



## QUOTIENT MEAN SERIES

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ABSTRACT. The well-known Mathieu series

$$S_M(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2} \quad (r > 0),$$

can be transformed into the form

$$S_M(r) = \frac{1}{2r} \sum_{n=1}^{\infty} \frac{\sqrt{nr^2}}{\left(\sqrt{\frac{n^2+r^2}{2}}\right)^4} = \frac{1}{2r} \sum_{n=1}^{\infty} \frac{G^2(n, r)}{Q^4(n, r)},$$

where  $G(n, r)$  and  $Q(n, r)$  denote the Geometric and Quadratic mean of  $n \in \mathbb{N}$  and  $r > 0$ . This connection leads us to the idea to introduce and research the so-called Quotient mean series as a be a generalizations of Mathieu's and Mathieu-type series. We give an integral representation of such series and their alternating variant, together with associated inequalities. Also, special cases of quotient mean series, involving Bessel function of the first kind, have been studied in detail.

### 1. INTRODUCTION AND PREPARATION

In his work on the solid state physics *Traite de Physique Mathématique*, VI-VII: *Théory de Élasticité des Corps Solides* (Part 2) in 1890. Émile Leonard

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Mathieu (1835.-1890.) defined the following series

$$S_M(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2} \quad (r > 0),$$

which we call the *Mathieu series*. This kind of series have their application in solid state physics, in mathematical models that describe the vibrations of planar figures, in boundary problems of biharmonic equations on rectangular domain [10, page. 258] and on two dimensional elastostatic problems [2].

The alternating variant of Mathieu series

$$\tilde{S}_M(r) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{(n^2 + r^2)^2} \quad (r > 0),$$

is considered in connection with a ODJ whose solution is the Butzer–Flocke–Hauss  $\Omega$ -function, see [1, 8].

Recently, the series  $S_M(r)$  and its various generalizations became the topic of interest of numerous mathematicians such as Acu, Alzer, Cerone, Dicu, Elezović, Lampret, Lenard, Gavrea, Guo, Hoorfar, Pogány, Qi, Srivastava, Tomovski, Trenčevski, following the pioneering work of Berg, Diananda, Emersleben and Makai among others. By investigating Mathieu series we mean deriving integral representations and sharp bilateral bounds, together with their application in numerical analysis, mathematical physics, number theory, special functions and so on.

It is straightforward to see that  $S_M(r)$  can be rewritten into

$$S_M(r) = \frac{1}{2r} \sum_{n=1}^{\infty} \frac{\sqrt{nr^2}}{\left(\sqrt{\frac{n^2+r^2}{2}}\right)^4} = \frac{1}{2r} \sum_{n=1}^{\infty} \frac{G^2(n, r)}{Q^4(n, r)},$$

where

$$G(n, r) = \sqrt{nr}, \quad Q(n, r) = \sqrt{\frac{n^2 + r^2}{2}}$$

are the well known Geometric and Quadratic mean of  $n \in \mathbb{N}$  and  $r > 0$ , respectively. This transformation leads as to study a series such that consists from quotients of power means so, that in deriving its closed integral representations we take the same mathematical tools as for Mathieu series in earlier papers, e.g. the Laplace–integral form of the Dirichlet series [4]

$$\mathcal{D}_\lambda(x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} = x \int_0^\infty e^{-xt} \left( \sum_{n=1}^{[\lambda^{-1}(t)]} a_n \right) dt, \quad (1.1)$$

mentioning that throughout the article  $[A]$  stands for the integer part of  $A$ ; the Gamma–function formula

$$a^{-s} \Gamma(s) = \int_0^\infty e^{-ax} x^{s-1} dx \quad (\Re\{s\} > 0)$$

and the Euler–Maclaurin summation formula written in the form [9]

$$\sum_{n=k+1}^l a_n = \int_k^l (a(x) + \{x\}a'(x))dx \equiv \int_k^l \mathfrak{d}_x a(x)dx. \quad (1.2)$$

The alternating variant of (1.2) reads as follows

$$\sum_{j=l+1}^m (-1)^{j-l-1} a_j = \int_l^m \mathfrak{d}_x^1 a(x)dx - 2 \int_{2l}^{2[\frac{1}{2}m]} \mathfrak{d}_x^{1/2} a(x)dx.$$

Here

$$\mathfrak{d}_x^q := 1 + \{qx\} \frac{\partial}{\partial x}, \quad q \in \{\frac{1}{2}, 1\}.$$

Let us recall that for a positive vector  $a = (a_1, \dots, a_n) \in \mathbb{R}_+^n$ ,  $r \in \mathbb{R} \setminus \{0\}$ , the *equal weights mean* of order  $r$  one defines as [6]

$$M_n^{[r]}(a) = \left( \frac{a_1^r + \dots + a_n^r}{n} \right)^{\frac{1}{r}}.$$

We will need in the sequel its well-known properties

$$\lim_{r \rightarrow 0^+} M_n^{[r]}(a) = \sqrt[n]{a_1 \cdots a_n} = G(a_1, \dots, a_n), \quad M_n^{[2]}(a) = Q(n, a).$$

Now we define the series

$$\begin{aligned} S_{p,q}^{s,t}(r_1, r_2) &:= \sum_{n=1}^{\infty} \frac{(M_2^{[s]}(n, r_1))^t}{(M_2^{[q]}(n, r_2))^p} = 2^{p/q-t/s} \sum_{n=1}^{\infty} \frac{(n^s + r_1^s)^{t/s}}{(n^q + r_2^q)^{p/q}} \\ \tilde{S}_{p,q}^{s,t}(r_1, r_2) &:= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(M_2^{[s]}(n, r_1))^t}{(M_2^{[q]}(n, r_2))^p} \quad (r_1, r_2, p, q, t, s > 0). \end{aligned}$$

Here, and in what follows,  $S_{p,q}$ ,  $\tilde{S}_{p,q}$  we call *quotient mean series* and *alternating quotient mean series*, respectively.

