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ON LEBESGUE TYPE DECOMPOSITION FOR COVARIANT COMPLETELY POSITIVE MAPS ON C^*-ALGEBRAS

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ABSTRACT. We show that there is an affine order isomorphism between completely positive maps from a C^*-algebra A to the C^*-algebra $L(H)$ of all bounded linear operators on a Hilbert space H, u-covariant with respect to a C^*-dynamical system (G, α, A) and u-covariant completely positive maps from the crossed product $A \times_\alpha G$ to $L(H)$, which preserves the Lebesgue decomposition.

1. INTRODUCTION AND PRELIMINARIES

This note is motivated by the applications of the theory of completely positive maps to quantum information theory (operator valued completely positive maps on C^*-algebras are used as mathematical model for quantum operations [9]) and quantum probability [8].

A completely positive map from a C^*-algebra A to the C^*-algebra $L(H)$ of all bounded linear operators on a Hilbert space H is a linear map $\varphi : A \to L(H)$ such that for all positive integers n, the maps $\varphi^{(n)} : M_n(A) \to L(H^n)$ defined by

$$\varphi^{(n)}\left([a_{ij}]_{i,j=1}^n\right) = [\varphi(a_{ij})]_{i,j=1}^n ,$$

where $M_n(A)$ denotes the C^*-algebra of all $n \times n$ matrices over A, are positive, that is $\varphi^{(n)}\left(\left([a_{ij}]_{i,j=1}^n\right)^* [a_{ij}]_{i,j=1}^n\right) \geq 0$ for all $[a_{ij}]_{i,j=1}^n \in M_n(A)$. In [11] it is

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shown that a completely positive map $\varphi : A \to L(H)$ is of the form

$$\varphi(a) = V_\varphi^* \Phi_\varphi(a) V_\varphi ,$$

where Φ_φ is a $*$-representation of A on a Hilbert space H_φ and V_φ is a bounded linear operator from H to H_φ. The cone $\mathcal{CP}(A,H)$ of completely positive maps from A to $L(H)$ defines a natural partial order relation and this relation is characterized by the Radon–Nikodým derivatives. In general, the Radon–Nikodým derivative is not a bounded linear operator. Two completely positive maps from A to $L(H)$ are comparable (with respect to the order relation) if and only if the Radon–Nikodým derivative is a bounded linear operator (see, [1, 3, 8]). But not all completely positive maps can be compared. In [1, 3, 4, 8] is introduced the notion of absolute continuity for completely positive maps and it is shown that given two completely positive maps φ and θ from A to $L(H)$, which are not comparable, then φ is absolutely continuous with respect to θ if and only if the Radon–Nikodým derivative of φ with respect to θ is a unbounded positive self-adjoint operator. In [8], Parthasarthy extended the classical Lebesgue decomposition theorem for the unital operator valued completely positive maps on C^*-algebras. In Section 2 of this note, we extend these results to the case of operator valued covariant completely positive maps on C^*-algebras.

A C^*-dynamical system is a triple (G, α, A), where G is a locally compact group, A is a C^*-algebra and α is a continuous action of G on A (this is, $g \longmapsto \alpha_g$ is a group morphism from G to the group of automorphisms of A and the map $g \longmapsto \alpha_g(a)$ from G to A is continuous for all $a \in A$). Let $g \longmapsto u_g$ be a unitary representation of G on a Hilbert space H. A completely positive linear map $\varphi : A \to L(H)$ is u-covariant with respect to the C^*-dynamical system (G, α, A) if

$$\varphi(\alpha_g(a)) = u_g \varphi(a) u_g^*$$

for all $g \in G$ and for all $a \in A$. Paulsen [7] obtained a covariant version of the Stinespring construction. Let $\varphi : A \to L(H)$ be a u-covariant completely positive map. If $(\Phi_\varphi, v^\varphi, H_\varphi, V_\varphi)$ is the covariant Stinespring construction associated to φ, the map $\widehat{\varphi} : C_c(G, A) \to L(H)$ defined by

$$\widehat{\varphi}(f) = \int_G \varphi(f(g)) u_g dg ,$$

where $C_c(G, A)$ denotes the vector space of all continuous functions from G to A with compact support and dg denotes a left Haar measure on G, extends to a completely positive map from the crossed product $A \times_\alpha G$ of A by α to $L(H)$, denoted also by $\widehat{\varphi}$ (see, for example, [6, 10]). Moreover, the Stinespring construction associated with $\widehat{\varphi}$ is unitarily equivalent with $(\Phi_\varphi \times v^\varphi, H_\varphi, V_\varphi)$, where $\Phi_\varphi \times v^\varphi$ is the integral form of the covariant representation $(\Phi_\varphi, v^\varphi, H_\varphi)$. In Section 3, we show that the map $\varphi \mapsto \widehat{\varphi}$ from u-covariant completely positive maps from A to $L(H)$ to u-covariant completely positive linear maps from $A \times_\alpha G$ to $L(H)$ is an affine order isomorphism, which preserves the Lebesgue decomposition.

2. Covariant completely positive maps

Let (G, α, A) be a C^*-dynamical system and let $g \longmapsto u_g$ be a unitary representation of G on a Hilbert space H and let $\mathcal{CP}((G,\alpha,A),H,u) = \{\varphi \in \mathcal{CP}(A,H); \varphi$ is u-covariant with respect to $(G,\alpha,A)\}$.

In [7, Theorem 2.1] and [2, Theorem 4] it is given a covariant version of the Stinespring construction. Let $\varphi \in \mathcal{CP}((G,\alpha,A),H,u)$. Then there is a quadruple $(\Phi_\varphi, v^\varphi, H_\varphi, V_\varphi)$ consisting of a covariant representation $(\Phi_\varphi, v^\varphi, H_\varphi)$ of (G,α,A) and an element V_φ in $L(H, H_\varphi)$ such that

(1) $V_\varphi u_g = v_g^\varphi V_\varphi$ for all $g \in G$;
(2) $\varphi(a) = V_\varphi^* \Phi_\varphi(a) V_\varphi$ for all $a \in A$;
(3) $[\Phi_\varphi(A) V_\varphi H] = H_\varphi$, where $[\Phi_\varphi(A) V_\varphi H]$ denotes the closed linear subspace of H_φ generated by $\{\Phi_\varphi(a) V_\varphi \xi; a \in A, \xi \in H\}$.

Moreover, the quadruple $(\Phi_\varphi, v^\varphi, H_\varphi, V_\varphi)$ is unique with the properties $(1)-(3)$ in the sense that if (Φ, v, K, V) is another quadruple consisting of a covariant representation (Φ, v, K) of (G,α,A) and an element V in $L(H,K)$, which verifies the relations $(1)-(3)$, then there is a unitary operator $U : K \to H_\varphi$ such that

(1) $\Phi(a) = U^* \Phi_\varphi(a) U$ for all $a \in A$;
(2) $v_g = U^* v_g^\varphi U$ for all $g \in G$;
(3) $V = U^* V_\varphi$.

