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On change-point detection in volatile series using GARCH models

Edoh KATCHEKPELE[1,*], **Kossi Essona GNEYOU**[2] **and Abdou Kâ DIONGUE**[3]

[1,2]Department of Mathematics, University of Lomé, Lomé, TOGO
[3]LERSTAD, University GASTON BERGER, Saint-Louis, SENEGAL

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Abstract. We study a Cumulative Sum (CUSUM)-type test to detect a change in the unconditional variance of GARCH models. We show that, under the null hypothesis (no change), the CUSUM test statistic converges to the supremum of a standard Brownian bridge. Using Monte Carlo simulation, we demonstrate that the asymptotic power of the test is almost 1 and compare the test result with existing results in the literature. Finally, the test procedure is applied to real-world situation namely stock market returns where we are able to detect a change in the unconditional variance at a very early stage of the financial crisis in comparison to other previous analyses of the same dataset.

Key words: GARCH model ; Change-point ; Squared cusum test ; Brownian bridge ; Weak convergence.
AMS 2010 Mathematics Subject Classification : 37M10, 62F03, 62F05.

Résumé. Nous étudions un test de type CUSUM pour la détection de rupture dans la variance inconditionnelle des modèles GARCH. Nous montrons que sous l'hypothèse nulle, notre statistique de test converge vers le supremum d'un pont Brownien standard. Utilisant des simulations de type Monte Carlo, nous démontrons que la puissance asymptotique du test est presqu'égale à 1 et comparons le résultat du test avec les résultats existants dans la littérature. Enfin, un exemple d'application sur les données réelles des rendements du marché boursier nous a permis de détecter une rupture dans la variance inconditionnelle à un stade très précoce de la crise financière par rapport à d'autres analyses précédentes du même ensemble de données.

*Corresponding author Edoh KATCHEKPELE : ekatchekpele@univ-lome.tg
Kossi Essona GNEYOU : kossi_gneyou@yahoo.fr
Abdou Kâ DIONGUE : abdou.diongue@ugb.edu.sn

1. Introduction

The autoregressive models introduced by Yule (1927) play a very important role in the study of the modelling and forecasting of temporal data. The Generalized Autoregressive Conditional Heteroscedasticity models (GARCH), introduced by Bollerslev (1986), which are special cases of autoregressive models, are applied to time series with no constant variance (hence the term heteroscedasticity) commonly present in the financial world (e.g. exchange rates, stock prices, etc). With heavy consequences sometimes, the distribution of the underlying stochastic process of the time series can change abruptly at a given time. In particular, the mean or covariance structure of this process may change abruptly at unknown dates. Several authors have pointed out the dangers faced if these moments of change are neither studied nor identified.

Therefore, since Page (1955), the problem of abrupt change detection has received much attention in the literature. A large number of articles have been published in various journals (See, e.g. Iclán and Tiao (1994), Csörgő and Horváth (1997), Chen and Gupta (2011), and the references therein). Recently, statistical theory on unique change-point in autoregressive models has been considered. We can cite, for example, Chalmond (1981) on the detection of a change in mean of an ARMA process. Bai (1994) obtained the weak convergence of the sequential empirical process of the estimated residuals in ARMA(p, q) models and applied the rusult to a change-point problem. Horváth *et al.* (2001) obtained the asymptotic law of the sequential empirical process of the squares of the residuals of an ARCH model and applied the results to change-point detection. Boldin (2002) established an asymptotic development of the residual empirical process of an ARCH model and constructed a Kolmogorov-Smirnov type test to detect change point in the residual distribution of the ARCH model. Kokoszka and Teyssière (2002) proposed two classes of tests to detect changes in volatility of a GARCH process. Procedures based on squared model residuals and on the likelihood ratio are considered. Lee *et al.* (2003) consider the problem of testing for a parameter change in GARCH(1,1) models based on the residual cusum test. Berkes *et al.* (2004) suggested a sequential monitoring scheme to detect changes in the parameters of a GARCH(p, q) sequence. Their procedure is based on quasi-likelihood scores and does not use model residuals.

The GARCH processes are uncorrelated, but the sequence of their squares are correlated. In pratice, using financial data, the squares of the returns are used to estimate the so-called volatility, which is an important parameter in asset pricing models. Unlike all the methods used for change-point detection by the aforementioned authors, the fact that the squares obtained from the GARCH are ARMA inspired us to construct our CUSUM-type test statistic based on squared observations of the GARCH, thus generalising the work of Chalmond (1981).

The remainder of the paper is organised as follows. In section 2 we present how the CUSUM test statistic is constructed. In Section 3, we establish the asymptotic distribution of the test statistic under the null hypothesis. In Section 4, we perform a simulation study and apply our test procedure to S&P 500 stock market returns and we conclude in Section 5.

