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HILBERT-SCHMIDT DIFFERENCES OF COMPOSITION OPERATORS BETWEEN THE WEIGHTED BERGMAN SPACES ON THE UNIT BALL

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Abstract. Let φ, ψ be the analytic self-maps of the unit ball $\mathbb{B}$, we characterize the Hilbert-Schmidt differences of two composition operator C_φ and C_ψ on weighted Bergman space A^2_α, and give some conclusions about the topological structure of $\mathcal{C}(A^2_\alpha)$, the space of all bounded composition operators on A^2_α endowed with operator norm.

1. Introduction

Let $\mathbb{B}$ be the unit ball in the N-dimensional complex space $\mathbb{C}^N$, with $\mathbb{D}$ for the unit disk of complex plane $\mathbb{C}$, $S(\mathbb{B})$ the collection of all holomorphic self-maps of $\mathbb{B}$ and let $H(\mathbb{B})$ be the space of all holomorphic functions on $\mathbb{B}$. Some function spaces, for instance, bounded mean oscillation class (BMO), vanishing mean oscillation class (VMO), Bergman space, Bloch space or other recent spaces, are treated by many authors (see e.g.[1, 16, 25, 26]). The inner product of $\mathbb{C}^N$ defined as

$$\langle z, w\rangle = \sum_{k=1}^{N} z_k \overline{w}_k,$$

where $z = (z_1, \cdots, z_N) \in \mathbb{C}^N$ and $w = (w_1, \cdots, w_N) \in \mathbb{C}^N$.

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For $\alpha > -1$, the weighted Bergman space $A^2_\alpha = A^2_\alpha(\mathbb{B})$ consists of holomorphic functions f on $\mathbb{B}$ satisfying

$$\|f\|^2_\alpha = \int_{\mathbb{B}} |f(z)|^2 d\nu_\alpha(z),$$

where

$$d\nu_\alpha(z) = \frac{\Gamma(N+\alpha+1)}{\Gamma(N+1)\Gamma(\alpha+1)}(1-|z|^2)^\alpha d\nu(z),$$

$d\nu$ denotes the normalized Lebesgue volume measure on $\mathbb{B}$ and Γ the usual Euler function, extension of the factorial function.

The Hardy space $H^2 = H^2(\mathbb{B})$ is the set of functions analytic on $\mathbb{B}$ such that

$$\|f\|^2_{H^2} = \sup_{0<r<1} \int_{\mathbb{S}} |f(r\zeta)|^2 d\sigma(\zeta) < \infty,$$

where $\mathbb{S}$ is the unit sphere and $d\sigma$ is the normalized measure on $\mathbb{S}$.

Let $\varphi \in S(\mathbb{B})$, the composition operator C_φ defined by $C_\varphi f = f \circ \varphi$. When $N = 1$, the Littlewood Subordination Theorem shows that C_φ is bounded on $A^2_\alpha(\mathbb{D})$ for any analytic self-map φ of $\mathbb{D}$, and many other properties of C_φ have been characterized, see, e.g. [3, 11, 13, 18, 25]. However, for $N > 1$, it is no longer the case that every composition operator is bounded on the weighted Bergman space of the ball (see Section 3.5 in [3]). We know that if C_φ maps A^2_α into A^2_α, then C_φ is a bounded operator by the closed graph theorem. So in this paper, for $\varphi, \psi \in S(\mathbb{B})$, we always suppose C_φ and C_ψ map A^2_α into A^2_α. The mapping properties of the differences of two composition operators, i.e. an operator of the form

$$T = C_\varphi - C_\psi$$

have also been studied. For related papers on the disk see [10, 12, 14, 15, 19, 23, 24], and on the unit ball [6, 7, 21, 22]. For the research of Hilbert-Schmidt operator we can see [2, 3, 4, 5, 17, 20]. The authors [2] studied the Hilbert-Schmidt differences on the weighted Bergman space $A^2_\alpha(\mathbb{D})$, the present paper continues this line of research, and characterizes the Hilbert-Schmidt differences on the unit ball. The paper is organized as follows: Section 3 is devoted to characterizing the conditions about Hilbert-Schmidt differences. Some conclusions about topological structure are given in section 4.

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next. The notation $a \preceq b$ means that there is a positive constant C such that $a \leq Cb$. We say $a \asymp b$, if both $a \preceq b$ and $b \preceq a$ hold.

2. Prerequisites

In this section, we will give some notations and well-known lemmas.

2.1. Weighted Bergman space and Hilbert-Schmidt operator.

Given $\alpha > -1$, the space A^2_α is a Hilbert space with inner product

$$\langle f, g\rangle_\alpha = \int_{\mathbb{B}} f\bar{g}d\nu_\alpha$$

for $f, g \in A^2_\alpha$. The reproducing kernel for the bounded linear functional of evaluation at $w \in \mathbb{B}$ in A^2_α is

$$K_w(z) = \frac{1}{(1 - \langle z, w\rangle)^{N+1+\alpha}}$$

such that

$$\langle f, K_w\rangle_\alpha = f(w),$$

and it has norm $(1 - |w|^2)^{-(N+1+\alpha)}$. We also have

$$K_w(z) = \sum_n e_n(z)\overline{e_n(w)}, \quad z, w \in \mathbb{B}$$

for any choice of an orthonormal basis $\{e_n\}$ for A^2_α.