*Remark 1.1.* The series  $S_{p,q}^{s,t}(r_1, r_2)$  converges for  $p - t > 1$ , namely

$$S_{p,q}^{s,t}(r_1, r_2) \sim \zeta(p - t).$$

Similarly,  $\tilde{S}_{p,q}^{s,t}(r_1, r_2)$  converges for  $p > t$  since

$$\tilde{S}_{p,q}^{s,t}(r_1, r_2) \sim \eta(p - t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{p-t}}.$$

The main topic of our interest is to derive an integral representation of such series and to give related bounding inequalities, making use of the derived integral expressions.

2. INTEGRAL REPRESENTATIONS FOR  $S_{p,q}^{s,t}$  AND  $\tilde{S}_{p,q}^{s,t}$ 

First, we give an integral representation for  $S_{p,q}^{s,t}$ .

**Theorem 2.1.** *Let  $r_1, r_2, p, q, t, s > 0$ ,  $p - t > 1$ . Then the quotient mean series possesses the closed integral expression*

$$S_{p,q}^{s,t}(r_1, r_2) = 2^{p/q-t/s} \frac{p}{q} \int_0^\infty \int_0^{[u^{1/q}]} \frac{\mathfrak{d}_w ((w^s + r_1^s)^{t/s})}{(u + r_2^q)^{p/q+1}} dw du.$$

*Proof.* We start by transforming  $S_{p,q}^{s,t}$  with the help of the Gamma function formula obtaining

$$\begin{aligned} S_{p,q}^{s,t}(r_1, r_2) &= \frac{2^{p/q-t/s}}{\Gamma(p/q)} \sum_{n=1}^\infty (n^s + r_1^s)^{t/s} \int_0^\infty e^{-(n^q+r_2^q)x} x^{\frac{p}{q}-1} dx \\ &= \frac{2^{p/q-t/s}}{\Gamma(p/q)} \int_0^\infty x^{p/q} e^{-r_2^q x} \left( \sum_{n=1}^\infty (n^s + r_1^s)^{t/s} e^{-n^q x} \right) dx. \end{aligned}$$

All considered sums are convergent by assumption, so the exchange of summation and integration order is legitimate. The inner sum is now a Dirichlet series

$$\mathcal{D}_{n^q}(x) = \sum_{n=1}^\infty (n^s + r_1^s)^{t/s} e^{-n^q x}.$$

By Laplace–integral formula, we get

$$\mathcal{D}_{n^q}(x) = x \int_0^\infty e^{-xu} \left( \sum_{n=1}^{[u^{1/q}]} (n^s + r_1^s)^{t/s} \right) du. \quad (2.1)$$

Applying the Euler–Maclaurin summation formula to the inner sum we get

$$A_n = \sum_{n=1}^{[u^{1/q}]} (n^s + r_1^s)^{t/s} = \int_0^{[u^{1/q}]} \mathfrak{d}_w ((w^s + r_1^s)^{t/s}) dw \quad (2.2)$$

Putting back (2.2) into (2.1), we have

$$\begin{aligned} \mathcal{D}_{n^q}(x) &= x \int_0^\infty e^{-xu} \left( \int_0^{[u^{1/q}]} \mathfrak{d}_w ((w^s + r_1^s)^{t/s}) dw \right) du \\ &= x \int_0^\infty \int_0^{[u^{1/q}]} e^{-xu} \mathfrak{d}_w ((w^s + r_1^s)^{t/s}) dw du. \end{aligned}$$

Consequently

$$\begin{aligned} S_{p,q}^{s,t}(r_1, r_2) &= \frac{2^{p/q-t/s}}{\Gamma(p/q)} \int_0^\infty \int_0^\infty \int_0^{[u^{1/q}]} x^{p/q} e^{-(u+r_2^q)x} \mathfrak{d}_w ((w^s + r_1^s)^{t/s}) dw du dx \\ &= \frac{2^{p/q-t/s}}{\Gamma(p/q)} \int_0^\infty \int_0^{[u^{1/q}]} \left( \int_0^\infty x^{p/q} e^{-(u+r_2^q)x} dx \right) \mathfrak{d}_w ((w^s + r_1^s)^{t/s}) dw du. \end{aligned}$$

The inner integral is equal to  $\Gamma(p/q + 1)(u + r_2^q)^{-1-p/q}$ , therefore

$$\begin{aligned} S_{p,q}^{s,t}(r_1, r_2) &= \frac{2^{p/q-t/s}\Gamma(p/q + 1)}{\Gamma(p/q)} \int_0^\infty \int_0^{[u^{1/q}]} \frac{\mathfrak{d}_w((w^s + r_1^s)^{t/s})}{(u + r_2^q)^{p/q+1}} dw du \\ &= 2^{p/q-t/s} \frac{p}{q} \int_0^\infty \int_0^{[u^{1/q}]} \frac{\mathfrak{d}_w((w^s + r_1^s)^{t/s})}{(u + r_2^q)^{p/q+1}} dw du. \end{aligned}$$

□

For the alternating variant of quotient mean series the proving procedure is the copy of the proof used in the previous theorem. The only exception is the alternating Dirichlet series such that appears in the integrand. Since  $[u^{1/q}] \equiv 0$  for  $0 \leq u < 1$ , the integration domain of all first integrals in integral representation becomes  $[1, \infty)$ .