2. CUSUM test statistic

Let us observe a stochastic phenomenon $(z_1, z_2, \dots, z_n)$ which is known to be generated by the square of a GARCH process. The model formally is as

$$z_t = u_t^2, \tag{1}$$

where

$$u_t = \sigma_t \varepsilon_t, \tag{2}$$

with

$$E(\varepsilon_t) = 0, \;\; E(\varepsilon_t^2) = 1,$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i u_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2, \;\; t = 1, 2, \dots \tag{3}$$

where the $\alpha_i, \;\; i = 1, 2, \dots, q$ and $\beta_j, \;\; j = 1, 2, \dots, p$ are nonnegative constants and α_0 is a (strictly) positive constant, which guarantees the strict positivity of the conditional variance, and

$$\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1 \tag{4}$$

which guarantees the stationarity of the process u_t (see Gouriéroux (1997)).

Let us introduce the innovation corresponding to the square of the process : $\nu_t = u_t^2 - \sigma_t^2$. Replacing σ_{t-j}^2 by $u_{t-j}^2 - \nu_{t-j}$ in (3), we obtain

$$u_t^2 = \alpha_0 + \sum_{i=1}^{\max(p,q)} \theta_i u_{t-i}^2 + \nu_t - \sum_{j=1}^{p} \beta_j \nu_{t-j}, \tag{5}$$

where $\theta_i = \alpha_i + \beta_i$ and with the convention $\theta_i = \beta_i$ (resp. $\theta_i = \alpha_i$) if $i > q$ (resp. $j > p$).

If L denotes the lag-operator $\left(L^j X_t = X_{t-j}\right)$, then (5) becomes

$$u_t^2 = \alpha_0 + \sum_{i=1}^{\max(p,q)} \theta_i L^i u_t^2 + \nu_t - \sum_{j=1}^{p} \beta_j L^j \nu_t$$

i.e.,

$$\left(1 - \sum_{i=1}^{\max(p,q)} \theta_i L^i\right) u_t^2 = \alpha_0 + \left(1 - \sum_{j=1}^{p} \beta_j L^j\right) \nu_t. \tag{6}$$

According to (4), the roots of the characteristic polynomial of

$$\Theta(L) = \left(1 - \sum_{i=1}^{\max(p,q)} \theta_i L^i\right)$$

are strictly outside the unit circle, thus $\Theta(L)$ is invertible (see Gouriéroux (1997)). We then obtain

$$u_t^2 = \omega + \nu_t + \sum_{j=1}^{\infty} b_j \nu_{t-j}, \tag{7}$$

with

$$\omega = \left(1 - \sum_{i=1}^{\max(p,q)} \theta_i\right)^{-1} \alpha_0$$

and the $b_j,\ j \geqslant 1$ are defined by the generating function

$$\left(1 - \sum_{i=1}^{\max(p,q)} \theta_i x^i\right)^{-1} \left(1 - \sum_{j=1}^{p} \beta_j x^j\right) = 1 + \sum_{j=1}^{\infty} b_j x^j.$$

It is therefore a question of knowing if the phenomenon $z_t = u_t^2$ has not undergone an abrupt change at a certain date t^* unknown, which can be interpreted as an abrupt change in the unconditional variance of the process (u_t).

Let us define the change-point problem by :

$$\mathbb{E}(z_t) = \begin{cases} \omega & \text{if } t < t^* \\ \omega + \delta & \text{if } t \geq t^*. \end{cases}$$

We then consider the testing problem of the null hypothesis given by

$$H_0 \ : \ \mathbb{E}(z_t) = \omega,$$

against the alternative hypothesis

$$H_1 : \ \mathbb{E}(z_t) = \omega + \delta,$$

where δ denotes a nonzero real number.

Denote, respectively by deviations and cumulative sum of the deviations, the two statistics below :

$$d_r = z_r - \overline{z}$$

where $\overline{z} = \frac{1}{n}\sum_{i=1}^{n} z_i = \frac{1}{n}\sum_{i=1}^{n} u_i^2$, and

$$S_r = \sum_{i=1}^{r} d_i.$$

Under H_0 the sums S_r have zero mean, whereas an abrupt change in the mean of z_t (which is equivalent to an abrupt change in the unconditionnal variance of the GARCH process u_t) propagates bias after t^*. Indeed, it is shown that

$$E(S_r) = t^*(r - t^*)(w + \delta).$$

This specifies the sensibility of the cumulative sum to a possible abrupt change. Thus we define the test statistic by

$$K_n = \sup_{0\leqslant t\leqslant 1} |X_n(t)|, \tag{8}$$

where

$$X_n(t) = \frac{1}{\sqrt{n}} S_{[nt]}$$

and $[x]$ denotes the largest integer smaller than or equal to x. Let $D[0;1]$ be the space of functions on $[0;1]$ that are right-continuous and have left-hand limits endowed with the Skorokhod topology. From Billingsley (1968), X_n is a random element of $D[0;1]$. In the next section, we study the asymptotic behaviour of the test statistic K_n. Before we go further, we stress that the theory of weak convergence we will be dealing takes places in the Skorokhod space. But later, we will have the opportunity to deal with it in the more concenient space of bounded functions $\ell^{+\infty}$ endowed with the uniform topology as made fashion by the book of van der Vaart and Wellner (1996). The paper by Lo (2014) and the book of Lo *et al.* (2016). would be interesting introductory readings.

