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LINEAR MAPS BETWEEN OPERATOR ALGEBRAS PRESERVING CERTAIN SPECTRAL FUNCTIONS

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Abstract. Let H be an infinite dimensional complex Hilbert space and let ϕ be a surjective linear map on $B(H)$ with $\phi(I)-I \in \mathcal{K}(H)$, where $\mathcal{K}(H)$ denotes the closed ideal of all compact operators on H. If ϕ preserves the set of upper semi-Weyl operators and the set of all normal eigenvalues in both directions, then ϕ is an automorphism of the algebra $B(H)$. Also the relation between the linear maps preserving the set of upper semi-Weyl operators and the linear maps preserving the set of left invertible operators is considered.

1. Introduction and preliminaries

Let H be an infinite dimensional complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H and $\mathcal{K}(H) \subseteq B(H)$ be the closed ideal of all compact operators. We write T^* for the conjugate operator of $T \in B(H)$. An operator $T \in B(H)$ is called upper semi-Fredholm if it has closed range $R(T)$ with finite dimensional null space $N(T)$ and if $R(T)$ has finite co-dimension, $T \in B(H)$ is called a lower semi-Fredholm operator. We call $T \in B(H)$ Fredholm if it has closed range with finite dimensional null space and its range of finite co-dimension. For a semi-Fredholm operator $T \in B(H)$ (upper semi-Fredholm operator or lower semi-Fredholm operator), let $n(T) = dimN(T)$ and $d(T) = dimH/R(T) = codimR(T)$. The index of a semi-Fredholm operator $T \in B(H)$ is given by $ind(T) = n(T) - d(T)$. The operator T is Weyl if it is Fredholm of index

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zero; T is called Browder if T is Fredholm with finite ascent and finite descent; $T \in B(H)$ is called upper semi-Weyl if T is upper semi-Fredholm with $ind(T) \leq 0$. Let $SF_+^-(H)$ denote the set of all upper semi-Weyl operators and let $\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+^-(H)\}$ be the essential approximate point spectrum of T. $\sigma(T)$, $\sigma_e(T)$, $\sigma_{SF_+}(T)$, $\sigma_{SF_-}(T)$, $\sigma_w(T)$ and $\sigma_b(T)$ denote the spectrum, the essential spectrum, the upper semi-Fredholm spectrum, the lower semi-Fredholm spectrum, the Weyl spectrum and the Browder spectrum respectively ([8, 9]). Let $\sigma_0(T) = \sigma(T) \backslash \sigma_b(T)$ denote the set of all normal eigenvalues.

Let $\Phi(H) \subseteq B(H)$ be the set of all Fredholm operators. We denote the Calkin algebra $B(H)/\mathcal{K}(H)$ by $\mathcal{C}(H)$. Let $\pi : B(H) \to \mathcal{C}(H)$ be the quotient map. A bijective linear map $\phi : B(H) \to B(H)$ is called a Jordan isomorphism if $\phi(A^2) = (\phi(A))^2$ for every $A \in B(H)$, or equivalently $\phi(AB+BA) = \phi(A)\phi(B) + \phi(B)\phi(A)$ for all A and B in $B(H)$. It is obvious that every isomorphism and every anti-isomorphism is a Jordan isomorphism. For further properties of Jordan homomorphisms, we refer the reader to [10] and [11].

In the last two decades there has been considerable interest in the so-called linear preserver problems (see [1, 5, 16]). The goal of studying linear preservers is to give structural characterizations of linear maps on algebras having some special properties such as leaving invariant a certain subset of the algebra, or leaving invariant a certain function on the algebra. One of the most famous problem in this direction is Kaplansky's problem([13]): Let ϕ be a surjective linear map between two semi-simple Banach algebras $\mathcal{A}$ and $\mathcal{B}$. Suppose that $\sigma(\phi(x)) = \sigma(x)$ for all $x \in \mathcal{A}$. Is it true that ϕ is Jordan isomorphism? This problem was first solved in the finite dimensional case. J.Dieudonnĕ ([7]) and Marcus and Purves ([15]) proved that every unital invertibility preserving linear map on a complex matrix algebra is either an inner automorphism or a linear anti-automorphism. This result was later extended to the algebra of all bounded linear operators on a Banach space by A.R.Sourour([22]) and to von Neumann algebra by B.Aupetit([1]). Many other linear preserver problems have been extended to the infinite dimensional case. For the most significant partial obtained in this direction, we refer the reader to ([1, 18, 22, 23]). New contributions to the study of linear preserver problem in $B(H)$ have been recently made by Mbekhta in [17], Mbekhta, Rodman and Šemrl in [18], Mbekhta and Šemrl in [16] and Bendaoud, Bourhim and Sarih in [4].

