorphic at any points satisfying all of the following inequalities;

$$\mathrm{Re}(s_{r+1}) > -N + \varepsilon,$$
$$\mathrm{Re}(s_j + s_r + s_{r+1}) > 1 - N + \varepsilon,$$

$$\mathrm{Re}(s_{j_1} + s_{j_2} + s_r + s_{r+1}) > 2 - N + \varepsilon \quad (1 \le j_1 < j_2 \le r-1),$$

$$\vdots$$

$$\mathrm{Re}(s_{j_1} + \cdots + s_{j_{r-2}} + s_r + s_{r+1}) > r - 2 - N + \varepsilon \quad (1 \le j_1 < \cdots < j_{r-2} \le r-1),$$

$$\mathrm{Re}(s_1 + \cdots + s_{r-1} + s_r + s_{r+1}) > r - 1 - N + \varepsilon.$$

Since N can be taken arbitrarily large, (20) implies the meromorphic continuation of $\hat{\zeta}_{MT,r-1,r}$ to the whole $\mathbf{C}^{r+1}$-space. The first, the second and the third terms on right-hand side of (20) have a possible singularities that are located only on the subsets of $\mathbf{C}^{r+1}$ defined by one of the following equations;

$$s_j + s_r + s_{r+1} + d = 1 - \ell \quad (1 \le j \le r-1, \ell \ge 0),$$

$$s_{j_1} + s_{j_2} + s_r + s_{r+1} + d = 2 - \ell \quad (1 \le j_1 < j_2 \le r-1, \ell \ge 0),$$

$$\vdots$$

$$s_{j_1} + \cdots + s_{j_{r-2}} + s_r + s_{r+1} + d = r - 2 - \ell \quad (1 \le j_1 < \cdots < j_{r-2} \le r-1, \ell \ge 0),$$

$$s_1 + \cdots + s_{r-1} + s_r + s_{r+1} + d = r - 1,$$

where $d = -1, 0, 1, 3, 5, 7, \ldots$ $(-1 \le d \le N-1)$. Here, we note that $\{\ell + d \mid \ell \in \mathbf{N}_0, d = -1, 0, 1, 3, 5, \ldots\} = \{\ell \in \mathbf{Z} \mid \ell \ge -1\}$. Since N can be arbitrarily large, we obtain the result in the case of $j = r-1$ in (ii).

When $j = r-2$ in (15), and we shift the path of integration to the right to $\mathrm{Re}(z) = N - \varepsilon$ to obtain

$$\begin{aligned}
&\hat{\zeta}_{MT,r-2,r}(s_1, \ldots, s_{r-2}; s_{r-1}, s_r, s_{r+1}) \\
&\quad = \frac{1}{s_{r+1} - 1} \hat{\zeta}_{MT,r-1,r}(s_1, \ldots, s_{r-2}; s_{r-1}, s_r + s_{r+1} - 1) \\
&\qquad - \frac{1}{2} \hat{\zeta}_{MT,r-1,r}(s_1, \ldots, s_{r-2}; s_{r-1}, s_r + s_{r+1}) \\
&\qquad + \sum_{n=1}^{[N/2]} \binom{-s_{r+1}}{2n-1} \hat{\zeta}_{MT,r-1,r}(s_1, \ldots, s_{r-2}; s_{r-1}, s_r + s_{r+1} + n) \zeta(1-2n) \\
&\qquad + \frac{1}{2\pi i} \int_{(N-\varepsilon)} \frac{\Gamma(s_{r+1} + z)\Gamma(-z)}{\Gamma(s_{r+1})} \\
&\qquad \times \hat{\zeta}_{MT,r-1,r}(s_1, \ldots, s_{r-2}; s_{r-1}, s_r + s_{r+1} + z) \zeta(-z) \, dz.
\end{aligned} \tag{21}$$

The possible singularities on the right-hand side of (21) are

$$
\begin{aligned}
& s_{r+1} = 1, \\
& s_r + s_{r+1} + n = 1, \\
& s_j + s_{r-1} + s_r + s_{r+1} + n = 1 - \ell \quad (1 \le j \le r-2, \ell \ge -1), \\
& s_{j_1} + s_{j_2} + s_{r-1} + s_r + s_{r+1} + n = 2 - \ell \quad (1 \le j_1 < j_2 \le r-2, \ell \ge -1), \\
& \quad \vdots \\
& s_{j_1} + \cdots + s_{j_{r-3}} + s_{r-1} + s_r + s_{r+1} + n = r - 3 - \ell \\
& \qquad (1 \le j_1 < \cdots < j_{r-3} \le r-2, \ell \ge -1), \\
& s_1 + \cdots + s_{r-2} + s_{r-1} + s_r + s_{r+1} + n = r - 2 - d,
\end{aligned}
$$