3. Asymptotic distribution of K_n

In all the sequel, we assume that $\sum_{j=0}^{\infty} b_j \neq 0$. From now on, let us make the following assumptions :

A.1 $E\left(u_t^4\right) < \infty$.

A.2 $\sum_{j=1}^{\infty} j|b_j| < \infty$.

A.3 There is a random variable (r.v.) Y dominating ν_i and satisfying

i) $\mathbb{E}Y^2 < +\infty$

ii) $\mathbb{E}(Y^2 \ln^+ |Y|) < +\infty$.

A.4 The sequence of r.v. (ν_i) is uniformly integrable such that

i) $P\left[|\nu_i| \geqslant x\right] \leqslant cP\left[|Y| \geqslant x\right]$, for each $x \geqslant 0$, $i \geqslant 1$ and for some nonnegative constant c.

ii) $\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(\nu_i^2|\nu_{\underline{i-1}}) \overset{\text{a.s.}}{\to} \sigma_\nu^2$, $n \to \infty$, where $\nu_{\underline{i}}$ denotes the $\sigma-$field genereted by $\nu_{i'}$, $i' \leqslant i$ and $\overset{\text{a.s.}}{\to}$ denotes the almost sure convergence.

iii) $\sup_k \mathbb{E}\left(|\nu_k|^{2+\eta}\right) < \infty$, $\forall \eta > 0$.

These assumptions are quite standard and are required to ensure the invariance principle in the case of a sequence of non independent r.v. as stated in Lemma 1 below.

Theorem 1. *Let* : $b = \sum_{j=0}^{\infty} b_j$ *and* $\tau = \sigma_\nu b$. *If Assumptions A.1 - A.4 hold then under* H_0, *in* $D[0;1]$,

$$\frac{1}{\tau} \sup_{0 \leqslant t \leqslant 1} X_n(t) \xrightarrow{d} \sup_{0 \leqslant t \leqslant 1} W^0(t)$$

and

$$\frac{1}{\tau} K_n \xrightarrow{d} \sup_{0 \leqslant t \leqslant 1} |W^0(t)|,$$

where $W^0(t)$ *is a standard Brownian bridge and* $\xrightarrow{d}$ *denotes the convergence in distribution.*

Proof

The proof of Theorem 1 is based on two lemmas, which we present after introducing a simple polynomial decomposition (see Phillips and Solo (1992)).

Let

$$B(L) = \sum_{j=0}^{\infty} b_j L^j.$$

Then

$$B(L) = B(1) - (1-L)\tilde{B}(L) = b - (1-L)\tilde{B}(L), \tag{9}$$

where

$$\tilde{B}(L) = \sum_{j=0}^{\infty} \tilde{b}_j L^j, \ \tilde{b}_j = \sum_{k=j+1}^{\infty} b_k.$$

Let

$$Z_i = z_i - \mathbb{E}(z_i),$$

from (7),

$$Z_i = B(L)\nu_i,$$

where ν_i is the innovation of the process u_i^2 with variance σ_ν^2. Then we have

$$X_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \left(Z_i - \overline{Z}\right). \tag{10}$$

From (7) and (9),

$$Z_i = b\nu_i + \tilde{Z}_{i-1} - \tilde{Z}_i, \tag{11}$$

where

$$\tilde{Z}_i = \tilde{B}(L)\nu_i = \sum_{j=0}^{\infty} \tilde{b}_j \nu_{i-j}, \quad \tilde{b}_j = \sum_{k=j+1}^{\infty} b_k.$$

Lemma 1. *If Assumptions A.1 - A.4 hold, then under H_0 we have in $D[0;1]$*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]} Z_i \xrightarrow{d} b\sigma_\nu W(t),$$

where σ_ν is the standard deviation of $\{\nu_i\}$ and $W(t)$ a standard Brownian motion.