In this article, we give the characterization of automorphism on $B(H)$. We get that: Let ϕ be a surjective linear maps on $B(H)$ with $\phi(I) - I \in \mathcal{K}(H)$ preserving the set of upper semi-Weyl operators and the set of all normal eigenvalues in both directions, then ϕ is an automorphism of the algebra $B(H)$. Also the relation between the linear maps preserving the set of upper semi-Weyl operators and the linear maps preserving the set of left invertible operators is considered.

2. Main results

An operator is left invertible if it has a left inverse. It turns out that an operator $T \in B(H)$ is left invertible if and only if it is bounded below, or equivalently, it is upper semi-Fredholm with $n(T) = 0$. Let $\sigma_a(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not left invertible}. We say that a linear map $\phi : B(H) \to B(H)$ preserves the set of upper semi-Weyl operators (left invertible operators) in both directions if $T \in SF_+^-(H)$ (T is left invertible)$\Leftrightarrow \phi(T) \in SF_+^-(H)$ ($\phi(T)$ is left invertible).

A linear map $\phi : B(H) \to B(H)$ is said to be surjective up to compact operators if for every $T \in B(H)$ there exists $T' \in B(H)$ such that $T - \phi(T') \in \mathcal{K}(H)$. It is clear that if ϕ is surjective, then it is surjective up to compact operators.

Remark 2.1. (1) If a linear map $\phi : B(H) \to B(H)$ preserves the set of upper semi-Weyl operators in both directions, we can not induce that ϕ preserves the set of left invertible operators in both directions. For example, let $A, B \in B(\ell_2)$ be defined by:

$$A(x_1, x_2, x_3, \cdots) = (x_2, x_3, \cdots),$$
$$B(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots),$$

and let $\phi(T) = ATB$, $T \in B(\ell_2)$. We can see that both A and B are Fredholm operators, and $ind(A) + ind(B) = 0$. By the properties of the index it follows that $T \in SF_+^-(B(\ell_2))$ if and only if $\phi(T) \in SF_+^-(B(\ell_2))$. For any $T \in B(\ell_2)$, let $T_1 = BTA$, then $\phi(T_1) = T$. Thus $\phi : B(\ell_2) \to B(\ell_2)$ is surjective and ϕ preserves the set of upper semi-Weyl operators in both directions. But ϕ does not preserve the set of left invertible operators in both directions. In fact, for an operator $T \in B(\ell_2)$ defined by:

$$T(x_1, x_2, x_3, \cdots) = (x_2 - x_1, x_2 - x_1, x_3, x_4 \cdots),$$

we can find that $\phi(T) = I$ is left invertible but T is not left invertible.

(2) If a linear map $\phi : B(H) \to B(H)$ preserves the set of left invertible operators in both directions, we can not induce that ϕ preserves the set of upper semi-Weyl operators in both directions. For example, let $A \in B(\ell_2)$ be defined by:

$$A(x_1, x_2, x_3, \cdots) = (0, 0, x_1, x_2, \cdots),$$

$B \in B(\ell_2)$ is invertible and let $\phi(T) = ATB$, $T \in B(\ell_2)$. We can see that A is left invertible, there exists $A_1 \in B(\ell_2)$ such that $A_1A = I$. Since $A \in B(\ell_2)$ is Fredholm, there are $A_2 \in B(\ell_2)$ and a compact operator K_0 satisfying $AA_2 = I + K_0$. For any $T \in B(\ell_2)$, let $T_0 = A_2TB^{-1}$ and $K = -K_0T$. Then K is compact and $T = \phi(T_0) + K$, which means that ϕ is surjective up to compact operators. For any left invertible operator $T \in B(\ell_2)$, suppose that $T_1T = I$. Then $B^{-1}T_1A_1\phi(T) = I$, this shows that $\phi(T)$ is left invertible. For the converse, if $\phi(T)$ is left invertible and suppose $D\phi(T) = I$. Then $BDAT = BDATBB^{-1} = BD\phi(T)B^{-1} = BB^{-1} = I$, thus $T \in B(\ell_2)$ is left invertible. It follows that ϕ preserves the set of left invertible operators in both directions. But ϕ does not preserve the set of upper semi-Weyl operators in both directions. In fact, let $T \in B(\ell_2)$ be defined as $T(x_1, x_2, x_3, \cdots) = (x_2, x_3, \cdots)$, then $\phi(T)$ is upper semi-Weyl with $ind(\phi(T)) = ind(A) + ind(T) + ind(B) = -2 + 1 + 0 = -2$ but T is not upper semi-Weyl.