Let T be the linear operator from Banach space X to Banch space Y, the operator norm define as follows:

$$\|T\| = \sup_{\|f\|_X=1} \|Tf\|_Y,$$

the notions $\|\cdot\|_X$ and $\|\cdot\|_Y$ denote the norm of X and Y, respectively.

A linear operator T on a separable Hilbert space H is Hilbert-Schmidt if

$$\|T\|^2_{HS} = \sum_{k=1}^{\infty} \|Te_k\|^2_H = \sum_{k,m=1}^{\infty} |\langle Te_k, e_m\rangle_H|^2 < \infty \tag{2.1}$$

for any (or some) orthonormal basis $\{e_k\}$ of H, $\|\cdot\|_H$ ($\langle\cdot,\cdot\rangle_H$) is the norm (respectively, inner product) of H. For an arbitrary linear operator T on H the (possibly infinite) sum on the right of (2.1) does not depend on the particular choice of $\{e_n\}$, and $\|T\| \le \|T\|_{HS}$. We know that if T is Hilbert-Schmidt operator, then T is compact operator.

Let $m = (m_1, \cdots, m_N)$ be a multi-index, since the function sequence $\{\frac{z^m}{\|z^m\|_\alpha}\}$ is an orthonormal basis of A^2_α, we have

$$\|C_\varphi\|^2_{HS} = \int_{\mathbb{B}} \frac{1}{(1 - |\varphi|^2)^{N+1+\alpha}} d\nu_\alpha.$$

2.2. **Pseudohyperbolic distance.** We will describe some automorphisms of $\mathbb{B}$ that are analogous to the disk automorphisms $(a - z)/(1 - \bar{a}z)$, for a in $\mathbb{D}$. Let $a \in \mathbb{B}$, and set

$$P_a(z) = \frac{\langle z, a\rangle}{|a|^2} a,$$

so P_a is projection onto the subspace $[a]$ spanned by a, and $Q_a = I - P_a$, projection onto the orthogonal complement of $[a]$. To simplify notation write $s_a = \sqrt{1 - |a|^2}$. Define $\varphi_a(z)$ by

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}.$$

Clearly φ_a is analytic in $\overline{\mathbb{B}}$, $\varphi_a(0) = a$ and $\varphi_a(a) = 0$.

For $a, b \in \mathbb{B}$, the Bergman metric defined as

$$\beta(a, b) = \frac{1}{2} \log \frac{1 + |\varphi_a(b)|}{1 - |\varphi_a(b)|}.$$

we denote by $\rho(a,b)$ the pseudohyperbolic distance between a and b, i.e.,

$$\rho(a,b) = |\varphi_a(b)|,$$

and we have the following equation

$$1-\rho^2(a,b) = \frac{(1-|a|^2)(1-|b|^2)}{|1-\langle a,b\rangle|^2}. \tag{2.2}$$

For $\varphi \in S(\mathbb{B})$ and $z,w\in\mathbb{B}$, by the Schwarz–Pick Theorem, we have

$$\rho(\varphi(z),\varphi(w)) \le \rho(z,w),$$

thus for $z\in\mathbb{B}$, we obtain

$$\begin{aligned} |z|^2 &= \rho^2(0,z) \ge \rho^2(\varphi(0),\varphi(z)) \\ &\ge 1-\frac{(1-|\varphi(0)|^2)(1-|\varphi(z)|^2)}{(1-|\varphi(z)\varphi(0)|)^2} = \frac{|\varphi(z)-\varphi(0)|^2}{(1-|\varphi(0)\varphi(z)|)^2}, \end{aligned}$$

then

$$|\varphi(z)| \le \frac{|z|+|\varphi(0)|}{1+|z||\varphi(0)|}.$$

Using the inequity above, we easily get

$$\frac{1-|\varphi(z)|}{1-|z|} \ge \frac{1-|\varphi(0)|}{1+|\varphi(0)|} > 0. \tag{2.3}$$

Now, let us recall some lemmas.

Lemma 2.1. *([26, Lemma 2.20]) For each $R>0$ there exists a positive constant C_R such that*

$$C_R^{-1} \le \frac{1-|a|^2}{1-|z|^2} \le C_R$$

and

$$C_R^{-1} \le \frac{1-|a|^2}{|1-\langle a,z\rangle|} \le C_R$$

for all a and z in $\mathbb{B}$ with $\beta(z,a)\le R$.

Lemma 2.2. *([26, Lemma 2.27]) For any $R>0$ and any real b there exists a constant $C_R>0$ such that*

$$\left|\frac{(1-\langle z,u\rangle)^b}{(1-\langle z,v\rangle)^b}-1\right| \le C_R\beta(u,v)$$

for all z,u and v in $\mathbb{B}$ with $\beta(u,v)\le R$.