Proof of Lemma 1

We check that (ν_i) is a martingale difference sequence (m.d.s.). Indeed, denoting by $\underline{u_i}$ the $\sigma-$field generated by $u_{i'}$, $i' \leqslant i$, it's clear that $\underline{\nu_{i-1}} \subset \underline{u_{i-1}}$ and that the sequence (ν_i) is adapted to the filtration $(\underline{\nu_i})$. By Assumption A.4 i), (ν_i) is uniformly integrable hence it is bounded in L^1 i.e. $\mathbb{E}|\nu_i| < \infty$, $\forall i$. Moreover we have

$$\begin{aligned}
\mathbb{E}\left(\nu_i|\underline{\nu_{i-1}}\right) &= \mathbb{E}\left(\mathbb{E}(\nu_i|\underline{u_{i-1}})|\underline{\nu_{i-1}}\right) \\
&= \mathbb{E}\left(\mathbb{E}((u_i^2-\sigma_i^2)|\underline{u_{i-1}})|\underline{\nu_{i-1}}\right) \\
&= \mathbb{E}\left((\mathbb{E}u_i^2|\underline{u_{i-1}}-\sigma_i^2)|\underline{\nu_{i-1}}\right) \\
&= \mathbb{E}((\sigma_i^2-\sigma_i^2)|\underline{\nu_{i-1}}) \\
&= 0.
\end{aligned}$$

Since (ν_i) is m.d.s., uniformly integrable, all the conditions in Theorem 3.15 in Phillips and Solo (1992) are satisfied under the assumptions A.1 - A.4. Hence Lemma 1 follows. □

Lemma 2. *. If the assumptions of Lemma 1 hold, then*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]} \overline{Z} \xrightarrow{d} b\sigma_\nu tW(1).$$

Proof of Lemma 2. We have,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]} \overline{Z} = \frac{1}{n}\sum_{i=1}^{[nt]}\frac{1}{\sqrt{n}}\sum_{j=1}^{n} Z_j = \frac{[nt]}{n}\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n} Z_j\right)$$

and Lemma 2 follows from Lemma 1 and the fact that $\dfrac{[nt]}{n} \longrightarrow t$ as $n \longrightarrow +\infty$. □

$\{X_n(t)\}$ satisfies the invariance principles in $D[0;1]$ under the null hypothesis. Thus by Kokoszka and Leipus (1999), page 183, the asymptotic theory for the standard statistic $\sup_{0\leqslant t\leqslant 1}|X_n(t)|$ and its modifications follows automatically. This ends the proof of Theorem 1. □

4. Simulations and applications

In practice, a large value of K_n implies a change in the variance. At the level $\alpha = 0.05$ we reject the null hypothesis, under which no variance change is assumed to occur, if $(K_n/\tau) > 1.358$ (see Iclán and Tiao (1994)). Also note that with $t = \dfrac{k}{n}$ we have $t^* = \dfrac{k^*}{n}$ and $\dfrac{K_n}{\tau} = \max_{1\leqslant k\leqslant n} C_k$.

4.1. Simulation study

In this subsection, we evaluate the performance of the test statistic K_n. In order to apply the asymptotic result of Theorem 1 we have to estimate τ^2 racalling that z_t has a MA(q) approximation. In the case of a $MA(q)$, $\gamma_h = 0$ if $h > q$. We estimate τ^2, as in Berkes *et al.* (2009), by

$$\hat{\tau}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(z_i - \overline{z})^2 + 2\sum_{j=1}^{q}\left(1 - \frac{j}{q+1}\right)\frac{1}{n-j}\sum_{i=1}^{n-j}(z_i - \overline{z})(z_{i+j} - \overline{z})$$

and it can be shown (see Giraitis *et al.* (2003)) that

$$|\hat{\tau}_n - \tau^2| = o_p(1).$$

According to Kokoszka and Leipus (1999), if $q(n)$ is a numerical sequence that $q(n) \to \infty$, and $\dfrac{q(n)}{n} \to 0$ $(n \to \infty)$ then by replacing τ with $\hat{\tau}_n$, Theorem 1 remains true.

In these simulations, we consider GARCH$(1,1)$ models. We denote by $\theta = (\alpha_0, \alpha, \beta)$ the parameter and by σ^2 the unconditional variance of the model considered obtained by $\sigma^2 = \dfrac{\alpha_0}{1-\alpha-\beta}$. σ_0^2 and σ_1^2 are respectively the unconditional variance under the null hypothesis (no change) and alternative hypothesis. We will take $q(n) = \left[(\ln(n))^2\right]$ as in Lee *et al.* (2003).