**Theorem 2.2.** *Let  $r_1, r_2, p, q, t, s > 0$ ,  $p > t$ . The alternating quotient mean series possesses the integral representation*

$$\begin{aligned} \tilde{S}_{p,q}^{s,t}(r_1, r_2) &= \frac{2^{p/q-t/s} p}{q} \left( \int_1^\infty \int_0^{[u^{1/q}]} \frac{\mathfrak{d}_w^1((w^s + r_1^s)^{t/s})}{(r_2^q + u)^{p/q+1}} dw du \right. \\ &\quad \left. - 2 \int_1^\infty \int_0^{2[\frac{1}{2}[u^{1/q}]]} \frac{\mathfrak{d}_w^{1/2}((w^s + r_1^s)^{t/s})}{(r_2^q + u)^{p/q+1}} dw du \right). \end{aligned}$$

### 3. BOUNDING INEQUALITIES

**3.1. Bilateral inequality for  $S_{p,q}^{s,t}$ .** From the definition of the operator  $\mathfrak{d}_x$  it is obvious that

$$a(x) + a'_-(x) < \mathfrak{d}_x a(x) = a(x) + \{x\}a'(x) < a(x) + a'_+(x),$$

where  $f_- = \min\{f, 0\}$ ;  $f_+ = \max\{f, 0\}$ . Then we have

$$\begin{aligned} a(x) &\leq \mathfrak{d}_x a(x) < a(x) + a'(x) && a \text{ monotone increasing,} \\ a(x) + a'(x) &< \mathfrak{d}_x a(x) \leq a(x) && a \text{ monotone decreasing.} \end{aligned} \tag{3.1}$$

Since in our case  $a(x)$  increases, we use (3.1) to derive inequalities for  $S_{p,q}^{s,t}$ .

**Theorem 3.1.** *Let  $\kappa \in \mathbb{N}_2 = \{2, 3, \dots\}$ ,  $t = \kappa s > 0$  and*

$$q > \max \left\{ s + \frac{1}{\kappa - 1}, \frac{\kappa s}{\kappa - 1} \right\}.$$

*Then we have*

$$\mathcal{L}_{p,q}^{s,t}(r_1, r_2) \leq S_{p,q}^{s,t}(r_1, r_2) < \mathcal{R}_{p,q}^{s,t}(r_1, r_2),$$

*where*

$$\begin{aligned} \mathcal{L}_{p,q}^{s,t}(r_1, r_2) &= \frac{2^{p/q-t/s} p}{q} \int_1^\infty \frac{[u^{1/q}] r_1^t {}_2F_1 \left( \begin{matrix} 1/s, -t/s \\ 1 + 1/s \end{matrix} \middle| -r_1^{-s} [u^{1/q}]^s \right)}{(u + r_2^s)^{t/s}} du, \\ \mathcal{R}_{p,q}^{s,t}(r_1, r_2) &= \frac{2^{p/q-t/s} p}{q} \int_1^\infty \frac{([u^{1/q}]^s + r_1^s)^{t/s} - r_1^t}{(u + r_2^s)^{t/s}} du + \mathcal{L}_{p,q}^{s,t}(r_1, r_2), \end{aligned}$$

with  $q^{-1} + t^{-1} < s^{-1}$ .

*Proof.* By (3.1) we have

$$(w^s + r_1^s)^{t/s} \leq \mathfrak{d}_w ((w^s + r_1^s)^{t/s}) < (w^s + r_1^s)^{t/s} + ((w^s + r_1^s)^{t/s})'$$

i.e.

$$\mathcal{I}_u \leq \int_0^{[u^{1/q}]} \mathfrak{d}_w ((w^s + r_1^s)^{t/s}) dw < \mathcal{I}_u + ([u^{1/q}]^s + r_1^s)^{t/s} - r_1^t,$$

where

$$\mathcal{I}_u = \int_0^{[u^{1/q}]} (w^s + r_1^s)^{t/s} dw.$$

Now, we have

$$\begin{aligned} \mathcal{I}_u &= r_1^{t+1} \int_0^{[u^{1/q}]/r} (1 + x^s)^{t/s} dx \\ &= r_1^{t+1} \int_0^{[u^{1/q}]/r} \sum_{n=0}^{\infty} \binom{t/s}{n} x^{sn} dx \\ &= r_1^{t+1} \sum_{n=0}^{\infty} \binom{t/s}{n} \frac{([u^{1/q}]/r)^{sn+1}}{sn+1} \\ &= [u^{1/q}] r_1^t \sum_{n=0}^{\infty} \frac{t/s(t/s-1) \cdots (t/s-n+1)}{sn+1} \cdot \frac{([u^{1/q}]/r)^{sn}}{n!}. \end{aligned}$$

Since

$$\begin{aligned} t/s(t/s-1) \cdots (t/s-n+1) &= (-1)^n (-t/s)(-t/s+1) \cdots (-t/s+n-1) \\ &\equiv (-1)_n (-t/s)_n \end{aligned}$$

and

$$\frac{1}{sn+1} = \frac{1/s(1/s+1) \cdots (1/s+n-1)}{(1/s+1)(1/s+2) \cdots (1/s+n)} \equiv \frac{(1/s)_n}{(1/s+1)_n},$$

where  $(a)_n$  stands for the *Pochhammer symbol* or shifted factorial, we obtain

$$\begin{aligned} \mathcal{I}_u &= [u^{1/q}] r_1^t \sum_{n=0}^{\infty} \frac{(-t/s)_n (1/s)_n}{(1/s+1)_n} \cdot \frac{-([u^{1/q}]/r)^{sn}}{n!} \\ &= [u^{1/q}] r_1^t {}_2F_1 \left( \begin{matrix} 1/s, -t/s \\ 1+1/s \end{matrix} \middle| -r_1^{-s} [u^{1/q}]^s \right), \end{aligned}$$

where  ${}_2F_1$  denotes the familiar Gaussian hypergeometric function.