It is well known that the set of left invertible operators is a subset of $SF_+^-(H)$, we need to study the relation between the linear maps preserving the set of upper semi-Weyl operators and the linear maps preserving the set of left invertible operators. Let's begin with a Theorem.

Theorem 2.2. *Let $\phi : B(H) \to B(H)$ be a surjective linear map preserving upper semi-Weyl operators in both directions and $\phi(I) - I \in \mathcal{K}(H)$. If $\sigma_0(K) = \sigma_0(\phi(K))$ for any Riesz operator K, then there is an invertible linear operator $A \in B(H)$ such that $\phi(T) = ATA^{-1}$ for any $T \in B(H)$.*

Proof. We will prove the Theorem by seven steps:

(i) For any $T \in B(H)$, $\sigma_{ea}(T) = \sigma_{ea}(\phi(T))$.

Let $\phi(I) = I + K$, where $K \in \mathcal{K}(H)$. Since $T - \lambda I \in SF_+^-(H) \Leftrightarrow \phi(T - \lambda I) = \phi(T) - \lambda\phi(I) = \phi(T) - \lambda I - \lambda K \in SF_+^-(H) \Leftrightarrow \phi(T) - \lambda I \in SF_+^-(H)$, it follows that $\sigma_{ea}(T) = \sigma_{ea}(\phi(T))$ for any $T \in B(H)$.

(ii) ϕ preserves compact operators in both directions.

First we claim that

$$\mathcal{K}(H) = \{K \in B(H) : K + SF_+^-(H) \in SF_+^-(H)\}$$

$$= \{K \in B(H) : \sigma_{ea}(T + K) = \sigma_{ea}(T) \text{ for all } T \in B(H)\}.$$

From the stability properties of index function, it is clear that $\mathcal{K}(H) \subseteq \{K \in B(H) : K + SF_+^-(H) \in SF_+^-(H)\} = \{K \in B(H) : \sigma_{ea}(T+K) = \sigma_{ea}(T) \text{ for all } T \in B(H)\}$.

Let ∂E and ηE denote the boundary and the polynomial convex hull of a compact subset E of $\mathbb{C}$ respectively. For any $T \in B(H)$, since

$$\partial\sigma_w(T) \subseteq \partial\sigma_e(T) \subseteq \sigma_e(T) \subseteq \sigma_w(T) \text{ and } \partial\sigma_w(T) \subseteq \partial\sigma_{ea}(T) \subseteq \sigma_{ea}(T) \subseteq \sigma_w(T),$$

it follows that $\eta\sigma_{ea}(T) = \eta\sigma_w(T) = \eta\sigma_e(T)$.

Now, let $K \in B(H)$ such that $\sigma_{ea}(T+K) = \sigma_{ea}(T)$ for all $T \in B(H)$. Then by Theorem 5.3.1 in [2], $\eta\sigma_e(T+K) = \eta\sigma_e(T)$ for all $T \in B(H)$. Taking into account the semisimplicity of $\mathcal{C}(H)$ and the spectral characterization of the radical, it is not difficult to prove that the $\mathcal{K}(H) = \{K \in B(H) : K + SF_+^-(H) \in SF_+^-(H)\} = \{K \in B(H) : \sigma_{ea}(T + K) = \sigma_{ea}(T) \text{ for all } T \in B(H)\}$.

Let $K \in \mathcal{K}(H)$, for any $T \in SF_+^-(H)$, since ϕ preserves upper semi-Weyl operators in both directions, there exists $T' \in SF_+^-(H)$ for which $T = \phi(T')$. Hence $T + \phi(K) = \phi(T') + \phi(K) = \phi(T' + K) \in SF_+^-(H)$. Then $\phi(K) \in \mathcal{K}(H)$. For the converse, let $\phi(K) \in \mathcal{K}(H)$, for any $T \in SF_+^-(H)$, $\phi(T + K) = \phi(T) + \phi(K) \in SF_+^-(H)$, then $T + K \in SF_+^-(H)$. It follows that $K \in \mathcal{K}(H)$. Now we prove that ϕ preserves compact operators in both directions.