Remark 2.3. Since $\beta(a,b)=\frac{1}{2}\log\frac{1+\rho(a,b)}{1-\rho(a,b)}$, then $\beta(a,b)\le R \Leftrightarrow \rho(a,b)\le r$, where $r=\frac{e^R-1}{e^R+1}$, and we can obtain that there exists a positive constant C_R such that $\beta(a,b)\le C_R\rho(a,b)$ for $a,b\in\mathbb{B}$ with $\rho(a,b)\le r$. So Lemma 2.1 and Lemma 2.2 also hold if $\beta(a,b)$ is replace by $\rho(a,b)$. In this paper, we always use $\rho(a,b)$ instead of $\beta(a,b)$.

If $\rho(a,b) < r < 1$, we can get the inequality

$$\frac{1-\rho(a,b)}{1+\rho(a,b)} \leq \frac{1-|a|^2}{1-|b|^2} \leq \frac{1+\rho(a,b)}{1-\rho(a,b)}. \tag{2.4}$$

To see this, for example, let $b = \varphi_a(w)$, using the (2.2), we have

$$1-|b|^2 = \frac{(1-|a|^2)(1-|w|^2)}{|1-\langle a,w\rangle|},$$

since

$$\frac{1-|w|}{1+|w|} \leq \frac{1-|w|^2}{|1-\langle a,w\rangle|} \leq \frac{1+|w|}{1-|w|},$$

and $w = \varphi_a(b)$, we get (2.4).

According to Lemma 2.2, we have

$$\Re\left(\frac{1-|a|^2}{1-\langle a,b\rangle}\right)^b \geq 1 - C\rho(a,b) \geq 0 \tag{2.5}$$

for all $a,b \in \mathbb{B}$ with $\rho(a,b)$ sufficiently small, where $\Re z$ denote the real part of $z \in \mathbb{C}$.

Lemma 2.4. *Given $s > 0$, there is a constant $C = C(s) > 0$ such that*

$$\frac{1}{|1-\langle a,b\rangle|^s} - \Re\left(\frac{1}{1-\langle a,b\rangle}\right)^s \leq C\frac{\rho^2(a,b)}{(1-|a|^2)^{s/2}(1-|b|^2)^{s/2}}$$

for all $a,b \in \mathbb{B}$.

Proof. Let $s > 0$ and fix $a,b \in \mathbb{B}$. Choose $\varepsilon = \varepsilon_s \in (0,1)$ such that (2.5) is satisfied. Put $z = (1-\langle a,b\rangle)^{-s} = x + iy$ where $x = \Re z$ and $y = \Im z$. Since

$$|z| = \frac{1}{|1-\langle a,b\rangle|^s} \leq \frac{1}{(1-|a|)^{s/2}(1-|b|)^{s/2}},$$

we have

$$|z| - x \leq 2|z| \leq \frac{2}{\varepsilon^2}\frac{\rho^2(a,b)}{(1-|a|)^{s/2}(1-|b|)^{s/2}}$$

for $\rho(a,b) \geq \varepsilon$.

When $\rho(a,b) < \varepsilon$, we have $x \geq 0$, so

$$|z| - x \leq \frac{|z|^2 - x^2}{|z|} = \frac{y^2}{|z|}.$$

By Lemma 2.2, we get

$$\begin{aligned}
|y| &= \frac{|z-\bar{z}|}{2} = \frac{1}{2}\left|\frac{1}{(1-\langle a,b\rangle)^s} - \frac{1}{(1-\langle b,a\rangle)^s}\right| \\
&\leq \frac{1}{2}\left(\left|\frac{1}{(1-\langle a,b\rangle)^s} - \frac{1}{(1-\langle a,a\rangle)^s}\right| + \left|\frac{1}{(1-\langle a,a\rangle)^s} - \frac{1}{(1-\langle b,a\rangle)^s}\right|\right) \\
&\leq \frac{1}{2|1-\langle a,b\rangle|^s}\left|\frac{(1-\langle a,b\rangle)^s}{(1-\langle a,a\rangle)^s} - 1\right| + \frac{1}{2|1-\langle b,a\rangle|^s}\left|\frac{(1-\langle b,a\rangle)^s}{(1-\langle a,a\rangle)^s} - 1\right| \\
&\leq \frac{C}{|1-\langle a,b\rangle|^s}\rho(a,b).
\end{aligned}$$

Thus,

$$\frac{y^2}{|z|} \leq C\frac{\rho^2(a,b)|1-\langle a,b\rangle|^s}{|1-\langle a,b\rangle|^{2s}} \leq C\frac{\rho^2(a,b)}{(1-|a|)^{s/2}(1-|b|)^{s/2}}.$$

Since $1-|z|^2 \asymp 1-|z|$, the proof is complete. $\square$

3. Hilbert-Schmidt differences

In this section we will use pseudohyperbolic distance to characterize Hilbert-Schmidt differences of composition operators on A^2_α.