One of the upper parameters in the hypergeometric function in  $\mathcal{I}$  is negative,  $t/s = \kappa \in \mathbb{N}$ , say. This means that  $(-\kappa)_n \equiv 0$  for all addends with indices  $n \geq \kappa$ . So

$${}_2F_1 \left( \begin{matrix} 1/s, -\kappa \\ 1+1/s \end{matrix} \middle| -r_1^{-s} [u^{1/q}]^s \right) = \mathcal{P}_{\kappa-1} (-r_1^{-s} [u^{1/q}]^s),$$

where  $\mathcal{P}_{\kappa-1}(\cdot)$  is a polynomial of degree  $\deg(\mathcal{P}) = \kappa - 1$ . Now we conclude:

$$\begin{aligned} [u^{1/q}]r_1^t \mathcal{P}_{\kappa-1}(-r_1^{-s}[u^{1/q}]^s) &\leq \int_0^{[u^{1/q}]} \mathfrak{d}_w((w^s + r_1^s)^{t/s}) dw \\ &< ([u^{1/q}]^s + r_1^s)^{t/s} - r_1^t + [u^{1/q}]r_1^t \mathcal{P}_{\kappa-1}(-r_1^{-s}[u^{1/q}]^s). \end{aligned}$$

It remains to test the convergence of the integrals in  $\mathcal{R}_{p,q}^{s,t}(r_1, r_2)$  and  $\mathcal{L}_{p,q}^{s,t}(r_1, r_2)$ . But, it is easy to see that  $\mathcal{R}_{p,q}^{s,t}(r_1, r_2)$  converges for

$$\frac{1}{q} < \left(1 - \frac{1}{\kappa}\right) \frac{1}{s}.$$

For  $\mathcal{L}_{p,q}^{s,t}(r_1, r_2)$  we have

$$\mathcal{L}_{p,q}^{s,t}(r_1, r_2) \leq \text{const.} \int_1^\infty \frac{u^{1/q} |\mathcal{P}_{\kappa-1}(-r_1^{-s}[u^{1/q}]^s)|}{(u + r_2^s)^\kappa} du = \text{const.} \int_1^\infty J(u) du,$$

where the behaviour of  $J(u)$  is critical for  $u$  large. Since

$$J(u) \sim u^{1/q} (u^{s/q})^{\kappa-1} u^\kappa = u^{(1+(\kappa-1)s)/q-\kappa} \quad (u \rightarrow \infty),$$

for the convergence of the integral  $\mathcal{L}_{p,q}^{s,t}(r_1, r_2)$  we need to have

$$\frac{1 + (\kappa - 1)s}{q} - \kappa < -1,$$

such that, in conjunction with (??) gives the condition:

$$0 < \frac{1}{q} < \min \left\{ \frac{\kappa - 1}{\kappa s}, \frac{\kappa - 1}{(\kappa - 1)s + 1} \right\}.$$

This finishes the proof. □

### 3.2. Bilateral inequality for $\tilde{S}_{p,q}^{s,t}$ .

**Theorem 3.2.** *For  $p/q > t/s$  we have*

$$\tilde{\mathcal{L}}_{p,q}^{s,t}(r_1, r_2) \leq \tilde{S}_{p,q}^{s,t}(r_1, r_2) < \tilde{\mathcal{R}}_{p,q}^{s,t}(r_1, r_2),$$

where

$$\begin{aligned} \tilde{\mathcal{L}}_{p,q}^{s,t}(r_1, r_2) &= \frac{2^{p/q-t/s} p}{q} \left( \int_1^\infty \int_{2[\frac{1}{2}[u^{1/q}]]}^{[u^{1/q}]} \frac{(w^s + r_1^s)^{t/s}}{(r_2^q + u)^{p/q+1}} dw du \right. \\ &\quad \left. - \int_0^\infty \frac{(2^s [\frac{1}{2}[u^{1/q}]]^s + r_1^s)^{t/s}}{(r_2^q + u)^{p/q+1}} du \right) - \frac{2^{p/q-t/s} r_1^t}{r_2^q + 1}, \end{aligned}$$

$$\begin{aligned} \tilde{\mathcal{R}}_{p,q}^{s,t}(r_1, r_2) &= \frac{2^{p/q-t/s} p}{q} \left( \int_1^\infty \int_{2[\frac{1}{2}[u^{1/q}]]}^{[u^{1/q}]} \frac{(w^s + r_1^s)^{t/s}}{(r_2^q + u)^{p/q+1}} dw du \right. \\ &\quad \left. + \int_0^\infty \frac{([u^{1/q}]^s + r_1^s)^{t/s}}{(r_2^q + u)^{p/q+1}} du \right) + \frac{2^{p/q-t/s} r_1^t}{r_2^q + 1}. \end{aligned}$$

*Proof.* Let us denote

$$I_1 = \int_0^{[u^{1/q}]} \mathfrak{d}_w^1((w^s + r_1^s)^{t/s})dw, \quad I_{1/2} = 2 \int_0^{2[\frac{1}{2}[u^{1/q}]]} \mathfrak{d}_w^{1/2}((w^s + r_1^s)^{t/s})dw.$$