Let $\varphi, \theta \in \mathcal{CP}((G,\alpha,A),H,u)$. We say that $\varphi \le \theta$ if $\theta - \varphi \in \mathcal{CP}((G,\alpha,A),H,u)$. This relation is a partial order relation on $\mathcal{CP}((G,\alpha,A),H,u)$. We say that φ is *uniformly dominated* by θ, and we write $\varphi \le_{\mathcal{U}} \theta$, if there is $\lambda > 0$ such that $\varphi \le \lambda\theta$. This relation is a partial preorder relation on $\mathcal{CP}((G,\alpha,A),H,u)$. Clearly, if $\varphi \le \theta$, then $\varphi \le_{\mathcal{U}} \theta$.

Suppose that $\varphi \le_{\mathcal{U}} \theta$. Then there is a bounded linear operator $J_\theta(\varphi) : H_\theta \to H_\varphi$ such that $J_\theta(\varphi)(\Phi_\theta(a) V_\theta \xi) = \Phi_\varphi(a) V_\varphi \xi$ for all $\xi \in A$. Moreover,

$$J_\theta(\varphi)\Phi_\theta(a) = \Phi_\varphi(a) J_\theta(\varphi) \text{ for all } a \in A$$

and

$$J_\theta(\varphi) v_g^\theta = v_g^\varphi J_\theta(\varphi) \text{ for all } g \in G.$$

Let $\Delta_\theta(\varphi) = J_\theta(\varphi)^* J_\theta(\varphi)$. Then $\Delta_\theta(\varphi)$ is a positive element in $\Phi_\theta(A)' \cap v^\theta(G)'$ and

$$\varphi(a) = V_\theta^* \Delta_\theta(\varphi) \Phi_\theta(a) V_\theta \text{ for all } a \in A.$$

Moreover, $\Delta_\theta(\varphi)$ is unique with these properties. If $\varphi \le \theta$, then $\Delta_\theta(\varphi) \le I_{H_\theta}$. The positive linear operator $\Delta_\theta(\varphi)$ is called the *Radon–Nikodým derivative* of φ with respect to θ.

In [5] it is shown that the map $\varphi \mapsto \Delta_\theta(\varphi)$ from $\{\varphi \in \mathcal{CP}((G,\alpha,A),H,u); \varphi \le \theta\}$ to $\left\{T \in \Phi_\theta(A)' \cap v^\theta(G)'; 0 \le T \le I_{H_\theta}\right\}$ is an affine order isomorphism and its inverse is given by $T \mapsto \theta_T$, where $\theta_T(a) = V_\theta^* T \Phi_\theta(a) V_\theta$ for all $a \in A$.

In the same manner, it can be shown that the map $\varphi \mapsto \Delta_\theta(\varphi)$ from $\{\varphi \in \mathcal{CP}((G,\alpha,A),H,u); \varphi \le_{\mathcal{U}} \theta\}$ to $\left\{T \in \Phi_\theta(A)' \cap v^\theta(G)'; 0 \le T\right\}$ is an affine order isomorphism.

Let $\varphi, \theta \in \mathcal{CP}\left(\left(G,\alpha,A\right),H,u\right)$ with $\varphi \leq \theta$. As in the case of completely positive maps from A to $L(H)$ we can recover the covariant Stinesprning construction of φ from the covariant Stinespring construction of θ (see, [3]). Since $\Delta_\theta\left(\varphi\right) \in \Phi_\theta\left(A\right)' \cap v^\theta\left(G\right)'$, $P_{\ker(\Delta_\theta(\varphi))}, P_{H_\theta \ominus \ker(\Delta_\theta(\varphi))} \in \Phi_\theta\left(A\right)' \cap v^\theta\left(G\right)'$ and $\left(\Phi_\theta|_{H_\theta \ominus \ker(\Delta_\theta(\varphi))}, v^\theta|_{H_\theta \ominus \ker(\Delta_\theta(\varphi))}, H_\theta \ominus \ker\left(\Delta_\theta\left(\varphi\right)\right)\right)$ is a covariant representation of (G,α,A). Moreover, $P_{H_\theta \ominus \ker(\Delta_\theta(\varphi))}\Delta_\theta\left(\varphi\right)^{\frac{1}{2}}V_\theta \in L(H, H_\theta \ominus \ker\left(\Delta_\theta\left(\varphi\right)\right)$. A simple calculus shows that

$$\varphi\left(a\right) = \left(P_{H_\theta \ominus \ker(\Delta_\theta(\varphi))}\Delta_\theta\left(\varphi\right)^{\frac{1}{2}}V_\theta\right)^* \ \Phi_\theta|_{H_\theta \ominus \ker(\Delta_\theta(\varphi))}\left(a\right)\left(P_{H_\theta \ominus \ker(\Delta_\theta(\varphi))}\Delta_\theta\left(\varphi\right)^{\frac{1}{2}}V_\theta\right)$$

for all $a \in A$ and

$$\left(P_{H_\theta \ominus \ker(\Delta_\theta(\varphi))}\Delta_\theta\left(\varphi\right)^{\frac{1}{2}}V_\theta\right)u_g = v_g^\theta|_{H_\theta \ominus \ker(\Delta_\theta(\varphi))}\left(P_{H_\theta \ominus \ker(\Delta_\theta(\varphi))}\Delta_\theta\left(\varphi\right)^{\frac{1}{2}}V_\theta\right)$$

for all $g \in G$, and since

$$\begin{aligned}
&\left[\Phi_\theta|_{H_\theta \ominus \ker(\Delta_\theta(\varphi))}\left(A\right)\left(P_{H_\theta \ominus \ker(\Delta_\theta(\varphi))}\Delta_\theta\left(\varphi\right)^{\frac{1}{2}}V_\theta\right)H\right] \\
= &\left[P_{H_\theta \ominus \ker(\Delta_\theta(\varphi))}\Phi_\theta\left(A\right)\Delta_\theta\left(\varphi\right)^{\frac{1}{2}}V_\theta H\right] \\
= &\left[P_{H_\theta \ominus \ker(\Delta_\theta(\varphi))}\Delta_\theta\left(\varphi\right)^{\frac{1}{2}}\Phi_\theta\left(A\right)V_\theta H\right] \\
= &\left[P_{H_\theta \ominus \ker(\Delta_\theta(\varphi))}\Delta_\theta\left(\varphi\right)^{\frac{1}{2}}H_\theta\right] \\
= &\ H_\theta \ominus \ker\left(\Delta_\theta\left(\varphi\right)\right)
\end{aligned}$$

$\left(\Phi_\theta|_{H_\theta \ominus \ker(\Delta_\theta(\varphi))}, v^\theta|_{H_\theta \ominus \ker(\Delta_\theta(\varphi))}, H_\theta \ominus \ker\left(\Delta_\theta\left(\varphi\right)\right), P_{H_\theta \ominus \ker(\Delta_\theta(\varphi))}\Delta_\theta\left(\varphi\right)^{\frac{1}{2}}V_\theta\right)$ is the covariant Stinespring construction associated to φ.

