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VOLTERRA COMPOSITION OPERATORS ON LOGARITHMIC BLOCH SPACES

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ABSTRACT. Let φ be a holomorphic self-map and g a fixed holomorphic function on the unit ball B. The boundedness and compactness of the Volterra composition operator

$$T_{g,\varphi}f(z) = \int_0^1 f(\varphi(tz))\Re g(tz)\frac{dt}{t}$$

on the logarithmic Bloch space and little logarithmic Bloch space are studied in this paper.

1. INTRODUCTION

Let B denote the unit ball of $\mathbb{C}^n$. Let $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ be points in $\mathbb{C}^n$, we write

$$\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n, \ |z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

Thus $B = \{z \in \mathbb{C}^n : |z| < 1\}$. We denote by $H(B)$ the space of all holomorphic functions in B. Let

$$\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z)$$

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represent the radial derivative of $f \in H(B)$. Recall that the Bloch space $\mathcal{B} = \mathcal{B}(B)$, is the space of all $f \in H(B)$ for which (see [14])

$$b(f) = \sup_{z\in B}(1-|z|^2)\,|\Re f(z)| < \infty.$$

The little Bloch space $\mathcal{B}_0 = \mathcal{B}_0(B)$, comprises of all $f \in H(B)$ such that

$$\lim_{|z|\to 1}(1-|z|^2)\,|\Re f(z)| = 0.$$

Under the norm $||f||_{\mathcal{B}} = |f(0)| + b(f)$, $\mathcal{B}$ is a Banach space. It is easy to see that $\mathcal{B}_0$ is a closed subspace of $\mathcal{B}$.

Let $\mathcal{LB} = \mathcal{LB}(B)$ stand for the class of all $f \in H(B)$ such that

$$\beta(f) = \sup_{z\in B}(1-|z|^2)\left(\ln\frac{e}{1-|z|^2}\right)|\Re f(z)| < \infty.$$

It is easy to see that $\mathcal{LB}$ is a Banach space with the norm $\|f\|_{\mathcal{LB}} = |f(0)| + \beta(f)$. $\mathcal{LB}$ is called the logarithmic Bloch space.

Let $\mathcal{LB}_0$ denote the class of $f \in \mathcal{LB}$ such that

$$\lim_{|z|\to 1}(1-|z|^2)\left(\ln\frac{e}{1-|z|^2}\right)|\Re f(z)| = 0.$$

In [13] (see also [14, Theorem 3.21]) was shown that f is a multiplier of $\mathcal{B}$ if and only if $f \in H^\infty$ and $f \in \mathcal{LB}$. Hence the space $\mathcal{LB}$ is appeared naturally.

Let φ be a holomorphic self-map of B. The composition operator C_φ is defined by

$$(C_\varphi f)(z) = (f\circ\varphi)(z),\ \ f \in H(B).$$

It is interesting to provide a function theoretic characterization of when φ induces a bounded or compact composition operator on various spaces. Recall that a linear operator is said to be bounded if the image of a bounded set is a bounded set, while a linear operator is compact if it takes bounded sets to sets with compact closure. The book [2] contains plenty of information on this topic.

Suppose that $g : B \to \mathbb{C}^1$ is a holomorphic map, define

$$T_g f(z) = \int_0^1 f(tz)\frac{dg(tz)}{dt} = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t}, \qquad f \in H(B),\ z \in B.$$

This operator is called the extended Cesàro operator (or the Riemann-Stieltjes operator), which was introduced in [3], and studied in [1, 3, 4, 6, 7, 8, 9, 11, 12].

Motivated by the definition of operators C_φ and T_g, in [15] we define a more general operator as follows

$$T_{g,\varphi} f(z) = \int_0^1 f(\varphi(tz))\Re g(tz)\frac{dt}{t}, \qquad f \in H(B),\ z \in B. \tag{1.1}$$

The operator $T_{g,\varphi}$ is called the Volterra composition operator. In the setting of the unit disk D, this operator has the following form

$$T_{g,\varphi} f(z) = \int_0^z (f\circ\varphi)(\xi)g'(\xi)d\xi, \qquad f \in H(D),\ z \in D,$$

which was first defined and studied in [5]. It is easy to see that $T_{g,z} = T_g$.