By the definition of  $\mathfrak{d}_w^1$  and  $\mathfrak{d}_w^{1/2}$  we have

$$\begin{aligned} \int_0^{[u^{1/q}]} (w^s + r_1^s)^{t/s} dw &\leq I_1 < \int_0^{[u^{1/q}]} (w^s + r_1^s)^{t/s} dw + ([u^{1/q}]^s + r_1^s)^{t/s} - r_1^t, \\ \int_0^{2[\frac{1}{2}[u^{1/q}]]} (w^s + r_1^s)^{t/s} dw &\leq I_{1/2} < \int_0^{2[\frac{1}{2}[u^{1/q}]]} (w^s + r_1^s)^{t/s} dw \\ &\quad + (2^s[\frac{1}{2}[u^{1/q}]]^s + r_1^s)^{t/s} - r_1^t. \end{aligned}$$

Since these bounds are positive, we can integrate them on  $\mathbb{R}^+$  with respect to the measure  $(r_2^q + u)^{-p/q-1}du$ , that is

$$\begin{aligned} \int_1^\infty \int_0^{[u^{1/q}]} \frac{(w^s + r_1^s)^{t/s}}{(r_2^q + u)^{p/q+1}} dw du &\leq \int_1^\infty \frac{I_1}{(r_2^q + u)^{p/q+1}} du \\ &< \int_1^\infty \int_0^{[u^{1/q}]} \frac{(w^s + r_1^s)^{t/s} + ([u^{1/q}]^s + r_1^s)^{t/s}}{(r_2^q + u)^{p/q+1}} dw du - \int_1^\infty \frac{r_1^t}{(r_2^q + u)^{p/q+1}} du \\ &= \int_1^\infty \int_0^{[u^{1/q}]} \frac{(w^s + r_1^s)^{t/s} + ([u^{1/q}]^s + r_1^s)^{t/s}}{(r_2^q + u)^{p/q+1}} dw du - \frac{r_1^t q}{p(r_2^q + 1)}, \end{aligned}$$

and

$$\begin{aligned} \int_1^\infty \int_0^{2[\frac{1}{2}[u^{1/q}]]} \frac{(w^s + r_1^s)^{t/s}}{(r_2^q + u)^{p/q+1}} dw du &\leq \int_1^\infty \frac{I_{1/2}}{(r_2^q + u)^{p/q+1}} du \\ &< \int_1^\infty \int_0^{2[\frac{1}{2}[u^{1/q}]]} \frac{(w^s + r_1^s)^{t/s} + (2^s[\frac{1}{2}[u^{1/q}]]^s + r_1^s)^{t/s}}{(r_2^q + u)^{p/q+1}} dw du - \int_1^\infty \frac{r_1^t du}{(r_2^q + u)^{p/q+1}} \\ &= \int_1^\infty \int_0^{2[\frac{1}{2}[u^{1/q}]]} \frac{(w^s + r_1^s)^{t/s} + (2^s[\frac{1}{2}[u^{1/q}]]^s + r_1^s)^{t/s}}{(r_2^q + u)^{p/q+1}} dw du - \frac{r_1^t q}{p(r_2^q + 1)}. \end{aligned}$$

Combining these two bounds we obtain the desired result, remarking that the integrals involved in  $\tilde{\mathcal{L}}_{p,q}^{s,t}$  and  $\tilde{\mathcal{R}}_{p,q}^{s,t}$  converge when  $t/s - p/q - 1 < -1$ , i.e.  $p/q > t/s$ .  $\square$

#### 4. QUOTIENT MEAN SERIES SUCH THAT CONTAIN BESSEL FUNCTION OF THE FIRST KIND

In the well known formula collection by Gradshteyn and Ryzhik we can find the following formula [3, eq. 6.623-1]:

$$\int_0^\infty e^{-\alpha x} J_\nu(\beta x) x^\nu dx = \frac{(2\beta)^\nu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}(\alpha^2 + \beta^2)^{\nu+1/2}} = \frac{\beta^\nu}{(\alpha^2 + \beta^2)^{\nu+1/2}} \cdot \frac{2^\nu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}}.$$

Putting

$$\alpha^2 = (1 - \lambda)n^q, \quad \beta^2 = \lambda n^q + r_2^q,$$



for some  $\lambda \in (0, 1)$ , we get

$$\begin{aligned} \frac{\beta^\nu}{(\alpha^2 + \beta^2)^{\nu+1/2}} &= \frac{(\lambda n^q + r_2^q)^{\nu/2}}{((1-\lambda)n^q + \lambda n^q + r_2^q)^{\nu/2}} = \frac{\lambda^{\nu/2} [(n^q + \frac{1}{\lambda} r_2^q)^{1/q}]^{\nu q/2}}{(n^q + r_2^q)^{\nu+1/2}} \\ &= \frac{\lambda^{\nu/2} [(n^q + \frac{1}{\lambda} r_2^q)^{1/q}]^{\nu q/2}}{[(n^q + r_2^q)^{1/q}]^{q(\nu+1/2)}} = \frac{\lambda^{\nu/2}}{2^{\frac{\nu+1}{2}}} \cdot \frac{(M^{[q]}(n, \lambda^{-1/q} r_2))^{\nu q/2}}{(M^{[q]}(n, r_2))^{q(\nu+1/2)}}. \end{aligned}$$

Now

$$\int_0^\infty e^{-\sqrt{(1-\lambda)n^q} x} J_\nu(\sqrt{\lambda n^q + r_2^q} x) x^\nu dx = \frac{\lambda^{\nu/2} \Gamma(\nu + \frac{1}{2})}{2^{\frac{1-\nu}{2}} \sqrt{\pi}} \cdot \frac{(M^{[q]}(n, \lambda^{-1/q} r_2))^{\nu q/2}}{(M^{[q]}(n, r_2))^{q(\nu+1/2)}},$$

i.e.