Since ϕ preserves compact operators in both directions, it follows that $\sigma(K) = \{0\} \cup \sigma_0(K) = \{0\} \cup \sigma_0(\phi(K)) = \sigma(\phi(K))$ for any compact operator K.

(iii) $N(\phi) \subseteq \mathcal{K}(H)$.

If $K \in N(\phi)$ and $T \in SF_+^-(H)$, then $\phi(T + K) = \phi(T) \in SF_+^-(H)$. Thus for all $T \in SF_+^-(H)$, $T + K \in SF_+^-(H)$. Thus $K \in \mathcal{K}(H)$.

(iv) Let $\varphi : \mathcal{C}(H) \to \mathcal{C}(H)$ be an induced linear map such that $\phi \circ \pi = \pi \circ \phi$, then φ is isomorphism.

ϕ induces a linear map $\varphi : \mathcal{C}(H) \to \mathcal{C}(H)$ such that $\varphi \circ \pi = \pi \circ \phi$. Clearly, φ is surjective since ϕ is surjective. By hypothesis and (ii), φ is $\eta\sigma$-preserving. From Corollary 2.3 in [5], φ is injective, and by Theorem 3.1 in [5], φ is either a homomorphism or an anti-homomorphism.

First we will prove that ϕ preserves upper semi-Fredholm operators in both directions. By Theorem 2.1 in [17], we know that ϕ preserves Fredholm operators in both directions. Let $T \in B(H)$ be an upper semi-Fredholm, there are two cases to consider: $d(T) = \infty$ and $d(T) < \infty$. If $d(T) = \infty$, using the fact that ϕ is a linear map preserving upper semi-Weyl operators in both directions, we know that $\phi(T)$ is upper semi-Fredholm. If $d(T) < \infty$, then T is Fredholm, thus $\phi(T)$ is Fredholm since ϕ preserves Fredholm operators in both directions. Using the same way, we can prove that T is upper semi-Fredholm if $\phi(T)$ is upper semi-Fredholm. By Corollary 3.6 in [3], φ is an isomorphism.

As ϕ preserves the essential spectrum, from Theorem 3.3 in [17] we deduce that $ind(\phi(T)) = ind(T)$ or $ind(\phi(T)) = -ind(T)$ for every Fredholm operator $T \in B(H)$. Since ϕ preserves upper semi-Weyl operators in both directions, it follows that $ind(\phi(T)) \cdot ind(T) \geq 0$ for any $T \in \Phi(H)$. Thus $ind(\phi(T)) = ind(T)$ for any $T \in \Phi(H)$. Also we can prove that $ind(\phi(T)) = ind(T)$ for any upper semi-Fredholm operator $T \in B(H)$. For lower semi-Fredholm operator $T \in B(H)$, we also have $ind(\phi(T)) = ind(T)$. In fact, since φ is an isomorphism, by Corollary 3.6 in [3], ϕ preserves lower semi-Fredholm operators in both directions. Let $T \in B(H)$ be a lower semi-Fredholm operator, then $\phi(T)$ is a lower semi-Fredholm operator. There are also two cases to consider: $n(T) = \infty$ and $n(T) < \infty$. If $n(T) = \infty$, using the fact that ϕ is a linear map preserving Fredholm operators in both directions, we know that $n(\phi(T)) = \infty$, then $ind(\phi(T)) = ind(T) = \infty$. If $n(T) < \infty$, then T is Fredholm, thus $\phi(T)$ is Fredholm since ϕ preserves Fredholm operators in both directions. Then $ind(\phi(T)) = ind(T)$ again.

(v) ϕ is injective.

If $\phi(T) = 0$, then T is compact and hence $\sigma(T) = \{0\} \cup \sigma_0(T) = \{0\} \cup \sigma_0(\phi(T)) = \{0\}$ since $\sigma_0(\phi(T)) = \emptyset$. This means that T is quasinipotent. Assume that $T \neq 0$, we can find $x \in H$ such that $Tx = y \neq 0$. Clearly, x and y are linear independent. Define a nilpotent operator $N \in B(H)$ by:

$$Nx = x - y,\ Ny = x - y,\ Nz = 0,\ \text{for } z \in \{x, y\}^{\perp}.$$

Then both N and $N+T$ are compact, thus $\phi(N+T) = \phi(N)$ is compact. From the condition we can find $\sigma(T+N) = \sigma(\phi(T+N))$, then $\sigma(T+N) = \sigma(\phi(T+N)) = \sigma(\phi(N)) = \sigma(N) = \{0\}$, which means that $T + N$ is quasinilpotent. This is in contraction to the fact that $1 \in \sigma(T + N)$.