where $n, d = -1, 0, 1, 3, 5, 7, \dots \ (-1 \le n \le N)$. Since

$$
\begin{aligned}
& \{\ell + d \,|\, \ell \in \{-1\} \cup \mathbf{N}_0,\ d = -1, 0, 1, 3, 5, \dots\} = \{\ell \in \mathbf{Z} \,|\, \ell \ge -2\}, \\
& \{d + n \,|\, d, n = -1, 0, 1, 3, 5, \dots \ (-1 \le n \le N)\} = \{\ell \in \mathbf{Z} \,|\, \ell \ge -2\},
\end{aligned}
$$

the above possible singularities can be rewritten as follows;

$$
\begin{aligned}
& s_{r+1} = 1, \\
& s_r + s_{r+1} = 1 - n, \\
& s_j + s_{r-1} + s_r + s_{r+1} = 1 - \ell \quad (1 \le j \le r-2, \ell \ge -2), \\
& s_{j_1} + s_{j_2} + s_{r-1} + s_r + s_{r+1} = 2 - \ell \quad (1 \le j_1 < j_2 \le r-2, \ell \ge -2), \\
& \quad \vdots \\
& s_{j_1} + \cdots + s_{j_{r-3}} + s_{r-1} + s_r + s_{r+1} = r - 3 - \ell \\
& \qquad (1 \le j_1 < \cdots < j_{r-2} \le r-2, \ell \ge -2), \\
& s_1 + \cdots + s_{r-2} + s_{r-1} + s_r + s_{r+1} = r - 2 - \ell \quad (\ell \ge -2).
\end{aligned}
$$

Since N can be taken arbitrarily large, we obtain the results of (ii) in the case of $j = r - 2$.

Let $k = r - j$ $(k \ge 2)$. Assume that the assertion of Theorem 1 is true in the case of $r - j = 2, 3, \dots, k-1$, and we prove by induction the assertion in the case of $r-j=2$. By Lemma 3, we obtain

$$\hat{\zeta}_{MT,r-k,r}(s_1,\dots,s_{r-k};s_{r-k+1},\dots,s_r,s_{r+1})$$
$$=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}$$
$$\times\hat{\zeta}_{MT,r-k,r-1}(s_1,\dots,s_{r-k};s_{r-k+1},\dots,s_{r-1},s_r+s_{r+1}+z)\zeta(-z)\,dz, \tag{22}$$

where $1\le j\le k-1$, $-\mathrm{Re}(s_{r+1})<c<-1$. By assumption of induction, we find that the possible singularities of $\hat{\zeta}_{MT,r-k,r-1}(s_1,\dots,s_{r-k};s_{r-k+1},\dots,s_{r-1},s_r+s_{r+1}+z)$ as a function in z are

$$z=-s_r-s_{r+1}+1,$$
$$z=-s_{r-1}-s_r-s_{r+1}+2-d \quad (d=-1,0,1,3,5,7,\dots),$$
$$z=-s_{r-2}-s_{r-1}-s_r-s_{r+1}+3-\ell \quad (\ell\in\mathbf{N}_0),$$
$$\vdots$$
$$z=-s_{r-k+2}-\cdots-s_{r-1}-s_r-s_{r+1}+k-1-\ell \quad (\ell\in\mathbf{N}_0),$$
$$z=-s_{j_1}-s_{r-k+1}-\cdots-s_{r-1}-s_r-s_{r+1}+1-\ell'$$
$$(1\le j_1\le r-k,\ell'\ge -k+2),$$
$$z=-s_{j_1}-s_{j_2}-s_{r-k+1}-\cdots-s_{r-1}-s_r-s_{r+1}+2-\ell'$$
$$(1\le j_1<j_2\le r-k,\ell'\ge -k+2),$$
$$\vdots$$
$$z=-s_{j_1}-\cdots-s_{j_{r-k-1}}-s_{r-k+1}-\cdots-s_{r-1}-s_r-s_{r+1}+r-k-\ell'$$
$$(1\le j_1<\cdots<j_{r-k-1}\le r-k,\ell'\ge -k+2),$$
$$z=-s_1-s_2-\cdots-s_{r-k+1}-\cdots-s_{r-1}-s_r-s_{r+1}+r-k+1-\ell'$$
$$(\ell'\ge -k+2),$$