$$\frac{(M^{[q]}(n, \lambda^{-1/q} r_2))^{\nu q/2}}{(M^{[q]}(n, r_2))^{q(\nu+1/2)}} = \frac{2^{\frac{1-\nu}{2}} \sqrt{\pi}}{\lambda^{\nu/2} \Gamma(\nu + \frac{1}{2})} \int_0^\infty e^{-\sqrt{(1-\lambda)n^q} x} J_\nu(\sqrt{\lambda n^q + r_2^q} x) x^\nu dx.$$

Summing up these terms over  $n \in \mathbb{N}$  we conclude

**Theorem 4.1.** *Let  $q, r, \nu > 0$ ,  $q(1 - \frac{\nu}{2}) > 1$ ,  $0 < \lambda < 1$ . Then we have*

$$S_{q, q(\nu+\frac{1}{2})}^{q, \frac{\nu q}{2}}(\lambda^{-1/q} r, r) = \frac{2^{\frac{1-\nu}{2}} \sqrt{\pi}}{\lambda^{\nu/2} \Gamma(\nu + \frac{1}{2})} \int_0^\infty \left( \sum_{n=1}^\infty e^{-\sqrt{(1-\lambda)n^q} x} J_\nu(\sqrt{\lambda n^q + r^q} x) \right) x^\nu dx \quad (4.1)$$

Now we will derive upper bounds for the quotient mean series (4.1). Obviously

$$\begin{aligned} \left| S_{q, q(\nu+\frac{1}{2})}^{q, \frac{\nu q}{2}}(\lambda^{-1/q} r, r) \right| &< \frac{2^{\frac{1-\nu}{2}} \sqrt{\pi}}{\lambda^{\nu/2} \Gamma(\nu + \frac{1}{2})} \int_0^\infty \left( \sum_{n=1}^\infty e^{-\sqrt{(1-\lambda)n^q} x} \right. \\ &\quad \left. \times \left| J_\nu(\sqrt{\lambda n^q + r^q} x) \right| \right) x^\nu dx; \end{aligned}$$

So, to obtain an upper bound it remains to estimate  $J_\nu(\cdot)$ . Consider two Landau's bounds [5]:

$$|J_\nu(x)| \leq B_L \nu^{-(1/3)}, \quad B_L := \sqrt[3]{2} \sup_{x \in \mathbb{R}^+} (\text{Ai}(x)) \quad (4.2)$$

and

$$|J_\nu(x)| \leq C_L |x|^{-(1/3)}, \quad C_L := \sup_{x \in \mathbb{R}^+} x^{1/3} (J_0(x)),$$

where  $\text{Ai}(x)$  is the Airy function

$$\text{Ai}(x) := \frac{\pi}{3} \sqrt{\frac{x}{3}} \left( J_{-(1/3)} \left( 2 \left( \frac{x}{3} \right)^{3/2} \right) + J_{1/3} \left( 2 \left( \frac{x}{3} \right)^{3/2} \right) \right),$$

and take the different kind of bound by Olenko [7, Theorem 1]

$$\sup_{x \geq 0} \sqrt{x} |J_\nu(x)| \leq B_L \sqrt{\nu^{1/3} + \frac{\alpha_1}{\nu^{1/3}} + \frac{3\alpha_1^2}{10\nu}} =: d_0 \quad (\nu > 0). \quad (4.3)$$

Here  $\alpha_1$  is the smallest positive zero of the Airy function  $\text{Ai}(x)$  and  $B_L$  is the first Landau constant.

Using the first Landau's bound, we get:

**Theorem 4.2.** *It holds holds true*

$$\left| S_{q,q(\nu+\frac{1}{2})}^{q,\frac{\nu q}{2}}(\lambda^{-1/q}r, r) \right| \leq \frac{2^{\frac{1-\nu}{2}} \sqrt{\pi} B_L \Gamma(\nu+2)}{\lambda^{\nu/2}(\nu-2/q+1)(1-\lambda)^{(\nu+1)/2} \Gamma(\nu+1/2) \nu^{1/3}}, \quad (4.4)$$

for all  $0 < \lambda < 1$ ,  $q > \frac{2}{3+\nu}$ ,  $\nu > -1$ .

*Proof.* With the help of (4.2) we get

$$\left| S_{q,q(\nu+\frac{1}{2})}^{q,\frac{\nu q}{2}}(\lambda^{-1/q}r, r) \right| \leq \frac{2^{\frac{1-\nu}{2}} \sqrt{\pi} B_L}{\lambda^{\nu/2} \Gamma(\nu+1/2) \nu^{1/3}} \int_0^\infty \left( \sum_{n=1}^\infty e^{-\sqrt{1-\lambda} n^{q/2} x} \right) x^\nu dx. \quad (4.5)$$

The sum in the integral is a Dirichlet series which, via Laplace integral formula, becomes

$$\begin{aligned} \sum_{n=1}^\infty e^{-\sqrt{1-\lambda} n^{q/2} x} &= x \int_0^\infty e^{-xt} \left( \sum_{n: (1-\lambda)^{1/2} n^{q/2} \leq t} 1 \right) dt \\ &= x \int_0^\infty e^{-xt} [t^{2/q} (1-\lambda)^{-1/q}] dt \\ &= x \int_{\sqrt{1-\lambda}}^\infty e^{-xt} [t^{2/q} (1-\lambda)^{-1/q}] dt. \end{aligned}$$