Let $\varphi, \theta \in \mathcal{CP}\left(\left(G,\alpha,A\right),H,u\right)$. We say that φ is *uniformly equivalent* to θ, and we write $\varphi \equiv_{\mathcal{U}} \theta$, if $\varphi \leq_{\mathcal{U}} \theta$ and $\theta \leq_{\mathcal{U}} \varphi$.

Proposition 2.1. *Let $\varphi, \theta \in \mathcal{CP}\left(\left(G,\alpha,A\right),H,u\right)$. If $\varphi \equiv_{\mathcal{U}} \theta$, then the covariant representations $\left(\Phi_\theta, v^\theta, H_\theta\right)$ and $\left(\Phi_\varphi, v^\varphi, H_\varphi\right)$ of (G,α,A) associated to θ and φ are unitarily equivalent.*

Proof. Since $\varphi \equiv_{\mathcal{U}} \theta$, $J_\theta\left(\varphi\right)$ is invertible [8]. Then $\Delta_\theta(\varphi)$ is invertible and so there is a unitary operator $U : H_\theta \to H_\varphi$ such that $J_\theta\left(\varphi\right) = U\Delta_\theta(\varphi)^{\frac{1}{2}}$. Moreover, $U\Phi_\theta\left(a\right) = \Phi_\varphi\left(a\right)U$ for all $a \in A$ [8]. Let $g \in G$. From

$$\begin{aligned}
\left(Uv_g^\theta\right)\left(\Delta_\theta(\varphi)^{\frac{1}{2}}\Phi_\theta\left(a\right)V_\theta\xi\right) &= U\Delta_\theta(\varphi)^{\frac{1}{2}}\Phi_\theta\left(\alpha_g\left(a\right)\right)v_g^\theta V_\theta\xi \\
&= J_\theta\left(\varphi\right)\Phi_\theta\left(\alpha_g\left(a\right)\right)V_\theta u_g\xi \\
&= \Phi_\varphi\left(\alpha_g\left(a\right)\right)V_\varphi u_g\xi \\
&= v_g^\varphi\Phi_\varphi\left(a\right)v_{g^{-1}}^\varphi V_\varphi u_g\xi = v_g^\varphi\Phi_\varphi\left(a\right)V_\varphi\xi \\
&= v_g^\varphi J_\theta\left(\varphi\right)\Phi_\theta\left(a\right)V_\theta\xi \\
&= \left(v_g^\varphi U\right)\left(\Delta_\theta(\varphi)^{\frac{1}{2}}\Phi_\theta\left(a\right)V_\theta\xi\right)
\end{aligned}$$

for all $a \in A$ and for all $\xi \in H$, and taking into account that $\Delta_\theta(\varphi)^{\frac{1}{2}}$ is surjective and $[\Phi_\theta(A) V_\theta H] = H_\theta$, we deduce that $U v_g^\theta = v_g^\varphi U$. Therefore, the covariant representations $\left(\Phi_\theta, v^\theta, H_\theta\right)$ and $(\Phi_\varphi, v^\varphi, H_\varphi)$ of (G, α, A) associated to θ and φ are unitarily equivalent. □

Let $\varphi, \theta \in \mathcal{CP}((G, \alpha, A), H, u)$. We say that φ is *θ-absolutely continuous*, and we write $\varphi \ll \theta$, if there is an increasing sequence $(\varphi_n)_n$ in $\mathcal{CP}((G, \alpha, A), H, u)$ such that $\varphi_n \leq_{\mathcal{U}} \theta$ for all positive integers n and the sequence $(\varphi_n(a))_n$ converges to $\varphi(a)$ with respect to the strong topology on $L(H)$ for each $a \in A$.

Lemma 2.2. *Let $\varphi \in \mathcal{CP}(A, H)$ and let $\{\varphi_n\}_n$ be an increasing sequence in $\mathcal{CP}(A, H)$. Then $\{\varphi_n(a)\}_n$ converges strongly to $\varphi(a)$ for each $a \in A$ if and only if $\varphi_n \leq \varphi$ for all positive integers n and the sequence $\{\Delta_\varphi(\varphi_n)\}_n$ converges strongly to I_H.*

Proof. Let $[a_{ij}]_{i,j=1}^m \in M_m(A)$. It is not difficult to check that the sequence $\left\{\varphi_n^{(m)}\left(\left([a_{ij}]_{i,j=1}^m\right)^* [a_{ij}]_{i,j=1}^m\right)\right\}_n$ converges strongly to $\varphi^{(m)}\left(\left([a_{ij}]_{i,j=1}^m\right)^* [a_{ij}]_{i,j=1}^m\right)$, and since $\left\{\varphi_n^{(m)}\left(\left([a_{ij}]_{i,j=1}^m\right)^* [a_{ij}]_{i,j=1}^m\right)\right\}_n$ is an increasing sequence of positive operators, $\varphi_n^{(m)}\left(\left([a_{ij}]_{i,j=1}^m\right)^* [a_{ij}]_{i,j=1}^m\right) \leq \varphi^{(m)}\left(\left([a_{ij}]_{i,j=1}^m\right)^* [a_{ij}]_{i,j=1}^m\right)$ for all positive integers n. Therefore, $\varphi - \varphi_n \in \mathcal{CP}(A, H)$ and so $\varphi_n \leq \varphi$ for all positive integers n. From

$$
\begin{aligned}
& \left\|\Delta_\varphi(\varphi_n) \Phi_\varphi(a) V_\varphi \xi - \Phi_\varphi(a) V_\varphi \xi\right\|^2 \\
= & \left\langle (I_H - \Delta_\varphi(\varphi_n)) \Phi_\varphi(a) V_\varphi \xi, (I_H - \Delta_\varphi(\varphi_n)) \Phi_\varphi(a) V_\varphi \xi \right\rangle \\
\leq & \left\langle (I_H - \Delta_\varphi(\varphi_n)) \Phi_\varphi(a) V_\varphi \xi, \Phi_\varphi(a) V_\varphi \xi \right\rangle \\
= & \left\langle V_\varphi^* (I_H - \Delta_\varphi(\varphi_n)) \Phi_\varphi(a^* a) V_\varphi \xi, \xi \right\rangle \\
= & \left|\left\langle (\varphi(a^* a) - \varphi_n(a^* a)) \xi, \xi \right\rangle\right| \\
\leq & \left\|(\varphi(a^* a) - \varphi_n(a^* a)) \xi\right\| \|\xi\|
\end{aligned}
$$

for all $\xi \in H$, for all $a \in A$ and taking into account that $[\Phi_\varphi(A) V_\varphi H] = H_\varphi$, we deduce that the sequence $\{\Delta_\varphi(\varphi_n)\}_n$ converges strongly to I_H.

Conversely, if $\varphi_n \leq \varphi$ for all positive integers n and if the sequence $\{\Delta_\varphi(\varphi_n)\}_n$ converges strongly to I_H, then it is easy to verify that $\{\varphi_n(a)\}_n$ converges strongly to $\varphi(a)$ for each $a \in A$. □

Remark 2.3. Let $\varphi, \theta \in \mathcal{CP}((G, \alpha, A), H, u)$. Then φ is θ-absolutely continuous if and only if there is an increasing sequence $\{\varphi_n\}_n$ in $\mathcal{CP}((G, \alpha, A), H, u)$ such that $\varphi_n \leq_{\mathcal{U}} \theta$ and $\varphi_n \leq \varphi$ for all positive integers n and the sequence $\{\Delta_\varphi(\varphi_n)\}_n$ converges strongly to I_H.