all of which are located to the left of $\mathrm{Re}(z)=c$. The other poles of the integrand on the right-hand side of (22) are $z=-s_{r+1}-n$ $(n\in\mathbf{N}_0)$, $z=n$ $(n\in\mathbf{N}_0)$ and $z=-1$. We shift the path of integration to the right to $\mathrm{Re}(z)=N-\varepsilon$, where N is a positive integer. Since the shift of the path of integration is possible as before, we obtain

$$
\begin{aligned}
&\hat{\zeta}_{MT,r-k,r}(s_1,\ldots,s_{r-k};s_{r-k+1},\ldots,s_{r+1})\\
&\quad=\frac{1}{s_{r+1}-1}\hat{\zeta}_{MT,r-k,r-1}(s_1,\ldots,s_{r-k};s_{r-k+1},\ldots,s_{r-1},s_r+s_{r+1}-1)\\
&\qquad-\frac{1}{2}\hat{\zeta}_{MT,r-k,r-1}(s_1,\ldots,s_{r-k};s_{r-k+1},\ldots,s_{r-1},s_r+s_{r+1})\\
&\qquad+\sum_{n=1}^{[N/2]}\binom{-s_{r+1}}{2n-1}\hat{\zeta}_{MT,r-k,r-1}(s_1,\ldots,s_{r-k};s_{r-k+1},\ldots,s_{r-1},s_r+s_{r+1}+2n-1)\\
&\qquad\times\zeta(1-2n)\\
&\qquad+\frac{1}{2\pi i}\int_{(N-\varepsilon)}\frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}\\
&\qquad\times\hat{\zeta}_{MT,r-k,r-1}(s_1,\ldots,s_{r-k};s_{r-k+1},\ldots,s_{r-1},s_r+s_{r+1}+z)\zeta(-z)\,dz. \qquad (23)
\end{aligned}
$$

The first, the second and the third terms on right-hand side of (23) have a possible singularities that are located only on the subsets of $\mathbf{C}^{r+1}$ defined by one of the following equations;

$$
\begin{aligned}
&s_{r+1}=1,\\
&s_r+s_{r+1}+n=1,\\
&s_{r-1}+s_r+s_{r+1}+n=2-d \quad (d=-1,0,1,3,5,7,\ldots),\\
&s_{r-2}+s_{r-1}+s_r+s_{r+1}+n=3-\ell \quad (\ell\in\mathbf{N}_0),\\
&\qquad\vdots\\
&s_{r-k+2}+s_{r-k+3}+\cdots+s_r+s_{r+1}+n=k-1-\ell \quad (\ell\in\mathbf{N}_0),\\
&s_{j_1}+s_{r-k+1}+\cdots+s_r+s_{r+1}+n=1-\ell' \quad (1\le j_1\le r-k,\ell'\ge-(k-1)), \quad (24)\\
&s_{j_1}+s_{j_2}+s_{r-k+1}+\cdots+s_r+s_{r+1}+n=2-\ell'\\
&\qquad(1\le j_1<j_2\le r-k,\ell'\ge-(k-1)),\\
&\qquad\vdots\\
&s_{j_1}+\cdots+s_{j_{r-k-1}}+s_{r-k+1}+\cdots+s_r+s_{r+1}+n=r-k-1-\ell'\\
&\qquad(1\le j_1<\cdots<j_{r-k-1}\le r-k,\ell'\ge-(k-1)),\\
&s_1+\cdots+s_{r-k}+s_{r-k+1}+\cdots+s_r+s_{r+1}+n=r-k-\ell' \quad (\ell'\ge-(k-1)),
\end{aligned}
$$

where $n = -1, 0, 1, 3, 5, 7, \ldots$ $(1 \le n \le N-1)$. The last integral of (23) is holomorphic at any satisfying all of the following inequalities;