(vi) $\phi(T)$ is an idempotent of rank one if and only if T is an idempotent of rank one.

Let $P \in B(H)$ be an idempotent of rank one and let $\phi(P) = Q$. Since both P and Q are compact operators, $\sigma(Q) = \sigma(P) = \{0, 1\}$. For any $K \in F_2(H)$, where $F_2(H)$ denotes the set of all operators in $B(H)$ with rank not greater than 2, there is $S \in B(H)$ such that $K = \phi(S)$ as ϕ is surjective. Thus by Theorem 1 in [12] we must have that $\sigma(S+P) \cap \sigma(S+2P) \subseteq \sigma(S)$. Since $S+P$, $S+2P$ and S are all compact operators, it follows that $\sigma(S+P) = \sigma(\phi(S+P)) = \sigma(K+Q)$, $\sigma(S+2P) = \sigma(\phi(S+2P)) = \sigma(K+2Q)$ and $\sigma(S) = \sigma(\phi(S)) = \sigma(K)$. Then $\sigma(K+Q) \cap \sigma(K+2Q) \subseteq \sigma(K)$. By Lemma 2.2 in [6], we know that $rankQ = 1$. This implies that Q satisfies a quadratic polynomial equation $p(Q) = 0$ ([14]).

Using the fact that $\sigma(Q)=\{0,1\}$, we know that p is of the form $p(\lambda)=\lambda(\lambda-1)$. Then $Q^2=Q$.

We get that ϕ preserves idempotent of rank one. The same must be true for ϕ^{-1}, and consequently, ϕ preserves idempotents of rank one in both directions. According to Proposition 2.6 in [19] there exists either an invertible $A\in B(H)$ such that $\phi(T)=ATA^{-1}$ for all finite rank operators $T\in B(H)$, or a bounded invertible conjugate-linear operator C on H such that $\phi(T)=CT^*C^{-1}$ for every $T\in B(H)$ of finite rank.

(vii) There is an invertible linear operator $A\in B(H)$ such that $\phi(T)=ATA^{-1}$ for any $T\in B(H)$.

Let $T\in B(H)$ such that $T^2=0$. Then $\sigma(T)=\{0\}$ and $\sigma_0(T)=\emptyset$. Since $T-\lambda I$ is Weyl for any $\lambda\neq 0$ and ϕ is a linear map preserving upper semi-Weyl operators in both directions, it follows that $\phi(T)-\lambda I$ is Weyl for any $\lambda\neq 0$. This implies that $\phi(T)$ is a Riesz operator. For every operator U of rank one, we know that both $T+U$ and $\phi(T)+\phi(U)$ are Riesz operators. Then $\sigma(T+U)=\sigma(\phi(T)+\phi(U))$. By assuming that $\phi(U)=AUA^{-1}$, this can be rewritten as $\sigma(T+U)=\sigma(A^{-1}\phi(T)A+U)$ for each rank one operator U. This gives directly that $T=A^{-1}\phi(T)A$, and hence $\phi(T)=ATA^{-1}$. Then $\phi(T)=ATA^{-1}$ for every $T\in B(H)$ by Theorem 2 in [20].

In the second case we show that similarly that $\phi(T)=CT^*C^{-1}$ for all $T\in B(H)$. It follows from that $ind(T)=ind(\phi(T))$ if T is Fredholm, we know that the second case cannot occur. The proof of the Theorem is complete. □

In the proof of Theorem 2.2, we use P.Šemrl's method in Theorem 4 in [21], but there are many differences in two proofs.

Similar to the proof of Lemma 1 in [12], we can get that: Let $A\in B(H)$. If $\sigma_a(T+A)\subseteq\sigma_a(T)$ for every rank one operator T, then $A=0$.

For surjective linear map $\phi: B(H)\to B(H)$, if $\sigma_a(T)\subseteq\sigma_a(\phi(T))$ for any $T\in B(H)$ and $\sigma_a(T)=\sigma_a(\phi(T))$ for any Riesz operator T, then $\phi(I)=I$. In fact, suppose that $\phi(S)=I$. For any rank one operator F, since $\sigma_a(F+S-I)=\sigma_a(F+S)-1\subseteq\sigma_a(\phi(F)+\phi(S))-1=\sigma_a(\phi(F)+I)-1=\sigma_a(\phi(F))=\sigma_a(F)$, we know that $S-I=0$, then $S=I$, which means that $\phi(I)=I$. In the proof of Theorem 2.2, we can see that if ϕ preserves Riesz operators in both directions and if $\sigma_0(T)=\sigma_0(\phi(T))$ for any Riesz operator T, then there exists either an invertible $A\in B(H)$ such that $\phi(T)=ATA^{-1}$ for every $T\in B(H)$, or a bounded invertible conjugate-linear operator C on H such that $\phi(T)=CT^*C^{-1}$ for every $T\in B(H)$.