Rewriting all these expressions in (4.5), we deduce:

$$\begin{aligned} \left| S_{q,q(\nu+\frac{1}{2})}^{q,\frac{\nu q}{2}}(\lambda^{-1/q}r, r) \right| &\leq \frac{2^{\frac{1-\nu}{2}} \sqrt{\pi} B_L}{\lambda^{\nu/2} \Gamma(\nu+1/2) \nu^{1/3}} \int_{\sqrt{1-\lambda}}^\infty \int_0^\infty x^{\nu+1} e^{-xt} \left[ \frac{t^{2/q}}{(1-\lambda)^{1/q}} \right] dt dx \\ &= \frac{2^{\frac{1-\nu}{2}} \sqrt{\pi} B_L}{\lambda^{\nu/2} \Gamma(\nu+1/2) \nu^{1/3}} \int_{\sqrt{1-\lambda}}^\infty \left[ \frac{t^{2/q}}{(1-\lambda)^{1/q}} \right] \left( \int_0^\infty x^{\nu+1} e^{-xt} dx \right) dt \\ &= \frac{2^{\frac{1-\nu}{2}} \sqrt{\pi} B_L \Gamma(\nu+2)}{\lambda^{\nu/2} \Gamma(\nu+1/2) \nu^{1/3}} \int_{\sqrt{1-\lambda}}^\infty \frac{[t^{2/q} (1-\lambda)^{-1/q}]}{t^{\nu+2}} dt. \end{aligned}$$

Since  $q > 2/(3+\nu)$  and

$$\begin{aligned} \int_{\sqrt{1-\lambda}}^\infty \frac{[t^{2/q} (1-\lambda)^{-1/q}]}{t^{\nu+2}} dt &\leq \frac{1}{(1-\lambda)^{1/q}} \int_{\sqrt{1-\lambda}}^\infty \frac{dt}{t^{\nu-2/q+2}} \\ &= \frac{1}{(1-\lambda)^{1/q}} \cdot \frac{1}{\nu-2/q+1} \cdot \frac{1}{((1-\lambda)^{1/2})^{\nu-2/q+1}} \\ &= \frac{1}{(\nu-2/q+1)(1-\lambda)^{\nu/2+1/2}} < \infty, \end{aligned}$$

the inequality (4.4) is not redundant, therefore the integral converges. So the result.  $\square$

By the second Landau's estimate we

**Theorem 4.3.** *For a special case of quotient mean series the following inequality holds true*

$$\left| S_{q,q(\nu+\frac{1}{2})}^{q,\frac{\nu q}{2}}(\lambda^{-1/q}r, r) \right| \leq \frac{2^{\frac{1-\nu}{2}} \sqrt{\pi} C_L \Gamma(\nu + 5/3)}{\lambda^{\nu/2} \Gamma(\nu + 1/2)} \times \int_{\sqrt{1-\lambda}}^{\infty} \frac{\int_0^{\left[\frac{t^{2/q}}{(1-\lambda)^{1/q}}\right]} \mathfrak{D}_w((\lambda w^q + r^q)^{-1/6}) dw}{t^{\nu+5/3}} dt,$$

where  $0 < \lambda < 1$ ,  $\nu > -2/3$ .

*Proof.* Using (??) we get

$$\left| S_{q,q(\nu+\frac{1}{2})}^{q,\frac{\nu q}{2}}(\lambda^{-1/q}r, r) \right| \leq \frac{2^{\frac{1-\nu}{2}} \sqrt{\pi} C_L}{\lambda^{\nu/2} \Gamma(\nu + 1/2)} \int_0^{\infty} \left( \sum_{n=1}^{\infty} \frac{e^{-\sqrt{1-\lambda} n^{q/2} x}}{(\lambda_n^q + r^q)^{1/6}} \right) x^{\nu-\frac{1}{3}} dx. \quad (4.6)$$

The sum in the integral is a Dirichlet series which, via Laplace integral formula and Euler–Maclaurin summation formula, becomes

$$\begin{aligned} \sum_{n=1}^{\infty} e^{-\sqrt{1-\lambda} n^{q/2} x} (\lambda_n^q + r^q)^{-1/6} &= x \int_0^{\infty} e^{-xt} \left( \sum_{n=1}^{\left[ \frac{t^{2/q} (1-\lambda)^{-1/q} \right]} (\lambda_n^q + r^q)^{-1/6} \right) dt \\ &= x \int_0^{\infty} e^{-xt} \int_0^{\left[ \frac{t^{2/q} (1-\lambda)^{-1/q} \right]} \mathfrak{D}_w((\lambda_w^q + r^q)^{-1/6}) dw dt. \end{aligned}$$

Putting all back to (4.6), we have

$$\begin{aligned} \left| S_{q,q(\nu+\frac{1}{2})}^{q,\frac{\nu q}{2}}(\lambda^{-1/q}r, r) \right| &\leq \frac{2^{\frac{1-\nu}{2}} \sqrt{\pi} C_L}{\lambda^{\nu/2} \Gamma(\nu + 1/2)} \\ &\quad \times \int_0^{\infty} x^{\nu+2/3} \int_0^{\infty} e^{-xt} \int_0^{\left[ \frac{t^{2/q}}{(1-\lambda)^{1/q}} \right]} \mathfrak{D}_w((\lambda_w^q + r^q)^{-1/6}) dw dt dx \\ &= \frac{2^{\frac{1-\nu}{2}} \sqrt{\pi} C_L}{\lambda^{\nu/2} \Gamma(\nu + 1/2)} \int_0^{\infty} \int_0^{\left[ \frac{t^{2/q} (1-\lambda)^{-1/q} \right]} \mathfrak{D}_w((\lambda_w^q + r^q)^{-1/6}) \\ &\quad \times \left( \int_0^{\infty} x^{\nu+\frac{2}{3}} e^{-xt} dx \right) dw dt \\ &= \frac{2^{\frac{1-\nu}{2}} \sqrt{\pi} C_L \Gamma(\nu + 5/3)}{\lambda^{\nu/2} \Gamma(\nu + 1/2)} \int_{\sqrt{1-\lambda}}^{\infty} \frac{\int_0^{\left[ \frac{t^{2/q}}{(1-\lambda)^{1/q}} \right]} \mathfrak{D}_w((\lambda w^q + r^q)^{-1/6}) dw}{t^{\nu+5/3}} dt. \end{aligned}$$