$$
\begin{aligned}
&\mathrm{Re}(s_{r+1}) > -N + \varepsilon, \\
&\mathrm{Re}(s_r + s_{r+1}) > 1 - N + \varepsilon, \\
&\mathrm{Re}(s_{r-1} + s_r + s_{r+1}) > 2 - N + \varepsilon, \\
&\quad \vdots \\
&\mathrm{Re}(s_{r-k+2} + s_{r-k+3} + \cdots + s_r + s_{r+1}) > k - 1 - N + \varepsilon, \\
&\mathrm{Re}(s_{j_1} + s_{r-k+1} + \cdots + s_r + s_{r+1}) > k - N + \varepsilon \quad (1 \le j_1 \le r-k), \\
&\mathrm{Re}(s_{j_1} + s_{j_2} + s_{r-k+1} + \cdots + s_r + s_{r+1}) > k + 1 - N + \varepsilon \\
&\qquad (1 \le j_1 < j_2 \le r-k), \\
&\quad \vdots \\
&\mathrm{Re}(s_{j_1} + \cdots + s_{j_{r-k-2}} + s_{r-k+1} + \cdots + s_r + s_{r+1}) > r - 2 - N + \varepsilon \\
&\qquad (1 \le j_1 < \cdots < j_{r-k-2} \le r-k), \\
&\mathrm{Re}(s_1 + \cdots + s_{r-k} + s_{r-k+1} + \cdots + s_r + s_{r+1}) > r - 1 - N + \varepsilon.
\end{aligned}
\tag{25}
$$

Since N can be taken arbitrarily large, (25) implies the meromorphic continuation of $\hat{\zeta}_{MT,r-k,r}$ to the whole $\mathbf{C}^{r+1}$ space. By the method similar to that as in the case of $j = r-2$, we obtain the result in the case of $2 \le j \le r$ in (ii).

Let

$$
\begin{aligned}
&\Phi_{r-k,r,N}(s_1, \ldots, s_{r-k}; s_{r-k+1}, \ldots, s_{r+1}) \\
&\quad = (s_{r+1} - 1) \prod_{\substack{-1 \le d \le N-1 \\ d:0 \text{ or odd}}} (s_r + s_{r+1} - 2 + d) \\
&\qquad \times \prod_{\ell=0}^{N} \{(s_{r-1} + s_r + s_{r+1} - 3 - \ell)(s_{r-2} + s_{r-1} + s_r + s_{r+1} - 4 - \ell) \\
&\qquad\qquad \times \cdots \times (s_{r-k+1} + \cdots + s_r + s_{r+1} - k - 1 + \ell)\} \\
&\qquad \times \prod_{\ell'=-k}^{N} \left\{ \prod_{j_1=1}^{r-k} (s_{j_1} + s_{r-k+1} + \cdots + s_r + s_{r+1} - 1 + \ell') \right.
\end{aligned}
$$

$$\times \prod_{1\le j_1<j_2\le r-k}^{r-k} (s_{j_1}+s_{j_2}+s_{r-k+1}+\cdots+s_r+s_{r+1}-1+\ell')$$

$$\times\cdots\times \prod_{1\le j_1<\cdots<j_{r-k-1}\le r-k}^{r-k} (s_{j_1}+\cdots+s_{j_{r-k-1}}$$

$$+s_{r-k+1}+\cdots+s_r+s_{r+1}-1+\ell')$$

$$\left.\times(s_1+\cdots+s_{r-k}+s_{r-k+1}+\cdots+s_r+s_{r+1}-r+k+\ell')\right\}$$

where N is positive integer. By (23) and (ii),

$$\hat{\zeta}_{MT,r-k,r}(s_1,\dots,s_{r-k};s_{r-k+1},\dots,s_{r+1})\Phi_{r-k,r,N}(s_1,\dots,s_{r-k};s_{r-k+1},\dots,s_{r+1})$$

is shown to be holomorphic, to obtain (iii). Finally we can also prove (iv) also by the induction assumption on the order $\hat{\zeta}_{MT,r-k,r-1}$ and Stirling's formula. Hence the proof of Theorem 1 is complete. □

4. Proof of Theorem 2

Theorem 5 (Wu [10]). *The function $L_{MT,r}(s_1,\dots,s_r;s_{r+1};\chi_1,\dots,\chi_r)$ can be meromorphically continued to the $\mathbf{C}^{r+1}$-space. If none of the characters $\chi_1,\dots,\chi_r$ are principal, then $L_{MT,r}$ is entire. If there are k principal characters $\chi_{t_1},\dots,\chi_{t_k}$ among them, then possible singularities are located only on the subsets of $\mathbf{C}^{r+1}$ defined by one of the following equations*;

$$s_{t_{u(1)}}+s_{r+1}=1-\ell \quad (1\le u(1)\le k,\ell\in\mathbf{N}_0),$$

$$s_{t_{u(1)}}+s_{t_{u(2)}}+s_{r+1}=2-\ell \quad (1\le u(1)<u(2)\le k,\ell\in\mathbf{N}_0),$$