Theorem 3.1. *Let $\alpha > -1$ and $J \in \mathbb{N}$, for $a_1, \cdots, a_J \in \mathbb{C}$ and $\varphi_1, \cdots, \varphi_J \in S(\mathbb{B})$, the identity*

$$\left\| \sum_{j=1}^{J} a_j C_{\varphi_j} \right\|_{HS}^2 = \int_{\mathbb{B}} \left\| \sum_{j=1}^{J} \overline{a_j} K_{\varphi_j(z)} \right\|_\alpha^2 d\nu_\alpha(z)$$

holds.

Proof. The proof is similar to Proposition 3.1 in [2], we omit the detail. $\square$

By the theorem above, we need to consider the quantity $\|K_z - K_w\|_\alpha$ in order to study Hilbert-Schmidt differences of composition operators.

Theorem 3.2. *Let $\alpha > -1$, for $z, w \in \mathbb{B}$, we have*

$$\|K_z - K_w\|_\alpha^2 \asymp (\|K_z\|_\alpha - \|K_w\|_\alpha)^2 + \rho^2(z,w)\|K_z\|_\alpha\|K_w\|_\alpha.$$

Proof. The reproducing property gives

$$\begin{aligned}
& \|K_z - K_w\|_\alpha^2 \\
= \; & (\|K_z\|_\alpha - \|K_w\|_\alpha)^2 + 2\|K_z\|_\alpha\|K_w\|_\alpha - 2\Re\langle K_z, K_w\rangle_\alpha \\
= \; & (\|K_z\|_\alpha - \|K_w\|_\alpha)^2 + 2(\|K_z\|_\alpha\|K_w\|_\alpha - |\langle K_z, K_w\rangle_\alpha|) \\
& + 2(|\langle K_z, K_w\rangle_\alpha| - \Re\langle K_z, K_w\rangle_\alpha) \\
=: \; & (\|K_z\|_\alpha - \|K_w\|_\alpha)^2 + 2F_\alpha + 2G_\alpha.
\end{aligned}$$

We estimate F_α, by (2.2) and $1 - t^{\frac{N+1+\alpha}{2}} \asymp 1 - t$ when $0 \leq t \leq 1$, we have

$$\begin{aligned}
F_\alpha &= \frac{1}{(1-|z|^2)^{\frac{N+1+\alpha}{2}}}\frac{1}{(1-|w|^2)^{\frac{N+1+\alpha}{2}}} - \frac{1}{|1-\langle z,w\rangle|^{N+1+\alpha}} \\
&\leq \frac{1}{(1-|z|^2)^{\frac{N+1+\alpha}{2}}}\frac{1}{(1-|w|^2)^{\frac{N+1+\alpha}{2}}}\left(1 - (1-\rho^2(z,w))^{\frac{N+1+\alpha}{2}}\right) \\
&\asymp \rho^2(z,w)\|K_z\|_\alpha\|K_w\|_\alpha.
\end{aligned}$$

By Lemma 2.4, we know $0 \leq G_\alpha \leq C\rho^2(z,w)\|K_z\|_\alpha\|K_w\|_\alpha$. Consequently, we have $F_\alpha + G_\alpha \asymp \rho^2(z,w)\|K_z\|_\alpha\|K_w\|_\alpha$. The proof is complete. $\square$

Theorem 3.3. *Given $\alpha > -1$, the estimate*

$$\|K_z - K_w\|_\alpha \asymp \rho(z,w)(\|K_z\|_\alpha + \|K_w\|_\alpha)$$

holds for all $z, w \in \mathbb{B}$.

Proof. For $z, w \in \mathbb{B}$, put

$$\Phi := \left(\|K_z\|_\alpha^2 + \|K_w\|_\alpha^2\right)\rho^2(z,w),$$

$$\Psi_1 := \left(\|K_z\|_\alpha - \|K_w\|_\alpha\right)^2$$

and

$$\Psi_2 := \|K_z\|_\alpha \|K_w\|_\alpha \rho^2(z,w).$$

We only need to establish the estimate

$$C^{-1}\Phi \le \Psi_1 + \Psi_2 \le C\Phi$$

on $\mathbb{B}^2$ by Theorem 3.2. We decompose $\mathbb{B}^2$ into three parts

$$E := \{(a,b) \in \mathbb{B}^2) : \rho(a,b) < 1/2\},$$

$$Q_1 := \left\{(a,b) \in \mathbb{B}^2 \backslash E : 1/2 \le \left(\frac{1-|a|^2}{1-|b|^2}\right)^{\frac{N+1+\alpha}{2}} \le 2\right\},$$