Corollary 2.3. *Let $\phi: B(H)\to B(H)$ be a surjective linear map preserving upper semi-Weyl operators in both directions. If $\sigma_a(T)\subseteq\sigma_a(\phi(T))$ for any $T\in B(H)$ and $\sigma_a(T)=\sigma_a(\phi(T))$ for any Riesz operator T, then there is an invertible linear operator $A\in B(H)$ such that $\phi(T)=ATA^{-1}$ for any $T\in B(H)$.*

Proof. Since $\phi(I)=I$ and $\phi: B(H)\to B(H)$ preserves upper semi-Weyl operators in both directions, we can prove that ϕ preserves Riesz operators in both directions. Then $\sigma(T)=\sigma_a(T)=\sigma_a(\phi(T))=\sigma(\phi(T))$ for any Riesz operator T.

Thus $\sigma_0(T) = \sigma_0(\phi(T))$ for any Riesz operator T. By Theorem 2.2, the result is true. □

Corollary 2.4. *Let $\phi : B(H) \to B(H)$ be a surjective linear map. If $\phi(I) - I \in \mathcal{K}(H)$ and $\sigma_0(T) = \sigma_0(\phi(T))$ for any Riesz operator $T \in B(H)$, then the following statements are equivalent:*

(1) $\sigma_a(T) = \sigma_a(\phi(T))$ *for any* $T \in B(H)$*;*
(2) $\sigma_{ea}(T) = \sigma_{ea}(\phi(T))$ *for any* $T \in B(H)$*;*
(3) $\sigma_e(T) = \sigma_e(\phi(T))$ *and* $ind(T) = ind(\phi(T))$ *if* T *is a Fredholm operator;*
(4) $\sigma_{SF_+}(T) = \sigma_{SF_+}(\phi(T))$ *and* $ind(T) = ind(\phi(T))$ *if* T *is an upper semi-Fredholm operator;*
(5) $\sigma_{SF_-}(T) = \sigma_{SF_-}(\phi(T))$ *and* $ind(T) = ind(\phi(T))$ *if* T *is a lower semi-Fredholm operator;*
(6) *There exists an invertible operator* $A \in B(H)$ *such that* $\phi(T) = ATA^{-1}$ *for every* $T \in B(H)$.

Proof. It follows from Theorem 2.2, Theorem 2.1 in [17], Theorem 4.8 in [3] and Corollary 3.6 in [3], that (2), (3), (4), (5) and (6) are equivalent. The implication (6) ⇒ (1) is clear, and the converse can be argued as in Theorem 4 in [21]. □

From the proof of Theorem 4 in [21], we know that if $\phi : B(H) \to B(H)$ be a surjective linear map and $\sigma_a(T) = \sigma_a(\phi(T))$ for any $T \in B(H)$, then (2), (3), (4) and (5) in Corollary 2.4 are true.

Remark 2.5. In Corollary 2.4, the condition "$\sigma_0(T) = \sigma_0(\phi(T))$ for any Riesz operator $T \in B(H)$" is essential. For example, let $A, B \in B(\ell_2)$ and ϕ be defined as in (1) in Remark 2.1. Then $\phi : B(H) \to B(H)$ is a surjective linear map preserving upper semi-Weyl operators in both directions and $\phi(I) = I$, which means that $\sigma_{ea}(T) = \sigma_{ea}(\phi(T))$ for any $T \in B(H)$ (from the proof of Theorem 2.2). Let $T_0 = BA$, then $T_0(x_1, x_2, x_3, \cdots) = (0, x_2, x_3, x_4, \cdots)$ and $\phi(T_0) = I$. Since $T_0 = T_0^2$ and $\phi(T_0)$ is invertible, we can see that $0 \in \sigma_0(T_0)$ but $0 \notin \sigma_0(\phi(T_0))$. Then we can not induce that ϕ preserves the set of left invertible operators in both directions from (1) in Remark 2.1.

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