Figure 1 shows two scenarios of GARCH$(1,1)$ model : a scenario without change (Figure 1(a) and Figure 1(c)) and the same series with change at $k^* = 500$ (Figure 1(b) and Figure 1(d)). Figure 1(a) represents a sample of GARCH$(1,1)$ model of size 1000 for $\sigma_0^2 = 4.255$ $(\theta_0 = (2, 0.03, 0.5))$ (without change) and Figure 1(b) the same series but now the variance, at $k^* = 500$, changes from $\sigma_0^2 = 4.255$ to $\sigma_1^2 = 6.666$ $(\theta_1 = (2, 0.2, 0.5))$. On the plot of C_k (Figure 1(c)) we can see that all values (then the maximum) of C_k are less than the limit of the critical region which is represented by the horizontal red line, confirming that there is no change. On the other hand, on Figure 1(d), around the point where the change occurs, C_k is greater than the critical value of the test and the place of the maximum is taken is the time k^* where the change occurs. As the maximum of C_k (which is our test statistic value) is large, we may conclude that a change occurs and reject the null hypothesis. Furthermore, on Figure 1(b) the change is not apparent but our proposed test statistic detects it ; which confirms its performance.
Figure 2 shows one scenario of GARCH$(1,1)$ model with change where the unconditional variance changes from $\sigma_0^2 = 10$ $(\theta_0 = (1, 0.1, 0.8))$ to $\sigma_1^2 = 3.333$ $(\theta_1 = (1, 0.1, 0.6))$ at different times. The first at $k^* = 300$ (Figure 2, (a) and (c)) and the second at $k^* = 700$ (Figure 2, (b) and (d)). In Figure 2(b) and Figure 2(d) we observe the same thing like in Figure 1(d) ; the maximum value of C_k which is our test statistic value has taken place at the change-point. This value is greater than the critical value of the test, hence confirms that a change occurs at these different points in time.

Table 1 presents the results of a simulation study intended to assess the performance of the change-point detection in the unconditional variance. The empirical levels computed

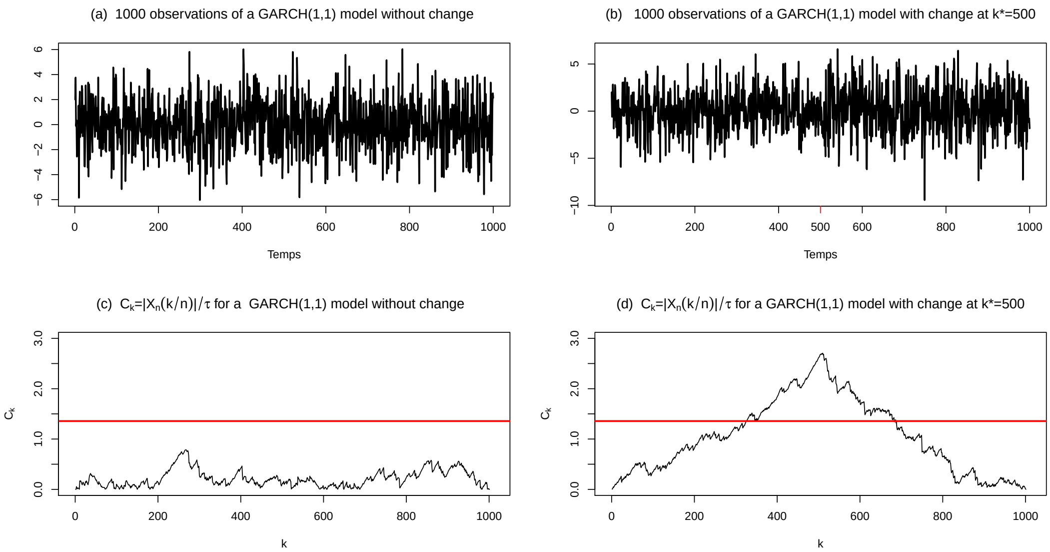

Fig. 1. Realisation of 1000 obsevations of a GARCH(1,1) process and the corresponding curve $C_k = \frac{1}{\hat{\tau}_n} |X_n(k/n)|$. **(a)** is a GARCH$(1,1)$ without change-point, where the unconditional variance $\sigma_0^2 = 4.255$. **(b)** is a GARCH$(1,1)$ with change-point where the unconditional variance $\sigma_0^2 = 4.255$ changes to $\sigma_1^2 = 6.666$ at $k^* = 500$. **(c)** and **(d)** are their corresponding curves C_k . The horizontal red line represents the limit of the critical region of the test.

when the unconditional variance is σ_0^2 (without change) and the empirical powers computed when when the unconditional variance σ_0^2 changes to σ_1^2 at $k^* = 0.3n, 0.5n, 0.7n$ where, $n = 500,\ 1000$ is the sample size. The results of the empirical levels and powers calculations are obtained after 200 replications. For the powers, we considered various data generating models named $S_i,\ i = 1, 2, 3, 4$ of GARCH (1,1).