$$Q_2 := \mathbb{B}^2 \backslash (E \cup Q_1).$$

To obtain the left inequality, that is

$$\Phi \le C(\Psi_1 + \Psi_2),$$

we proceed similarly to Proposition 3.5 in [2]. Then, we only prove

$$\Psi_1 + \Psi_2 \le C\Phi.$$

It is easy to see that $2\Psi_2 \le \Phi$ on $\mathbb{B}^2$, and $\Psi_1 \le 4\Phi$ on $\mathbb{B}^2 \backslash E$. Now, when $(z,w) \in E$, by Lemma 2.2, we have

$$\begin{aligned}
&\Psi_1 \\
= \ &\left(\frac{1}{(1-|z|^2)^s} - \frac{1}{(1-|w|^2)^s}\right)^2 \\
\le \ &\left[\frac{1}{(1-|z|^2)^s}\left|1 - \left(\frac{1-|z|^2}{1-\langle z,w\rangle}\right)^s\right| + \frac{1}{(1-|w|^2)^s}\left|1 - \left(\frac{1-|w|^2}{1-\langle z,w\rangle}\right)^s\right|\right]^2 \\
\le \ &\left[\frac{C\rho(z,w)}{(1-|z|^2)^s} + \frac{C\rho(z,w)}{(1-|w|^2)^s}\right]^2 \\
\le \ &C\rho^2(z,w)\left[\frac{1}{(1-|z|^2)^{2s}} + \frac{1}{(1-|w|^2)^{2s}}\right],
\end{aligned}$$

here $s = \frac{N+1+\alpha}{2}$.

Thus, we obtain $\Psi_1 \le C\Phi$ on E. This completes the proof. □

Now, we are now ready to estimate the quantity $\|C_\varphi - C_\psi\|_{HS}$.

Theorem 3.4. *Assume that $\varphi, \psi \in S(\mathbb{B})$, then the following estimate*

$$\|C_\varphi - C_\psi\|_{HS}^2 \asymp \int_{\mathbb{B}} \left(\frac{1}{1-|\varphi(z)|^2} + \frac{1}{1-|\psi(z)|^2}\right)^{N+1+\alpha} \rho^2(\varphi(z),\psi(z))\,d\nu_\alpha(z)$$

holds.

Proof. By Theorem 3.1 and Theorem 3.3, we have

$$\begin{aligned}\|C_\varphi - C_\psi\|_{HS}^2 &= \int_{\mathbb{B}} \|K_{\varphi(z)} - K_{\psi(z)}\|_\alpha^2 d\nu_\alpha(z)\\ &\asymp \int_{\mathbb{B}} (\|K_{\varphi(z)}\|_\alpha + \|K_{\psi(z)}\|_\alpha)^2 \rho^2(\varphi(z),\psi(z)) d\nu_\alpha(z)\\ &\asymp \int_{\mathbb{B}} \left(\frac{1}{1-|\varphi(z)|^2} + \frac{1}{1-|\psi(z)|^2}\right)^{N+1+\alpha} \rho^2(\varphi(z),\psi(z)) d\nu_\alpha(z).\end{aligned}$$

Thus, we finish the proof. □

As a corollary of Theorem 3.4, we get an equivalent condition for the differences $C_\varphi - C_\psi$ to be Hilbert-Schmidt. This result will provide some heuristcs for the proof of our theorem in section 4.

Corollary 3.5. *Assume that $\varphi, \psi \in S(\mathbb{B})$, then $C_\varphi - C_\psi$ is Hilbert-Schmidt on A_α^p if and only if*

$$\int_{\mathbb{B}} \frac{\rho^2(\varphi(z),\psi(z)) d\nu_\alpha(z)}{(1-|\varphi(z)|^2)^{N+1+\alpha}} < \infty$$

and

$$\int_{\mathbb{B}} \frac{\rho^2(\varphi(z),\psi(z)) d\nu_\alpha(z)}{(1-|\psi(z)|^2)^{N+1+\alpha}} < \infty.$$

Using (2.3), we can get another corollary of Theorem 3.4.

Corollary 3.6. *Let $\alpha > 1$ and $\varphi, \psi \in S(\mathbb{B})$, If $C_\varphi - C_\psi$ is Hilbert-Schmidt on A_α^2, then $C_\varphi - C_\psi$ is Hilbert-Schmidt on A_β^2 for $\beta > \alpha$.*

On the disk, when composition operators C_φ and C_ψ are not Hilbert-Schmidt, we know that the linear combinations $aC_\varphi + bC_\psi$ is Hilbert-Schmidt if and only if $a + b = 0$ and $C_\varphi - C_\psi$ is Hilbert-Schmidt, where $a, b \in \mathbb{C}\backslash\{0\}$. Here, we can get the same result, for the purpose, we need the following estimate, it is easy to get from Lemma 3.9 in [2] and Theorem 3.2, we omit the proof.

Theorem 3.7. *For $z, w \in \mathbb{B}$ and $\lambda \in \mathbb{C}$, we have the following estimate*

$$(\|K_z\|_\alpha - |\lambda|\|K_w\|_\alpha)^2 + |\lambda|\rho^2(z,w)\|K_z\|_\alpha\|K_w\|_\alpha \prec \|K_z - \lambda K_w\|_\alpha^2.$$

when $|\lambda| = 1$, $\|K_z - K_w\|_\alpha^2 \prec \|K_z - \lambda K_w\|_\alpha^2$.