$$\vdots$$

$$s_{t_{u(1)}}+\cdots+s_{t_{u(k-1)}}+s_{r+1}=k-1-\ell$$

$$(1\le u(1)<\cdots<u(k-1)\le k,\ell\in\mathbf{N}_0),$$

$$s_{t_1}+s_{t_2}+\cdots+s_{t_k}+s_{r+1}=k-\ell\left(1-\left[\frac{k}{r}\right]\right) \quad (\ell\in\mathbf{N}_0),$$

where $1\le h\le k,\ 1\le u(1)<\cdots<u(h)\le k,\ \ell\in N_0.$

Proof of Theorem 2. For (iii) and (iv) the method is exactly the same as in the proof of Theorem 1. When $j = r$ the assertion is nothing but Theorem 5. If $j = r-1$, (16) implies

$$\begin{aligned}
&\hat{L}_{MT,r-1,r}(s_1,\dots,s_{r-1};s_r,s_{r+1};\chi_1,\dots,\chi_r)\\
&\quad = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}\\
&\qquad \times L_{MT,r-1}(s_1,\dots,s_{r-1};s_r+s_{r+1}+z;\chi_1,\dots,\chi_{r-1})L(-z,\chi_r)\,dz \qquad (26)
\end{aligned}$$

where $-\mathrm{Re}(s_{r+1}) < c < -1$, and $L(\cdot,\chi_r)$ is Dirichlet L-function. By Theorem 5, the poles of

$$L_{MT,r-1}(s_1,\dots,s_{r-1};s_r+s_{r+1}+z;\chi_1,\dots,\chi_{r-1})$$

as in the z-plane are located to the left of $\mathrm{Re}(z) = c$. The other poles of the integrand on the right-hand side of (26) are $z = -s_{r+1} - n$ $(n \in \mathbf{N}_0)$, $z = n$ $(n \in \mathbf{N}_0)$. Also, when χ_r is principal, $z = -1$ is a simple pole. We shift the path of integration of (26) to the right to $\mathrm{Re}(z) = N - \varepsilon$, to obtain

$$\begin{aligned}
&\hat{L}_{MT,r-1,r}(s_1,\dots,s_{r-1};s_r,s_{r+1};\chi_1,\dots,\chi_r)\\
&\quad = \frac{1}{s_{r+1}-1}L_{MT,r-1}(s_1,\dots,s_{r-1};s_r+s_{r+1}-1;\chi_1,\dots,\chi_{r-1})\cdot\frac{\varphi(q)}{q}\cdot\delta_r\\
&\qquad + \sum_{n=0}^{N-1}\binom{-s_{r+1}}{n}L_{MT,r-1}(s_1,\dots,s_{r-1};s_r+s_{r+1}+n;\chi_1,\dots,\chi_{r-1})L(-n,\chi_r)\\
&\qquad + \frac{1}{2\pi i}\int_{(N-\varepsilon)} \frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}\\
&\qquad \times L_{MT,r-1}(s_1,\dots,s_{r-1};s_r+s_{r+1}+z;\chi_1,\dots,\chi_{r-1})L(-z,\chi_r)\,dz \qquad (27)
\end{aligned}$$

where δ_r is defined in the statement of Theorem 2. Futher, if $\chi_{t_1},\dots,\chi_{t_k}$ $(1 \le t_1 < \cdots < t_k \le r-1)$ are principal and the others are non-principal, possible singularities of (27) are

$$s_{t_{u(1)}} + s_r + s_{r+1} = 1 - \ell \quad (1 \le u(1) \le k, \ell \ge -\delta_r),$$

$$s_{t_{u(1)}} + s_{t_{u(2)}} + s_r + s_{r+1} = 2 - \ell \quad (1 \le u(1) < u(2) \le k, \ell \ge -\delta_r),$$

$$\vdots$$

$$s_{t_{u(1)}} + \cdots + s_{t_{u(k-1)}} + s_r + s_{r+1} = k - 1 - \ell$$

$$(1 \le u(1) < \cdots < u(k-1) \le k, \ell \ge -\delta_r),$$

$$s_{t_1} + \cdots + s_{t_k} + s_r + s_{r+1} = k - \ell \quad (\ell \ge -\delta_r), \tag{28}$$

moreover, if $\chi_{t_1}, \ldots, \chi_{t_k}$ $(1 \le t_1 < \cdots < t_k \le r-1)$ and χ_r are principal and the others are non-principal, then

$$s_{r+1} = 1$$

is also a possible singularity. Proof in the case of $1 \le j \le r-2$ is the same as the proof of Theorem 1; we can prove the assertion using the induction on k with $k = r - j$. Also, how to deal with Dirichlet characters is similar to the case of $j = r - 1$. □

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