S_1 : GARCH(1,1) where the unconditional variance remains constant ($\sigma^2 = 0.222$)

$$\sigma_0^2 = 0.222 \text{ for } \theta_0 = (0.100, 0.050, 0.500)$$
$$\sigma_1^2 = 0.222 \text{ for } \theta_1 = (0.150, 0.030, 0.295).$$

S_2 : GARCH(1,1) where the unconditional variance changes (small change) from $\sigma_0^2 = 0.222$ (for $\theta_0 = (0.10, 0.05, 0.50)$) to $\sigma_1^2 = 0.400$ (for $\theta_1 = (0.10, 0.05, 0.70)$).
S_3 : GARCH(1,1) where the unconditional variance changes (large change) from $\sigma_0^2 = 4.255$ (for $\theta_0 = (2.00, 0.03, 0.50)$) to $\sigma_1^2 = 2.127$ (for $\theta_1 = (1.00, 0.03, 0.50)$).
S_4 : GARCH(1,1) where the unconditional variance changes (large change) from $\sigma_0^2 = 4.255$ for ($\theta_0 = (2.00, 0.03, 0.50)$) to $\sigma_1^2 = 6.666$ (for $\theta_1 = (2.00, 0.2, 0.50)$).

It can be observed that empirical levels approach the nominal one when n increases and the power increases as n increases. The power of the test is close to the nominal level of

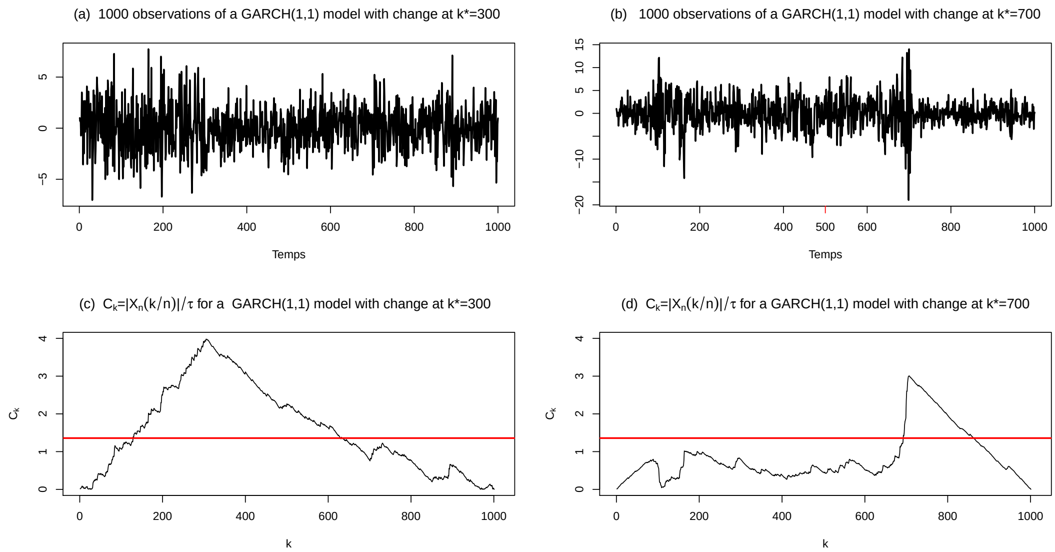

Fig. 2. Realisation of 1000 obsevations of a GARCH(1,1) with change-point and their corresponding curve $C_k = \frac{1}{\hat{\tau}_n} |X_n(k/n)|$. **(a)** curve with change-point where the unconditional variance $\sigma_0^2 = 10$ changes to $\sigma_1 = 3.333$ at $k^* = 300$. **(b)** curve with change-point where the unconditional variance $\sigma_0^2 = 10$ changes to $\sigma_1^2 = 3.333$ at $k^* = 700$. **(c)** and **(d)** are their corresponding curves C_k. The horizontal red line represents the limit of the critical region of the test.

the test when the unconditional variance remains unchanged even if the parameters of the model change. It can also be observed that our procedure is sensitive to a slight variation of the unconditional variance. Moreover, without surprise, if the change in unconditional variance is greater, then it is easier to detect it. The powers diminish compared to the case in which the change point is at the middle of the series. It can also be observed that when α_0 and β change, the test also achieves good powers. These results enable us to conclude that the test performs well.

In Table 2 we present the results of a simulations study intended to compare the performance of our change point-detection procedure to that of Lee *et al.* (2003) (method based on the Residual CUSUM test), even though their work does not rely directly on the detection of change in the unconditional variance of the model but rather on the detection of changes in the parameters of the model. The results of the empirical power calculations are obtained after 1000 replications. Our results are those in bold in the table. In regard of the results, our methode perform better than the one proposed by Lee *et al.* (2003).

Table 1. Empirical levels and powers at the nominal level 0.05 of test for unconditional variance changes of GARCH(1,1) model with one change-point.