As in the case of completely positive maps on C^*-algebras [3, Theorem 2.11] or [8], we have the following theorem.

Theorem 2.4. *Let $\varphi, \theta \in \mathcal{CP}((G, \alpha, A), H, u)$ and let $\left(\Phi_\theta, v^\theta, H_\theta, V_\theta\right)$ be the Stinespring construction associated to θ. Then φ is θ-absolutely continuous if and only if there is a unique positive selfadjoint linear operator $\Delta_\theta(\varphi)$ in H_θ such that*

(1) $\Delta_\theta(\varphi)$ *is affiliated with* $\Phi_\theta(A)'$ *and* $v^\theta(G)'$;
(2) $\Phi_\theta(A)V_\theta H$ *is a core for* $\Delta_\theta(\varphi)^{\frac{1}{2}}$;
(3) $\Delta_\theta(\varphi)^{\frac{1}{2}}V_\theta \in L(H, H_\theta)$;
(4) $\varphi(a) = \left(\Delta_\theta(\varphi)^{\frac{1}{2}}V_\theta\right)\Phi_\theta(a)\left(\Delta_\theta(\varphi)^{\frac{1}{2}}V_\theta\right)$ *for all* $a \in A$.

Proof. By [3, Theorem 2.11], there is a unique positive selfadjoint linear operator $\Delta_\theta(\varphi)$ in H_θ such that

(1) $\Delta_\theta(\varphi)$ is affiliated with $\Phi_\theta(A)'$;
(2) $\Phi_\theta(A)V_\theta H$ is a core for $\Delta_\theta(\varphi)^{\frac{1}{2}}$;
(3) $\Delta_\theta(\varphi)^{\frac{1}{2}}V_\theta \in L(H, H_\theta)$;
(4) $\varphi(a) = \left(\Delta_\theta(\varphi)^{\frac{1}{2}}V_\theta\right)\Phi_\theta(a)\left(\Delta_\theta(\varphi)^{\frac{1}{2}}V_\theta\right)$ for all $a \in A$.

To prove the theorem, it remains to show that $\Delta_\theta(\varphi)$ is affiliated with $v^\theta(G)'$. By the proof of Theorem 7.13 [3], $\Delta_\theta(\varphi) = \Delta_\rho(\theta)^{-\frac{1}{2}}\Delta_\rho(\varphi)\Delta_\rho(\theta)^{-\frac{1}{2}}$, where $\rho = \varphi + \theta$ and $\Delta_\rho(\theta)$ is supposed to be injective. Then, modulo a unitary equivalence, $H_\theta = H_\rho$ and $v^\theta = v^\rho$. From $v_g^\rho\Delta_\rho(\theta)^{\frac{1}{2}} = \Delta_\rho(\theta)^{\frac{1}{2}}v_g^\rho$ for all $g \in G$, we deduce that $\Delta_\rho(\theta)^{-\frac{1}{2}}v_g^\rho\Delta_\rho(\theta)^{\frac{1}{2}} = v_g^\rho$ for all $g \in G$, and then

$$\begin{aligned} v_g^\theta\Delta_\theta(\varphi)\left(\Delta_\rho(\theta)^{\frac{1}{2}}\Phi_\rho(a)V_\rho\xi\right) &= v_g^\theta\Delta_\rho(\theta)^{-\frac{1}{2}}\Delta_\rho(\varphi)\Delta_\rho(\theta)^{-\frac{1}{2}}\Delta_\rho(\theta)^{\frac{1}{2}}\Phi_\rho(a)V_\rho\xi \\ &= v_g^\rho\Delta_\rho(\theta)^{-\frac{1}{2}}\Delta_\rho(\varphi)\Phi_\rho(a)V_\rho\xi \\ &= \Delta_\rho(\theta)^{-\frac{1}{2}}\Delta_\rho(\varphi)v_g^\rho\Phi_\rho(a)V_\rho\xi \\ &= \Delta_\rho(\theta)^{-\frac{1}{2}}\Delta_\rho(\varphi)\Delta_\rho(\theta)^{-\frac{1}{2}}v_g^\rho\Delta_\rho(\theta)^{\frac{1}{2}}\Phi_\rho(a)V_\rho\xi \\ &= \Delta_\theta(\varphi)v_g^\rho\left(\Delta_\rho(\theta)^{\frac{1}{2}}\Phi_\rho(a)V_\rho\xi\right) \end{aligned}$$

for all $a \in A$, for all $\xi \in H$ and for all $g \in G$. Therefore, $\Delta_\theta(\varphi)$ is affiliated with $v^\theta(G)'$ and the theorem is proved. □

Let $\varphi, \theta \in \mathcal{CP}((G, \alpha, A), H, u)$. We say that φ is θ-*singular* if the only $\psi \in \mathcal{CP}((G, \alpha, A), H, u)$ such that $\psi \leq \varphi$ and $\psi \leq \theta$ is 0.

The following theorem extends [3, Theorem 3.1].

Theorem 2.5. *Let* $\varphi, \theta \in \mathcal{CP}((G, \alpha, A), H, u)$. *Then there are* φ_{ac} *and* φ_s *in* $\mathcal{CP}((G, \alpha, A), H, u)$ *such that*

(1) φ_{ac} *is* θ*-absolutely continuous and* φ_s *is* θ*-singular;*
(2) $\varphi = \varphi_{ac} + \varphi_s$;
(3) φ_{ac} *is maximal in the sense that if* ψ *is* θ*-absolutely continuous and* $\psi \leq \varphi_{ac}$, *then* $\psi = \varphi_{ac}$.

Proof. By [3, Theorem 3.1], there are φ_{ac}, $\varphi_{\mathrm{s}} \in \mathcal{CP}(A, H)$ such that $\varphi = \varphi_{\mathrm{ac}} + \varphi_{\mathrm{s}}$, φ_{ac} is θ-absolutely continuous and maximal, in the sense that if σ is a completely positive map from A to $L(H)$, θ-absolutely continuous and $\sigma \leq \varphi_{\mathrm{ac}}$, then $\sigma = \varphi_{\mathrm{ac}}$, and φ_{s} is θ-singular. Moreover,

$$\varphi_{\mathrm{ac}}(a) = V_\rho^*\Delta_\rho(\varphi)P_{H_\rho \ominus \ker\Delta_\rho(\varphi)}\Phi_\rho(a)V_\rho$$

and

$$\varphi_{\mathrm{s}}(a) = V_\rho^* P_{\ker \Delta_\rho(\varphi)} \Phi_\rho(a) V_\rho$$