Theorem 3.8. *Let $\alpha > -1$ and $a, b \in \mathbb{C}\backslash\{0\}$. Suppose that C_φ and C_ψ are not Hilbert-Schmidt on A_α^2. Then $aC_\varphi + bC_\psi$ is Hilbert-Schmidt on A_α^2 if and only if $a + b = 0$ and $C_\varphi - C_\psi$ is Hilbert-Schmidt on A_α^2.*

Proof. The proof is similar to Theorem 3.10 in [2], we also omit the proof. □

4. TOPOLOGY STRUCTURE

We will give some conclusions about the topology structure in this section. Let $\mathcal{C}(A^2_\alpha)$ be the space of all bounded composition operators on A^2_α endowed with norm topology.

Write $C_\varphi \sim C_\psi$ if C_φ and C_ψ are in the same path component of $\mathcal{C}(A^2_\alpha)$. For $t \in [0,1]$, put $\varphi_t = (1-t)\varphi + t\psi$, it is easy to see $\varphi_t \in S(\mathbb{B})$.

Theorem 4.1. *Let $\alpha > 1$ and assume that $C_\varphi, C_\psi \in \mathcal{C}(A^2_\alpha)$, $C_\varphi - C_\psi$ is Hilbert-Schmidt on A^2_α. Then $C_{\varphi_s} - C_{\varphi_t}$ is Hilbert-Schmidt for any $s,t \in [0,1]$.*

Proof. Since $\|C_{\varphi_s} - C_{\varphi_t}\|_{HS} \le \|C_\varphi - C_{\varphi_s}\|_{HS} + \|C_\varphi - C_{\varphi_t}\|_{HS}$, it is sufficient to prove $C_\varphi - C_{\varphi_s}$ is Hilbert-Schmidt for $s \in [0,1]$.

From the definition of ρ, we have

$$\begin{aligned}\rho(\varphi_s(z),\varphi(z)) &= \left|\frac{\varphi(z) - P_{\varphi(z)}(\varphi_s(z)) - s_{\varphi(z)}Q_{\varphi(z)}(\varphi_s(z))}{1-\langle\varphi_s(z),\varphi(z)\rangle}\right| \\ &= s\left|\frac{\varphi(z) - P_{\varphi(z)}(\psi(z)) - s_{\varphi(z)}Q_{\varphi(z)}(\psi(z))}{1-\langle\varphi_s(z),\varphi(z)\rangle}\right| \\ &= \frac{s\rho(\varphi(z),\psi(z))|1-\langle\psi(z),\varphi(z)\rangle|}{|1-\langle\varphi_s(z),\varphi(z)\rangle|}.\end{aligned}$$

So, if $\rho(\varphi(z),\psi(z)) \ge 1/2$, $\rho(\varphi_s(z),\varphi_(z)) \le 1 \le 2\rho(\varphi(z),\psi(z))$.

If $\rho(\varphi(z),\psi(z)) < 1/2$, by Lemma 2.1, we have

$$\begin{aligned}\rho(\varphi_s(z),\varphi(z)) &\le \frac{s\rho(\varphi(z),\psi(z))|1-\langle\psi(z),\varphi(z)\rangle|}{1-|\varphi(z)|} \\ &\le \frac{2s\rho(\varphi(z),\psi(z))|1-\langle\psi(z),\varphi(z)\rangle|}{1-|\varphi(z)|^2} \\ &\le 2Cs\rho(\varphi(z),\psi(z)),\end{aligned}$$

thus, we get $\rho(\varphi_s(z),\varphi(z)) \le C\rho(\varphi(z),\psi(z))$ for all $z \in \mathbb{B}$. And

$$|\varphi_s(z)| = |(1-s)\varphi(z) + s\psi(z)| \le max\{|\varphi(z)|,|\psi(z)|\},$$

then

$$\frac{1}{1-|\varphi_s(z)|^2} \le \frac{1}{1-|\varphi(z)|^2} + \frac{1}{1-|\psi(z)|^2},$$

hence we get

$$\begin{aligned}&\int_{\mathbb{B}} \frac{\rho^2(\varphi_s(z),\varphi(z))d\nu_\alpha(z)}{(1-|\varphi_s(z)|^2)^{N+1+\alpha}} \\ \le\ & C\int_{\mathbb{B}}\left(\frac{1}{1-|\varphi(z)|^2} + \frac{1}{1-|\psi(z)|^2}\right)^{N+1+\alpha} \rho^2(\varphi(z),\psi(z))d\nu_\alpha(z) \\ \asymp\ & \|C_\varphi - C_\psi\|^2_{HS} < \infty.\end{aligned}$$

Similarly,

$$\int_{\mathbb{B}} \frac{\rho^2(\varphi_s(z),\varphi(z))d\nu_\alpha(z)}{(1-|\varphi(z)|^2)^{N+1+\alpha}} < \infty,$$

according to Corollary 3.5, we have $C_\varphi - C_{\varphi_s}$ is Hilbert-Schmidt. This completes the proof. $\square$

Since $\|C_{\varphi_s}\| \le \|C_{\varphi_s} - C_\varphi\| + \|C_\varphi\| \le \|C_{\varphi_s} - C_\varphi\|_{HS} + \|C_\varphi\|$, the composition operator C_{φ_s} belongs to $\mathcal{C}(A^2_\alpha)$ for $s \in [0,1]$ by the theorem above.