It is easy to show that the righthandside integral converges for  $\nu > -2/3$ , and since  $\left[ \frac{t^{2/q} (1-\lambda)^{-1/q} \right] \equiv 0$  for  $0 \leq \frac{t^{2/q} (1-\lambda)^{-1/q} < 1$ , we integrate from  $\sqrt{1-\lambda}$  over  $t$ .  $\square$

**Theorem 4.4.** *For a special case of quotient mean series the following inequality holds true*

$$\begin{aligned} \left| S_{q,q(\nu+\frac{1}{2})}^{q,\frac{\nu q}{2}}(\lambda^{-1/q}r, r) \right| &\leq \frac{2^{\frac{1-\nu}{2}}\sqrt{\pi}d_0}{\lambda^{\nu/2}\Gamma(\nu+1/2)} \\ &\quad \times \int_{\sqrt{(1-\lambda)}}^{\infty} \int_0^{\left[\frac{t^{2/q}}{(1-\lambda)^{1/q}}\right]} \frac{\mathfrak{d}_w((\lambda w^q + r^q)^{-1/2})}{t^2} dw dt. \end{aligned}$$

*Proof.* Making use of (4.3), we get:

$$x^\nu |J_\nu(\sqrt{\lambda n^q + r^q}x)| = x^{\nu-1/2} x^{1/2} \left| J_\nu((\lambda n^q + r^q)x) \right| \leq x^{\nu-1/2} \frac{d_0}{(\lambda n^q + r^q)^{1/2}},$$

that is

$$\begin{aligned} \left| S_{q,q(\nu+\frac{1}{2})}^{q,\frac{\nu q}{2}}(\lambda^{-1/q}r, r) \right| &\leq \frac{2^{\frac{1-\nu}{2}}\sqrt{\pi}}{\lambda^{\nu/2}\Gamma(\nu+1/2)} \int_0^\infty \sum_{n=1}^\infty \frac{d_0 e^{-\sqrt{1-\lambda}n^{q/2}x}}{(\lambda n^q + r^q)^{1/2}} x^{\nu-q/2} dx \\ &= \frac{2^{\frac{1-\nu}{2}}\sqrt{\pi}d_0}{\lambda^{\nu/2}\Gamma(\nu+1/2)} \int_0^\infty \left( \sum_{n=1}^\infty \frac{1}{(\lambda n^q + r^q)^{1/2}} e^{-\sqrt{1-\lambda}n^{q/2}x} \right) x^{\nu-q/2} dx. \end{aligned} \quad (4.7)$$

The inner Dirichlet series we apply the Laplace integral formula (1.1), which gives us

$$\sum_{n=1}^\infty \frac{e^{-\sqrt{1-\lambda}n^{q/2}x}}{(\lambda n^q + r^q)^{1/2}} = x \int_0^\infty e^{-xt} \left( \sum_{n=1}^{\lfloor t^{2/q}(1-\lambda)^{-1/q} \rfloor} (\lambda n^q + r^q)^{-1/2} \right) dt.$$

We write the sum in the integrand by Euler–Maclaurin summation formula

$$\sum_{n=1}^\infty \frac{e^{-\sqrt{1-\lambda}n^{q/2}x}}{(\lambda n^q + r^q)^{1/2}} = x \int_0^\infty \int_0^{\lfloor t^{2/q}(1-\lambda)^{-1/q} \rfloor} e^{-xt} \mathfrak{d}_w((\lambda w^q + r^q)^{-1/2}) dw dt. \quad (4.8)$$

Putting (4.8) in (4.7) results with

$$\begin{aligned} \left| S_{q,q(\nu+\frac{1}{2})}^{q,\frac{\nu q}{2}}(\lambda^{-1/q}r, r) \right| &\leq \frac{2^{\frac{1-\nu}{2}}\sqrt{\pi}d_0}{\lambda^{\nu/2}\Gamma(\nu+1/2)} \\ &\quad \times \int_0^\infty \int_0^\infty \int_0^{\left[\frac{t^{2/q}}{(1-\lambda)^{1/q}}\right]} x e^{-xt} \mathfrak{d}_w((\lambda w^q + r^q)^{-1/2}) dw dt dx \\ &= \frac{2^{\frac{1-\nu}{2}}\sqrt{\pi}d_0}{\lambda^{\nu/2}\Gamma(\nu+1/2)} \\ &\quad \times \int_0^\infty \int_0^{\left[\frac{t^{2/q}}{(1-\lambda)^{1/q}}\right]} \mathfrak{d}_w((\lambda w^q + r^q)^{-1/2}) \left( \int_0^\infty x e^{-xt} dx \right) dw dt \\ &= \frac{2^{\frac{1-\nu}{2}}\sqrt{\pi}d_0}{\lambda^{\nu/2}\Gamma(\nu+1/2)} \int_{(1-\lambda)^{1/2}}^\infty \int_0^{\left[\frac{t^{2/q}}{(1-\lambda)^{1/q}}\right]} \mathfrak{d}_w((\lambda w^q + r^q)^{-1/2}) t^{-2} dw dt, \end{aligned}$$

we finish the proof.  $\square$

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