for all $a \in A$, where $\rho = \varphi + \theta$. Then

$$\begin{aligned} \varphi_{\mathrm{ac}}(\alpha_g(a)) &= V_\rho^* \Delta_\rho(\varphi) P_{H_\rho \ominus \ker \Delta_\rho(\varphi)} \Phi_\rho(\alpha_g(a)) V_\rho \\ &= V_\rho^* \Delta_\rho(\varphi) P_{H_\rho \ominus \ker \Delta_\rho(\varphi)} v_g^\rho \Phi_\rho(a) v_{g^{-1}}^\rho V_\rho \\ &= V_\rho^* v_g^\rho \Delta_\rho(\varphi) P_{H_\rho \ominus \ker \Delta_\rho(\varphi)} \Phi_\rho(a) v_{g^{-1}}^\rho V_\rho \\ &= u_g V_\rho^* \Delta_\rho(\varphi) P_{H_\rho \ominus \ker \Delta_\rho(\varphi)} \Phi_\rho(a) V_\rho u_{g^{-1}} = u_g \varphi_{\mathrm{ac}}(a) u_{g^{-1}} \end{aligned}$$

and

$$\begin{aligned} \varphi_{\mathrm{s}}(\alpha_g(a)) &= V_\rho^* P_{\ker \Delta_\rho(\varphi)} \Phi_\rho(\alpha_g(a)) V_\rho = V_\rho^* P_{\ker \Delta_\rho(\varphi)} v_g^\rho \Phi_\rho(a) v_{g^{-1}}^\rho V_\rho \\ &= V_\rho^* v_g^\rho P_{\ker \Delta_\rho(\varphi)} \Phi_\rho(a) v_{g^{-1}}^\rho V_\rho \\ &= u_g V_\rho^* P_{\ker \Delta_\rho(\varphi)} \Phi_\rho(a) V_\rho u_{g^{-1}} = u_g \varphi_{\mathrm{s}}(a) u_{g^{-1}} \end{aligned}$$

for all $g \in G$ and for all $a \in A$. Therefore, $\varphi_{\mathrm{ac}}, \varphi_{\mathrm{s}} \in \mathcal{CP}((G, \alpha, A), H, u)$.

Let $\psi \in \mathcal{CP}((G, \alpha, A), H, u)$ such that ψ is θ-absolutely continuous and $\psi \leq \varphi_{\mathrm{ac}}$. Then, by [3, Theorem 3.1], $\psi = \varphi_{\mathrm{ab}}$ and the theorem is proved. □

Let $\varphi, \theta \in \mathcal{CP}((G, \alpha, A), H, u)$. The decomposition $\varphi = \varphi_{\mathrm{ac}} + \varphi_{\mathrm{s}}$ is called *the θ-Lebesgue decomposition of* φ, φ_{ac} is called the absolutely continuos part and φ_{s} is the singular part of φ with respect to θ.

As in the case of completely positive maps on C^*-algebras [3], we have

Corollary 2.6. *Let $\varphi, \psi, \theta \in \mathcal{CP}((G, \alpha, A), H, u)$. If $\varphi = \varphi_{ac} + \varphi_s$ and $\psi = \psi_{ac} + \psi_s$ are the θ-Lebesgue decomposition of φ and θ. Then*

1. *φ is θ-singular if and only if $\varphi_{ac} = 0$*
2. *φ is θ-absolutely continuous if and only if $\varphi_s = 0$*
3. *$(t\varphi)_{ac} = t\varphi_{ac}$ for each positive number t*
4. *$\varphi_{ac} + \psi_{ac} \leq (\varphi + \psi)_{ac}$, where $(\varphi + \psi)_{ac}$ is the absolutely continuous part of $\varphi + \psi$ with respect to θ*
5. *If $\psi \leq \varphi$, then $\psi_{ac} \leq \varphi_{ac}$.*

3. Covariant completely positive maps and crossed products

Let $\varphi \in \mathcal{CP}((G, \alpha, A), H, u)$. If $(\Phi_\varphi, v^\varphi, H_\varphi, V_\varphi)$ is the covariant Stinespring construction associated to φ, the map $\widehat{\varphi} : C_c(G, A) \to L(H)$ defined by

$$\widehat{\varphi}(f) = \int_G \varphi(f(g)) u_g dg\,,$$

where $C_c(G, A)$ denotes the vector space of all continuous functions from G to A with compact support, extends to a completely positive map from $A \times_\alpha G$ to $L(H)$, denoted also by $\widehat{\varphi}$ (see, for example, [6, 10]). Moreover, the Stinespring construction associated with $\widehat{\varphi}$ is unitarily equivalent with $(\Phi_\varphi \times v^\varphi, H_\varphi, V_\varphi)$, where $\Phi_\varphi \times v^\varphi$ is the integral form of the covariant representation $(\Phi_\varphi, v^\varphi, H_\varphi)$.

Proposition 3.1. *Let $\theta \in \mathcal{CP}((G, \alpha, A), H, u)$.*

(1) *The map* $\varphi \rightarrow \widehat{\varphi}$ *is an affine order isomorphism from* $\{\varphi \in \mathcal{CP}((G,\alpha,A), H,u);\ \varphi \leq \theta\}$ *to* $\{\rho \in \mathcal{CP}\left(\left(A\times_{\alpha}G\right),H\right);\rho \leq \widehat{\theta}\}$.

(2) *The map* $\varphi \rightarrow \widehat{\varphi}$ *is an affine order isomorphism from* $\{\varphi \in \mathcal{CP}((G,\alpha,A), H,u);\ \varphi \leq_{\mathcal{U}} \theta\}$ *to* $\{\rho \in \mathcal{CP}\left(\left(A\times_{\alpha}G\right),H\right);\rho \leq_{\mathcal{U}} \widehat{\theta}\}$.

Proof. (1) Let $\left(\Phi_{\theta},v^{\theta},H_{\theta},V_{\theta}\right)$ be the covariant Stinespring construction associated to θ and let $\varphi \in \mathcal{CP}\left(\left(G,\alpha,A\right),H,u\right)$ with $\varphi \leq \theta$. Then