In this paper, we study the operator $T_{g,\varphi}$ on the logarithmic Bloch space and little logarithmic Bloch space. The sufficient and necessary conditions for the operator $T_{g,\varphi}$ to be bounded and compact are given. As a corollary, we obtain the characterization of the boundedness and compactness of the extended Cesàro operator on the logarithmic Bloch space and little logarithmic Bloch space.

Throughout the paper, constants are denoted by C, they are positive and may not be the same in every occurrence.

2. Auxiliary results

In order to prove the main results of this paper, we need some auxiliary results, which are incorporated in the lemmas which follows.

Lemma 2.1. *Let φ be a holomorphic self-map of B and $g \in H(B)$. Then $T_{g,\varphi} : \mathcal{LB}(or\ \mathcal{LB}_0) \to \mathcal{LB}$ is compact if and only if $T_{g,\varphi} : \mathcal{LB}(or\ \mathcal{LB}_0) \to \mathcal{LB}$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{LB}(or\ \mathcal{LB}_0)$ which converges to zero uniformly on compact subsets of B as $k \to \infty$, we have $\|T_{g,\varphi} f_k\|_{\mathcal{LB}} \to 0$ as $k \to \infty$.*

Proof. The result follows by standard arguments similar to those outlined in Proposition 3.11 of [2]. We omit the details. □

Lemma 2.2. *A closed set K in $\mathcal{LB}_0$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} (1-|z|^2) \ln \frac{e}{1-|z|^2} |\Re f(z)| = 0.$$

Proof. The proof is similar to the proof of [10, Lemma 1]. We omit the details. □

Lemma 2.3. *Let $f \in \mathcal{LB}$. Then there exists a positive constant C such that*

$$|f(z)| \le C \ln\ln \frac{4}{1-|z|^2} \|f\|_{\mathcal{LB}}.$$

Proof. Assume that $f \in \mathcal{LB}$. Let $|z| > 1/2,\ z = r\zeta$ and $\zeta \in \partial B$. We have

$$|f(z) - f(r\zeta/2)| \le \int_{1/2}^{1} \left| \frac{\Re f(tz)}{t} \right| dt \le 4\|f\|_{\mathcal{LB}} \int_0^1 \frac{|z|dt}{(1-t|z|) \ln \frac{e}{1-t|z|}}.$$

By the standard estimate of the last integral, we have

$$|f(z)| \le \max_{|z|\le 1/2} |f(z)| + 4\|f\|_{\mathcal{LB}} \ln\ln \frac{e}{1-|z|^2}. \tag{2.1}$$

Let $|z| \le 1/2$. From [7] we see that

$$\max_{|z|\le 1/2} |f(z) - f(0)| \le C \max_{|z|\le 3/4} |\Re f(z)|.$$

Hence,

$$\max_{|z|\le 1/2} |f(z)| \le |f(0)| + C\|f\|_{\mathcal{LB}}. \tag{2.2}$$

Then the result follows from (2.1) and (2.2). □

Lemma 2.4. *Let $f \in \mathcal{LB}_0$. Then*

$$\lim_{|z|\to 1} \frac{|f(z)|}{\ln\ln \frac{e}{1-|z|^2}} = 0.$$

Proof. Since $f \in \mathcal{LB}_0$, it follows that for any $\varepsilon > 0$ there is a $\delta \in (1/2, 1)$ such that

$$(1-|z|)\ln\frac{e}{1-|z|}|\Re f(z)| < \varepsilon, \tag{2.3}$$

whenever $\delta < |z| < 1$.

From (2.3), when $1/2 < \delta < |z| < 1$, we have that

$$\begin{aligned}
|f(z)| &= \left| f(z/2|z|) + \int_{1/(2|z|)}^{1} \Re f(tz)\frac{dt}{t}\right| \\
&\le M_\infty(f,1/2) + 2\int_{1/(2|z|)}^{\frac{\delta}{|z|}} |\Re f(tz)|\,|z|dt + 2\int_{\frac{\delta}{|z|}}^{1} |\Re f(tz)|\,|z|dt \\
&\le M_\infty(f,1/2) + 2\|f\|_{\mathcal{LB}}\int_0^{\frac{\delta}{|z|}} \frac{|z|dt}{(1-t|z|)\ln\frac{e}{1-t|z|}} + 2\varepsilon\int_{\frac{\delta}{|z|}}^{1} \frac{|z|dt}{(1-t|z|)\ln\frac{e}{1-t|z|}} \\
&\le M_\infty(f,1/2) + 2\|f\|_{\mathcal{LB}}\ln\ln\frac{e}{1-\delta} + 2\varepsilon\ln\ln\frac{e}{1-|z|} - 2\varepsilon\ln\ln\frac{e}{1-\delta},
\end{aligned}$$