Now, we give the sufficient condition of path connected. For the process of proof, please refer to [2, Theorem 4.2].

Theorem 4.2. *Let $\alpha > 1$ and $\varphi, \psi \in S(\mathbb{B})$. Assume that $C_\varphi, C_\psi \in \mathcal{C}(A^2_\alpha)$, and $C_\varphi - C_\psi$ is Hilbert-Schmidt operator, then $C_\varphi \sim C_\psi$.*

Proof. According to Theorem 4.1, we obtain $C_{\varphi_s} \in \mathcal{C}(A^2_\alpha)$, we need to show that $s \in [0,1] \to C_{\varphi_s}$ is a continuous path in $\mathcal{C}(A^2_\alpha)$. Here, it is sufficient to consider the case $\lim\limits_{s\to 0} \|C_\varphi - C_{\varphi_s}\| = 0$.

Given $s \in (0,1]$, put

$$\Phi_s = \left(\frac{1}{1-|\varphi|^2} + \frac{1}{1-|\varphi_s|^2}\right)^{N+1+\alpha} \rho(\varphi, \varphi_s)$$

for short, since $\rho(\varphi,\varphi_s) \le 2\rho(\varphi,\psi)$ and $\frac{1}{1-|\varphi_s|^2} \le \frac{1}{1-|\varphi|^2} + \frac{1}{1-|\psi|^2}$, we have

$$\Phi_t \le C\left(\frac{1}{1-|\varphi|^2} + \frac{1}{1-|\psi|^2}\right)^{N+1+\alpha} \rho(\varphi, \psi)$$

for all s. Because $C_\varphi - C_\psi$ is Hilbert-Schmidt operator, $\int_{\mathbb{B}} \Phi_t d\nu_\alpha$ is integrable. Since $\rho(\varphi_s, \varphi) \to 0$ as $s \to 0$, so we get

$$\lim_{s\to 0} \|C_\varphi - C_{\varphi_s}\|_{HS} = 0,$$

and

$$\lim_{s\to 0} \|C_\varphi - C_{\varphi_s}\| = 0.$$

This proof is complete. $\square$

By the theorem above, we can obtain the following consequences.

(1) Given $C_\varphi \in \mathcal{C}(A^2_\alpha)$, the set

$$N(\varphi) = \{C_\psi : \|C_\varphi - C_\psi\|_{HS} < \infty\}$$

is the path-connected set in $\mathcal{C}(A^2_\alpha)$ containing C_φ.

(2) Let $\mathcal{HS}(A^2_\alpha) \subset \mathcal{C}(A^2_\alpha)$ be the set of all Hilbert-Schmidt composition operators on A^2_α, then $\mathcal{HS}(A^2_\alpha)$ belongs to a path component of $\mathcal{C}(A^2_\alpha)$.

(3) $N(\varphi)$ is "convex" in the sense that if $C_\psi \in N(\varphi)$, then $\{C_{(1-t)\varphi+t\psi}\}_{t\in[0,1]} \in N(\varphi)$.

Next, we will study the isolation using the extreme point, for related papers see [2, 8, 9, 14]. It is easy to see that the set $S(\mathbb{B})$ is a convex set. For the set $S(\mathbb{B})$, we define the extreme point as following: If $\varphi \in S(\mathbb{B})$ is not proper convex combination of two distinct elements of $S(\mathbb{B})$, we call φ is an extreme point. It is easy to see that φ is an extreme point if and only if $\varphi = \frac{f+g}{2}$ for $f, g \in S(\mathbb{B})$, implies $f = g = \varphi$.

Example 4.3. For $N = 2$, let $\varphi(z_1, z_2) = (A\varphi_1, B\varphi_2)$, where $A, B \leq 0$, $A^2+B^2 = 1$ and φ_i are inner functions on unit ball $\mathbb{B}_2$, then φ is an extreme point.