$$
\begin{aligned}
&\widehat{\varphi}^{(m)}\left(\left([f_{ij}]_{i,j=1}^{m}\right)^{*}[f_{ij}]_{i,j=1}^{m}\right)\\
=&\left[\widehat{\varphi}\left(\sum_{k=1}^{m}f_{ik}^{\#}*f_{kj}\right)\right]_{i,j=1}^{m}=\left[\int_{G}\varphi\left(\left(\sum_{k=1}^{m}f_{ik}^{\#}*f_{kj}\right)(g)\right)u_{g}dg\right]_{i,j=1}^{m}\\
=&\left[\int_{G}V_{\theta}^{*}\Delta_{\theta}\left(\varphi\right)\Phi_{\theta}\left(\left(\sum_{k=1}^{m}f_{ik}^{\#}*f_{kj}\right)(g)\right)V_{\theta}u_{g}dg\right]_{i,j=1}^{m}\\
=&\left[\int_{G}V_{\theta}^{*}\Delta_{\theta}\left(\varphi\right)\Phi_{\theta}\left(\left(\sum_{k=1}^{m}f_{ik}^{\#}*f_{kj}\right)(g)\right)v_{g}^{\theta}V_{\theta}dg\right]_{i,j=1}^{m}\\
=&\left[V_{\theta}^{*}\Delta_{\theta}\left(\varphi\right)\int_{G}\Phi_{\theta}\left(\left(\sum_{k=1}^{m}f_{ik}^{\#}*f_{kj}\right)(g)\right)v_{g}^{\theta}V_{\theta}dg\right]_{i,j=1}^{m}\\
=&\left[V_{\theta}^{*}\Delta_{\theta}\left(\varphi\right)\left(\Phi_{\theta}\times v^{\theta}\right)\left(\sum_{k=1}^{m}f_{ik}^{\#}*f_{kj}\right)V_{\theta}\right]_{i,j=1}^{m}\\
=&\left[\sum_{k=1}^{m}\left(\Delta_{\theta}\left(\varphi\right)\left(\Phi_{\theta}\times v^{\theta}\right)\left(f_{ik}\right)V_{\theta}\right)^{*}\left(\Phi_{\theta}\times v^{\theta}\right)\left(f_{kj}\right)V_{\theta}\right]_{i,j=1}^{m}\\
=&\left(\left[\left(\Phi_{\theta}\times v^{\theta}\right)\left(f_{ij}\right)V_{\theta}\right]_{i,j=1}^{m}\right)^{*}\left[\delta_{ij}\Delta_{\theta}\left(\varphi\right)\right]_{i,j=1}^{m}\left[\left(\Phi_{\theta}\times v^{\theta}\right)\left(f_{ij}\right)V_{\theta}\right]_{i,j=1}^{m}\\
\leq&\left(\left[\left(\Phi_{\theta}\times v^{\theta}\right)\left(f_{ij}\right)V_{\theta}\right]_{i,j=1}^{m}\right)^{*}\left[\left(\Phi_{\theta}\times v^{\theta}\right)\left(f_{ij}\right)V_{\theta}\right]_{i,j=1}^{m}\\
=&\left[V_{\theta}^{*}\sum_{k=1}^{m}\left(\left(\Phi_{\theta}\times v^{\theta}\right)\left(f_{ik}\right)\right)^{*}\left(\Phi_{\theta}\times v^{\theta}\right)\left(f_{kj}\right)V_{\theta}\right]_{i,j=1}^{m}\\
=&\left[V_{\theta}^{*}\left(\Phi_{\theta}\times v^{\theta}\right)\left(\sum_{k=1}^{m}f_{ik}^{\#}*f_{kj}\right)V_{\theta}\right]_{i,j=1}^{m}\\
=&\left[\widehat{\theta}\left(\sum_{k=1}^{m}f_{ik}^{\#}*f_{kj}\right)\right]_{i,j=1}^{m}=\widehat{\theta}^{(m)}\left(\left([f_{ij}]_{i,j=1}^{m}\right)^{*}[f_{ij}]_{i,j=1}^{m}\right)_{i,j=1}^{m}
\end{aligned}
$$

for all $[f_{ij}]_{i,j=1}^{n}\in M_{m}\left(C_{c}(G,A)\right)$ and so $\widehat{\theta}-\widehat{\varphi}\in\mathcal{CP}\left(\left(A\times_{\alpha}G\right),H\right)$. Therefore, the map $\varphi\rightarrow\widehat{\varphi}$ is well defined.

Clearly,

$$\widehat{\varphi+\sigma}=\widehat{\varphi}+\widehat{\sigma}$$

and

$$\widehat{\lambda\varphi} = \lambda\widehat{\varphi}$$

for all $\varphi, \sigma \in \mathcal{CP}\left(\left(G,\alpha,A\right),H,u\right)$ and for all positive numbers λ.

Let $\varphi \in \mathcal{CP}\left(\left(G,\alpha,A\right),H,u\right)$. If $\widehat{\varphi}=0$, then

$$V_\varphi^*\left(\Phi_\varphi \times v^\varphi\right)(x)^*\left(\Phi_\varphi \times v^\varphi\right)(x)V_\varphi = \widehat{\varphi}\left(x^*x\right) = 0$$

for all $x \in G \times_\alpha A$ and so $\left(\Phi_\varphi \times v^\varphi\right)(x)V_\varphi = 0$ for all $x \in G \times_\alpha A$. But

$$\varphi\left(a\right)\xi = V_\varphi^*\Phi_\varphi(a)V_\varphi\xi = V_\varphi^*\lim_i\left(\Phi_\varphi \times v^\varphi\right)\left(e_i\mathrm{i}_A\left(a\right)\right)V_\varphi\xi = 0\,,$$

where $\{e_i\}_i$ is an approximate unit for $A \times_\alpha G$ and i_A is a non-degenerate faithful homomorphism from A to the multiplier algebra of $A \times_\alpha G$, $\left(i_A(a)f\right)(g) = af(g)$ for all $g \in G$ and for all $f \in C_c(G,A)$ (see, for example, [12, Proposition 2.40]), for all $a \in A$ and for all $\xi \in H$. Therefore $\varphi = 0$, and so the map $\varphi \to \widehat{\varphi}$ from $\mathcal{CP}\left(\left(G,\alpha,A\right),H,u\right)$ to $\mathcal{CP}\left(\left(A\times_\alpha G\right),H\right)$ is injective.

To prove the assertion (1) it remains to show that the map $\varphi \to \widehat{\varphi}$ is surjective. Let $\rho \in \mathcal{CP}\left(\left(A\times_\alpha G\right),H\right)$, $\rho \leq \widehat{\theta}$. Then

$$\rho\left(x\right) = V_\theta^*\Delta_{\widehat{\theta}}\left(\rho\right)\left(\Phi_\theta \times v^\theta\right)(x)V_\theta$$

for all $x \in G \times_\alpha A$. Consider the map $\varphi : A \to L(H)$ defined by

$$\varphi\left(a\right) = V_\theta^*\Delta_{\widehat{\theta}}\left(\rho\right)\Phi_\theta(a)V_\theta.$$

Since $\Delta_{\widehat{\theta}}\left(\rho\right) \in \left(\Phi_\theta \times v^\theta\right)\left(G\times_\alpha A\right)'$ and since $\left(\Phi_\theta \times v^\theta\right)\left(A\times_\alpha G\right)' = \Phi_\theta\left(A\right)' \cap v^\theta(G)'$, $\Delta_{\widehat{\theta}}\left(\rho\right) \in \Phi_\theta\left(A\right)'$ and so $\varphi \in \mathcal{CP}\left(A,H\right)$. Moreover,

$$\begin{aligned}\varphi\left(\alpha_g\left(a\right)\right) &= V_\theta^*\Delta_{\widehat{\theta}}\left(\rho\right)\Phi_\theta(\alpha_g\left(a\right))V_\theta = V_\theta^*\Delta_{\widehat{\theta}}\left(\rho\right)v_g^\theta\Phi_\theta(a)\left(v_g^\theta\right)^*V_\theta \\ &= V_\theta^*v_g^\theta\Delta_{\widehat{\theta}}\left(\rho\right)\Phi_\theta(a)\left(v_g^\theta\right)^*V_\theta = u_gV_\theta^*\Delta_{\widehat{\theta}}\left(\rho\right)\Phi_\theta(a)V_\theta u_g^* = u_g\varphi(a)u_g^*\end{aligned}$$