				$n=500$	$n=1000$
Empirical levels :					
	$\sigma^2=0.222$			0.060	0.055
	$\sigma^2=4.255$			0.080	0.045
Empirical powers :					
	$\sigma_0^2=0.222;$	$\sigma_1^2=0.222;$	$k^*=0.3n$	0.065	0.060
			$k^*=0.5n$	0.055	0.050
			$k^*=0.7n$	0.045	0.055
	$\sigma_0^2=0.222;$	$\sigma_1^2=0.400;$	$k^*=0.3n$	0.940	0.980
			$k^*=0.5n$	0.975	1.000
			$k^*=0.7n$	0.95	0.995
	$\sigma_0^2=4.255;$	$\sigma_1^2=2.127;$	$k^*=0.3n$	1.000	1.000
			$k^*=0.5n$	1.000	1.000
			$k^*=0.7n$	0.980	1.000
	$\sigma_0^2=4.255;$	$\sigma_1^2=6.666;$	$k^*=0.3n$	0.700	0.950
			$k^*=0.5n$	0.855	0.990
			$k^*=0.7n$	0.735	0.905

Table 2. GARCH(1,1) process with abrupt change-point in the middle of the sample (small and large change in unconditional variance) ; the unconditional variance $\sigma_0^2 = 0.500$ (for $\theta_0 = (0.10, 0.40, 0.40)$) changes to $\sigma_1^2 = 0.200$ (for $\theta_1 = (0.10, 0.10, 0.40)$) ; the unconditional variance $\sigma_0^2 = 0.833$ (for $\theta_0 = (0.50, 0.20, 0.20)$) changes to $\sigma_1^2 = 2.500$ (for $\theta_1 = (0.50, 0.60, 0.20)$).

			$n=500$	$n=1000$
Empirical powers				
	$\sigma_0^2=0.500$;	$\sigma_1^2=0.200$	**0.974** (0.526)	**0.999** (0.928)
	$\sigma_0^2=0.833$;	$\sigma_1^2=2.500$	**0.953** (0.493)	**0.993** (0.901)

4.2. Real data analysis

In this subsection, we intend to demonstrate the validity of our method in actual practice. For this task, we analyse the daily returns of S&P 500 stock market from September 16, 1980 to January 31, 2008 which were also analysed recently by Kouamo *et al.* (2010). Recall that the $\tilde{T}^{k_1,k_2}_{J_1,J_2}(k)$ plot, defined in Kouamo *et al.* (2010), is a useful tool to detect multiple changes. In our case, we are only focused on a single and most significant change in the data. Our procecdure detects a change at the vertical blue line in Figure 3. It corresponds to a volatility change held in March 1997. Remark that change volatility coincides with the Asian Crisis in July 1997 which turns into an economic crisis. These results are in accordance with those obtained by Kouamo *et al.* (2010).

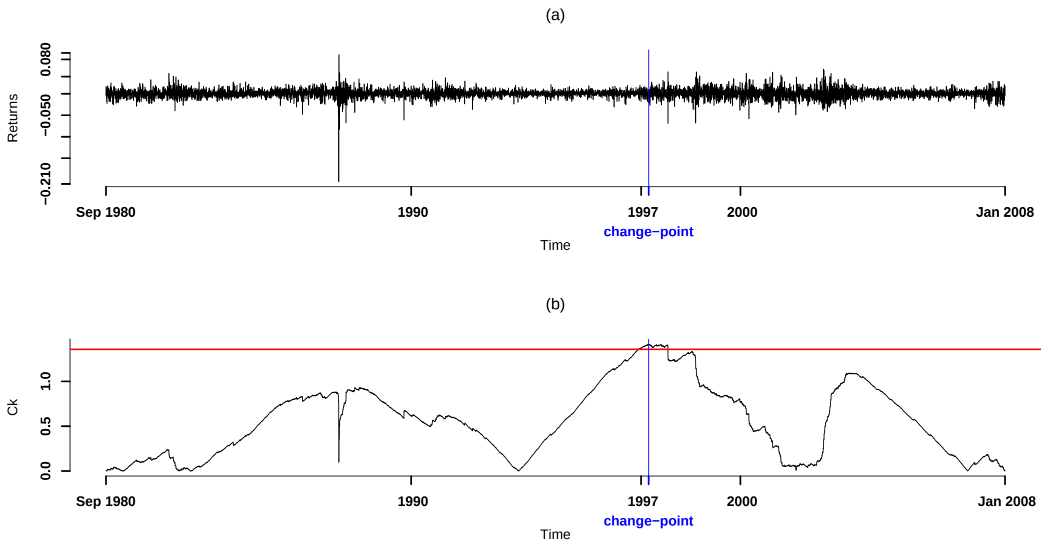

Fig. 3. **(a)** is a Standard and Poor daily return 09/16/1980-01/31/2008. **(b)** is the corresponding curve $C_k = \frac{1}{\hat{\tau}_n} |X_n(k/n)|$. The horizontal red line represents the limit of the critical region of the test. The vertical blue line represents the time where the change occurs.