To prove that this example of extreme point is correct, let us observe at first that, because of $A^2 + B^2 = 1$, it follows $|\varphi(\zeta)| = 1$ for $\zeta \in \mathbb{S}_2$ almost everywhere. If φ is not an extreme point, then there exist two distinct maps f and g such that $\varphi = \frac{f+g}{2}$. when $|\varphi(\zeta)| = 1$, we have $f(\zeta) = g(\zeta) \in \mathbb{S}_2$, thus the components $\varphi^i = f^i = g^i$ on $\mathbb{S}_2$ almost everywhere for $i = 1, 2$. Since $\varphi, f, g \in S(\mathbb{B}_2)$, $\varphi^i, f^i, g^i \in H^2$, and $\|\varphi^i\|_{H^2} = \|f^i\|_{H^2} = \|g^i\|_{H^2}$, so $\varphi^i = f^i = g^i$ on $\mathbb{B}_2$, then $\varphi = f = g$ on $\mathbb{B}$, this contradicts with the choose of f, g. Thus, φ is an extreme point.

Using the similar method, we can prove that every automorphisms φ_a is an extreme point on $\mathbb{B}$. When φ is a linear-fractional self-map of $\mathbb{B}$, C_φ is bounded on A^2_α, moreover, φ_a is linear-fractional, so C_{φ_a} belongs to $\mathcal{C}(A^2_\alpha)$, thus there is composition operator induce by extreme point of $S(\mathbb{B})$ in $\mathcal{C}(A^2_\alpha)$.

Now, we give a equivalent condition for extreme point, and research the non-isolation using the extreme point.

Theorem 4.4. *Let $\varphi \in S(\mathbb{B})$, then φ is not an extreme point if and only if there exists some $\omega \in S(\mathbb{B})$ such that $\omega \neq 0$ and $|\varphi| + |\omega| \leq 1$ on $\mathbb{B}$.*

Proof. If φ is not an extreme point, then there are two distinct functions $f, g \in S(\mathbb{B})$ such that $\varphi = \frac{f+g}{2}$. Put $\psi = \frac{f-g}{2}$, $\psi = (\psi_1, \cdots, \psi_N)$, it is obvious that $|\varphi|^2 + |\psi|^2 \leq 1$, so $\psi \in S(\mathbb{B})$, let $\omega = \frac{(\psi_1^2, \cdots, \psi_N^2)}{2}$, then $\omega \in S(\mathbb{B})$, and

$$|\omega| + |\varphi| \leq \frac{\sqrt{\sum\limits_{i=1}^{N} |\psi_i|^4}}{2} + |\varphi| \leq \frac{|\psi|^2}{2} + |\varphi| \leq 1.$$

If there is a non-zero function ω, such that $|\varphi| + |\omega| \leq 1$, then $|\varphi \pm \omega| \leq |\omega| + |\varphi| \leq 1$, so $\varphi \pm \omega \in S(\mathbb{B})$, and $\varphi = \frac{\varphi+\omega}{2} + \frac{\varphi-\omega}{2}$, thus φ is not an extreme point of $S(\mathbb{B})$. □

Theorem 4.5. *Let $\varphi \in S(\mathbb{B})$ and $C_\varphi \in \mathcal{C}(A^2_\alpha)$. If φ is not the extreme point of $S(\mathbb{B})$, then C_φ is not isolated in $\mathcal{C}(A^2_\alpha)$.*

Proof. Suppose that φ is not extreme point, we know that there is some non-zero element $\omega \in S(\mathbb{B})$, such that $|\varphi| + |\omega| \leq 1$. Let $s = \frac{N+3+\alpha}{2}$ and $\psi = \varphi + \frac{(\omega_1^s, \cdots, \omega_N^s)}{2}$, since

$$1 - |\psi| > 1 - |\varphi| - \frac{|(\omega_1^s, \cdots, \omega_N^s)|}{2} \geq 1 - |\varphi| - \frac{|\omega|^s}{2} \geq \frac{1 - |\varphi|}{2} \geq 0,$$

we have $\psi \in S(\mathbb{B})$. Moreover, it is easy to observe that

$$\begin{aligned}
&\rho^2(\varphi(z),\psi(z))\\
= \;& 1-\frac{(1-|\varphi(z)|^2)(1-|\psi(z)|^2)}{|1-\langle\varphi(z),\psi(z)\rangle|^2}\\
= \;& \frac{|\langle\varphi(z),\psi(z)\rangle|^2-2\Re\langle\varphi(z),\psi(z)\rangle+|\varphi(z)|^2+|\psi(z)|^2-|\varphi(z)|^2|\psi(z)|^2}{|1-\langle\varphi(z),\psi(z)\rangle|^2}\\
\leq \;& \frac{|\psi(z)-\varphi(z)|^2}{(1-|\varphi(z)|)^2} \leq \frac{|\omega(z)|^{2s}}{4(1-|\varphi(z)|)^2} \leq (1-|\varphi(z)|)^{2s-2},
\end{aligned}$$

so we have

$$\left(\frac{1}{1-|\varphi(z)|^2}+\frac{1}{1-|\psi(z)|^2}\right)^{N+1+\alpha}\rho^2(\varphi(z),\psi(z)) \leq 3^{N+1+\alpha},$$

then $C_\varphi - C_\psi$ is Hilbert–Schmidt by Theorem 3.4, and $C_\psi \in \mathcal{C}(A^2_\alpha)$, thus C_φ is not isolated. The proof is complete. □

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