for all $a \in A$ and for all $g \in G$. Therefore, $\varphi \in \mathcal{CP}\left(\left(G,\alpha,A\right),H,u\right)$, and $\varphi \leq \theta$. Moreover,

$$\begin{aligned}\widehat{\varphi}\left(f\right) &= \int\limits_G \varphi(f(g))u_g dg = \int\limits_G V_\theta^*\Delta_{\widehat{\theta}}\left(\rho\right)\Phi_\theta(f(g))V_\theta u_g dg \\ &= V_\theta^*\Delta_{\widehat{\theta}}\left(\rho\right)\int_G \Phi_\theta(f(g))v_g^\theta V_\theta dg \\ &= V_\theta^*\Delta_{\widehat{\theta}}\left(\rho\right)\left(\Phi_\theta \times v^\theta\right)(f)V_\theta = \rho\left(f\right)\end{aligned}$$

for all $f \in C_c(G,A)$, and so the map $\varphi \to \widehat{\varphi}$ from $\{\varphi \in \mathcal{CP}\left(\left(G,\alpha,A\right),H,u\right);\ \varphi \leq \theta\}$ to $\{\rho \in \mathcal{CP}\left(\left(A\times_\alpha G\right),H\right);\rho \leq \widehat{\theta}\}$ is surjective.

(2) It follows in the same manner as assertion (1). $\square$

Corollary 3.2. *Let $\varphi, \theta \in \mathcal{CP}\left(\left(G,\alpha,A\right),H,u\right)$ such that $\varphi \leq \theta$ or $\varphi \leq_{\mathcal{U}} \theta$. Then $\Delta_{\widehat{\theta}}\left(\widehat{\varphi}\right) = \Delta_\theta\left(\varphi\right)$.*

Corollary 3.3. *Let $\varphi, \theta \in \mathcal{CP}\left(\left(G,\alpha,A\right),H,u\right)$. Then $\varphi =_{\mathcal{U}} \theta$ if and only if $\widehat{\varphi} =_{\mathcal{U}} \widehat{\theta}$.*

Let $f \in C_c(G, A)$ and $g \in G$. Then the map $f_g : G \to A$ defined by $f_g(t) = f(g^{-1}t)$ is an element in $C_c(G, A)$.

A completely positive map $\rho : A \times_\alpha G \to L(H)$ is *u-covariant* if $\rho(\alpha_g \circ f_g) = u_g \rho(f)$ for all $f \in C_c(G, A)$ and for all $g \in G$ (see [10]).

Let $\mathcal{CP}((A \times_\alpha G), H, u) = \{\rho \in \mathcal{CP}((A \times_\alpha G), H); \rho$ is u-covariant$\}$. In [10] it is shown that there is an isomorphism between the unital completely positive maps from a unital C^*-algebra A to $L(H)$, u-covariant with respect to the C^*-dynamical system (G, α, A) and the normalized u-covariant completely positive maps from $A \times_\alpha G$ to $L(H)$. In the following theorem we extend this result.

Theorem 3.4. *The map $\varphi \to \widehat{\varphi}$ is an affine order isomorphism from $\{\mathcal{CP}((G, \alpha, A), H, u); \leq\}$ to $\{\mathcal{CP}((A \times_\alpha G), H, u); \leq\}$ respectively from $\{\mathcal{CP}((G, \alpha, A), H, u); \leq_\mathcal{U}\}$ to $\{\mathcal{CP}((A \times_\alpha G), H, u); \leq_\mathcal{U}\}$.*

Proof. Let $\varphi \in \mathcal{CP}((G, \alpha, A), H, u)$, $f \in C_c(G, A)$ and $g \in G$. Then

$$\begin{aligned} \widehat{\varphi}(\alpha_g \circ f_g) &= \int_G \varphi(\alpha_g(f_g(s))) u_s ds = \int_G u_g \varphi\left(f\left(g^{-1}s\right)\right) u_{g^{-1}} u_s ds \\ &= u_g \int_G \varphi(f(t)) u_t dt = u_g \widehat{\varphi}(f). \end{aligned}$$

Therefore, the map $\varphi \to \widehat{\varphi}$ is well defined.

According to Proposition 3.1, to prove the theorem it is sufficient to show that the map is surjective. Let $\rho \in \mathcal{CP}((A \times_\alpha G), H, u)$ and let $(\Phi_\rho, H_\rho, V_\rho)$ be the Stinespring construction associated to ρ. By [12, Proposition 2.40] there is a covariant representation (Φ, v, H_ρ) of (G, α, A) such that $\Phi \times v = \Phi_\rho$.

Consider the map $\varphi : A \to L(H)$ defined by

$$\varphi(a) = V_\rho^* \Phi(a) V_\rho.$$

Clearly, φ is completely positive. To show that φ is u-covariant with respect to (G, α, A) it is sufficient to show that $V_\rho u_g = v_g V_\rho$ for all $g \in G$, since

$$\begin{aligned} \varphi(\alpha_g(a)) &= V_\rho^* \Phi(\alpha_g(a)) V_\rho = V_\rho^* v_g \Phi(a) v_{g^{-1}} V_\rho \\ &= u_g V_\rho^* \Phi(a) V_\rho u_{g^{-1}} = u_g \varphi(a) u_{g^{-1}}. \end{aligned}$$

By the Stinesprig construction, H_ρ is the completion of the pre-Hilbert space $(A \times_\alpha G) \otimes_{\text{alg}} H$ with the pre-innner product given by

$$\langle x \otimes \xi, y \otimes \eta \rangle = \langle \rho(y^* x) \xi, \eta \rangle.$$

Moreover, $V_\rho^*(x \otimes \xi + \mathcal{N}) = \rho(x) \xi$, where $\mathcal{N} = \{x \otimes \xi \in (A \times_\alpha G) \otimes_{\text{alg}} H; \langle x \otimes \xi, x \otimes \xi \rangle = 0\}$, and $\Phi_\rho(x)(y \otimes \xi + \mathcal{N}) = xy \otimes \xi + \mathcal{N}$ for all $x, y \in A \times_\alpha G$ and for all $\xi \in H$.