5. Conclusion

In this article, we proposed a CUSUM-type test based on the square of GARCH to detect a change in the unconditional variance of a GARCH. Using a polynomial decomposition like Phillips and Solo, we showed that our test statistic converges to the supremum of a standard Brownian bridge. The results of the simulations enabled us to confirm the performance of our procedure. This method, applied to real data of type S&P 500 stock market returns (09/16/1980 to 01/31/2008), permit to detect a change in the data ; which can be interpreted as the financial crisis in Asia in March 1997. For future study, we will extend our procedure to detect multiple changes in GARCH models and achieve a CUSUM type test based on the absolute value of the GARCH to detect changes in the parameters of GARCH models.

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References

Bai, J. (1994). Weak convergence of the sequential empirical processes of residuals in ARMA models. *Ann. Statis.*, 22, 2051-2061

Berkes, I., Gombay, E., Horváth, L. and Kokoszka, P. (2004). Sequential change-point detection in GARCH(p,q) models. *Econometric Theory*, 20, 1140-1167.

Berkes, I., Gombay, E. and Horváth, L. (2009). Testing for changes in the covariance structure of linear processes. *Journal of Statistical Planing and Inference*, 139, 2044-2063.

Billingsley, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

Boldin, M. V. (2002). Sequential residue empirical processes in the ARCH model. *in Communication of the Moscow Mathematical Society.*, pp, 413-414.

Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroskedasticitydel. *Journal of Econometrics.*, 31, 307-327.

Chalmond, B. (1981). Détection d'une rupture de moyenne en un point inconnu d'un processus ARMA. *Revue de statistique appliquée.*, 29, 7-20.

Chen, J. et Gupta, A.K. (2011). *Parametric Statistical Change Point Analysis With Applications to Genetics, Medicine, and Finance.* Birkhäuser Verlag, Basel.

Csörgő, M. et Horváth, L. (1997). *Limit Theorems in Change–Point Analysis.* Wiley.

Engle, R. (1982). Autoregressive Conditional Heteroskedasticity with Estimate of the Variance U.K. Inflation. *Econometrica*, 20, 987-1008.

Francq, C. and Zakoïan,J.-M. (2010). *GARCH Models. Structure, Statistical Inference and Financial Applications.* Wiley.

Giraitis, L., Kokoszka, P.S., Leipus, R., Teyssière, G. (2003). Rescaled variance and related tests for long memory in volatility and levels. *J. Econometrics*, 112, 265-294.

Gouriéroux, C. (1997). *ARCH Models and Financial Applications.* Springer, New York.

Hall, P. and Heyde, C. C. (1980). *Martingale Limit Theory and Its Application.* Academic, New York.

Horváth, L., B. L., Kokoszka, P., and Teyssière, G. (2001). Empirical processes of the squared residuals of an ARCH sequence. *The Annals of Statistics*, 29, 445-469.

Inclán, C. and Tiao, G. C. (1994). Use of cumulative sums of squares for retrospective detection of changes of variances. *Stoch. Proc. Appl.*, 89, 913-923.

Kokoszka, P. and Leipus, R. (1999). Testing for parameter changes in ARCH. *Lietuvos Matematikos Rinkinys.*, 39, 231–247.

Kokoszka, P. and Teyssière, G. (2002). Change-point detection in GARCH models: asymptotic and booststrap tests. *Biometrika.*, 42, 523-526.

Kouamo, O., Moulines, V. and Rouef, E. (2010). Test for homogeneity of variance in wavelet domain. *Lecture Notes in Statistics, Springer Verlag.*, 175-205

Lee, S., Tokutsu, Y. and Maekawa, K. (2003). The residual cusum test for the constancy of parameters in GARCH (1,1) models. *Working papers at Hiroshima University.*

Lo, G.S., Kpanzou T.A., Ngom M. (2016). Weak Convergence (IA). Sequences of random vectors. SPAS Books Series.(2016). Doi : 10.16929/sbs/2016.0001. *Arxiv : 1610.05415.* ISBN : 978-2-9559183-1-9

Lo, G.S. (2014). A remark on the asymptotic tightness in $\ell^\infty([a,b])$. *Arxiv : 405.6342.* Doi : 10.16929/jmfsp/2014.01

Page, E. S. (1955). A test for a change in a parameter occurring at an unknown point. *Biometrika*, 42, 523-526.

Phillips, C.B. and Solo, V. (1992). Asymptotics for linear processes. *The annals of statistics.*, 20, 971-1001.

van der Vaart A. W. and Wellner J. A.(1996). *Weak Convergence and Empirical Processes With Applications to Statistics.* Springer, New-York.

Yule, G. U. (1927). On a Method of Investigating Periodicities in Disturbed Series with Special Reference to Wolfer's Sunspot Numbers. *Philos. Trans. Royal Society, London, Serie A*, 226, 267-298