Let $f \in C_c(G, A)$, $g \in G$, $\xi \in H$ and $\{e_i\}_{i \in I}$ an approximate unit for $A \times_\alpha G$. Then

$$\left(u_g V_\rho^*\right)(f \otimes \xi + \mathcal{N}) = u_g \rho(f) \xi$$

and

$$\begin{aligned} \left(V_\rho^* v_g\right)(f \otimes \xi + \mathcal{N}) &= V_\rho^* \left(\lim_i \Phi_\rho\left(e_i \mathrm{i}_G(g)\right)(f \otimes \xi + \mathcal{N})\right) \\ &= V_\rho^* \left(\lim_i e_i \mathrm{i}_G(g) f \otimes \xi + \mathcal{N}\right) \\ &= V_\rho^* \left(\lim_i e_i \left(\alpha_g \circ f_g\right) \otimes \xi + \mathcal{N}\right) \\ &= V_\rho^* \left(\alpha_g \circ f_g \otimes \xi + \mathcal{N}\right) = \rho\left(\alpha_g \circ f_g\right) \xi\,, \end{aligned}$$

where i_G is an injective strictly continuous homomorphism from G to the unitary group from the multiplier algebra of $A \times_\alpha G$ such that $\mathrm{i}_G(g) f = \alpha_g \circ f_g$ for all $f \in C_c(G, A)$ (see, for example, [12, Proposition 2.40]). But

$$\rho\left(\alpha_g \circ f_g\right) \xi = u_g \rho(f) \xi$$

and so $u_g V_\rho^* = V_\rho^* v_g$. Therefore, $V_\rho u_g = v_g V_\rho$ for all $g \in G$, and so $\varphi \in \mathcal{CP}\left((G, \alpha, A), H, u\right)$. Moreover,

$$\begin{aligned} \widehat{\varphi}(f) &= \int_G \varphi(f(g)) u_g dg = \int_G V_\rho^* \Phi(f(g)) V_\rho u_g dg \\ &= \int_G V_\rho^* \Phi(f(g)) v_g V_\rho dg = V_\rho^* (\Phi \times v)(f) V_\rho \\ &= V_\rho^* \Phi_\rho(f) V_\rho = \rho(f) \end{aligned}$$

for all $f \in C_c(G, A)$. Therefore, $\widehat{\varphi} = \rho$ and the map $\varphi \to \widehat{\varphi}$ from $\mathcal{CP}(\,(G, \alpha, A)\,,$ $H, u)$ to $\mathcal{CP}((A \times_\alpha G)\,, H, u)$ is surjective. □

Theorem 3.5. *Let $\varphi, \theta \in \mathcal{CP}\left((G, \alpha, A), H, u\right)$. Then*

(1) *φ is θ-absolutely continuous if and only if $\widehat{\varphi}$ is $\widehat{\theta}$-absolutely continuous;*
(2) *φ is θ-singular if and only if $\widehat{\varphi}$ is $\widehat{\theta}$-singular.*

Proof. (1) First, we suppose that φ is θ-absolutely continuous. Then, by Remark 2.3, there is an increasing sequence $\{\varphi_n\}_n$ in $\mathcal{CP}\left((G, \alpha, A), H, u\right)$ such that $\varphi_n \leq_{\mathcal{U}} \theta$ and $\varphi_n \leq \varphi$ for all positive integers n and the sequence $\{\Delta_\varphi(\varphi_n)\}_n$ converges strongly to I_H. By Proposition 3.1, $\{\widehat{\varphi_n}\}_n$ is an increasing sequence in $\mathcal{CP}(A \times_\alpha G, H)$ such that $\widehat{\varphi_n} \leq_{\mathcal{U}} \widehat{\theta}$ and $\widehat{\varphi_n} \leq \widehat{\varphi}$ for all positive integers n. But, for each positive integer n, $\Delta_\varphi(\varphi_n) = \Delta_{\widehat{\varphi}}(\widehat{\varphi_n})$ (Corollary 3.2), and then the sequence $\{\widehat{\varphi_n}(x)\}_n$ converges strongly to $\widehat{\varphi}(x)$ for all $x \in G \times_\alpha A$. Therefore, $\widehat{\varphi}$ is $\widehat{\theta}$-absolutely continuous.

Conversely, suppose that $\widehat{\varphi}$ is $\widehat{\theta}$-absolutely continuous. Then there is an increasing sequence $\{\rho_n\}_n$ in $\mathcal{CP}(A \times_\alpha G, H)$ such that $\rho_n \leq_{\mathcal{U}} \widehat{\theta}$ and $\rho_n \leq \widehat{\varphi}$ for all positive integers n, and the sequence $\{\Delta_\varphi(\rho_n)\}_n$ converges strongly to I_H. Since $\{\rho_n\}_n$ is an increasing sequence in $\mathcal{CP}(A \times_\alpha G, H)$, by Proposition 3.1, there is an increasing sequence $\{\varphi_n\}_n$ in $\mathcal{CP}\left((G, \alpha, A), H, u\right)$ such that $\widehat{\varphi_n} = \rho_n$ for all positive integers n. Moreover, $\varphi_n \leq_{\mathcal{U}} \theta$ and $\varphi_n \leq \varphi$ for all positive integers n, and since $\Delta_\varphi(\varphi_n) = \Delta_{\widehat{\varphi}}(\widehat{\varphi_n}) = \Delta_{\widehat{\varphi}}(\rho_n)$ for all positive integers n, the sequence $\{\Delta_\varphi(\varphi_n)\}_n$ converges strongly to I_H. Therefore, φ is θ-absolutely continuous.

(2) Suppose that φ is θ-singular. Let $\rho \in \mathcal{CP}(A \times_\alpha G, H)$ such that $\rho \leq \widehat{\varphi}, \widehat{\theta}$. Then $\rho \in \mathcal{CP}(A \times_\alpha G, H, u)$ and there is $\psi \in \mathcal{CP}((G, \alpha, A), H, u)$ such that $\widehat{\psi} = \rho$. By Theorem 3.4, $\psi \leq \varphi, \theta$ and then $\psi = 0$ and so $\rho = 0$.

Conversely, suppose that $\widehat{\varphi}$ is $\widehat{\theta}$-singular. If $\psi \leq \varphi, \theta$, then $\widehat{\psi} \leq \widehat{\varphi}, \widehat{\theta}$, whence it follows that $\widehat{\psi} = 0$ and so $\psi = 0$. □

Corollary 3.6. *The map $\varphi \to \widehat{\varphi}$ from $\mathcal{CP}((G, \alpha, A), H, u)$ to $\mathcal{CP}((A \times_\alpha G), H, u)$ preserves the Lebesgue decomposition.*

Proof. Let $\varphi, \theta \in \mathcal{CP}((G, \alpha, A), H, u)$ and let $\varphi = \varphi_{\rm ac} + \varphi_{\rm s}$ be the Lebesgue decomposition of φ with respect to θ. Then $\widehat{\varphi} = \widehat{\varphi_{\rm ac}} + \widehat{\varphi}_{\rm s}$ and moreover, $\widehat{\varphi_{\rm ac}}$ is $\widehat{\theta}$-absolutely continuous. Let $\rho \in \mathcal{CP}((A \times_\alpha G), H)$, $\widehat{\theta}$-absolutely continuous such that $\rho \leq \widehat{\varphi_{\rm ac}}$. Then $\rho \in \mathcal{CP}((A \times_\alpha G), H, u)$ and so there is a $\psi \in \mathcal{CP}((G, \alpha, A), H, u)$ such that $\widehat{\psi} = \rho$. By Theorem 3.5, ψ is θ-absolutely continuous and $\psi \leq \varphi_{\rm ac}$ and by the uniqueness of the Lebesgue decomposition, $\psi = \varphi_{\rm ac}$. Therefore, $\rho = \widehat{\varphi_{\rm ac}}$ and then $\widehat{\varphi}_{\rm ac} = \widehat{\varphi_{\rm ac}}$ and the corollary is proved. □

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