where $M_\infty(f,r) = \sup_{z\in B}|f(rz)|$. Dividing the above inequality by $\ln\ln\frac{e}{1-|z|}$, using the fact that the quantity $M_\infty(f,1/2)$ is finite, and letting $|z| \to 1$, we obtain

$$\frac{|f(z)|}{\ln\ln\frac{e}{1-|z|}} \le 2\varepsilon, \text{ as } |z| \to 1,$$

from which the lemma follows. □

3. Main results and proofs

Now we are in a position to state our main results and proofs in this paper.

Theorem 3.1. *Let φ be a holomorphic self-map of B and $g \in H(B)$. Then the following statements are equivalent.*

(1) $T_{g,\varphi} : \mathcal{LB} \to \mathcal{LB}$ *is bounded;*
(2) $T_{g,\varphi} : \mathcal{LB}_0 \to \mathcal{LB}$ *is bounded;*
(3)

$$M = \sup_{z\in B}(1-|z|^2)\ln\frac{e}{1-|z|^2}\ln\ln\frac{4}{1-|\varphi(z)|^2}|\Re g(z)| < \infty. \tag{3.1}$$

Proof. (3) ⇒ (1). Suppose that (3.1) holds. A calculation with (1.1) gives the following fundamental and useful formula (see e.g. [3])

$$\Re[T_{g,\varphi}(f)](z) = f(\varphi(z))\Re g(z).$$

For any $f \in \mathcal{LB}$, using Lemma 2.3 we have

$$\begin{aligned}
&(1-|z|^2)\ln\frac{e}{1-|z|^2}|\Re(T_{g,\varphi}f)(z)| \\
=\;& (1-|z|^2)\ln\frac{e}{1-|z|^2}|\Re g(z)||f(\varphi(z))| \\
\leq\;& C(1-|z|^2)\ln\frac{e}{1-|z|^2}\ln\ln\frac{4}{1-|\varphi(z)|^2}|\Re g(z)|\|f\|_{\mathcal{LB}}. \qquad (3.2)
\end{aligned}$$

Using the fact that $T_{g,\varphi}(f)(0) = 0$ and the condition (3.1), the boundedness of the operator $T_{g,\varphi} : \mathcal{LB} \to \mathcal{LB}$ follows by taking the supremum in (3.2) over B.

(1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3). Assume that $T_{g,\varphi} : \mathcal{LB}_0 \to \mathcal{LB}$ is bounded. For $a \in B$, set

$$f_a(z) = \ln\ln\frac{4}{1-\langle z, a\rangle}.$$

It is easy to check that $f_a \in \mathcal{LB}$ and $\sup_{a\in B}\|f_a\|_{\mathcal{LB}} < \infty$. Moreover,

$$\begin{aligned}
(1-|z|)\ln\frac{e}{1-|z|}|\Re f_a(z)| &= (1-|z|)\ln\frac{e}{1-|z|}\frac{1}{\ln\frac{4}{|1-\langle z,a\rangle|}}\frac{|\langle z,a\rangle|}{|1-\langle z,a\rangle|} \\
&\leq (1-|z|)\ln\frac{e}{1-|z|}\frac{1}{\ln 2}\frac{1}{1-|a|} \to 0
\end{aligned}$$

as $|z| \to 1$. Therefore $f_a \in \mathcal{LB}_0$. For $b \in B$, we have

$$\begin{aligned}
\infty > \|T_{g,\varphi}f_{\varphi(b)}\|_{\mathcal{LB}} &= \sup_{z\in B}(1-|z|^2)\ln\frac{e}{1-|z|^2}|\Re(T_{g,\varphi}f_{\varphi(b)})(z)| \\
&= \sup_{z\in B}(1-|z|^2)\ln\frac{e}{1-|z|^2}|\Re g(z)||f_{\varphi(b)}(\varphi(z))| \\
&\geq (1-|b|^2)\ln\frac{e}{1-|b|^2}\ln\ln\frac{4}{1-|\varphi(b)|^2}|\Re g(b)|,
\end{aligned}$$

from which we obtain that (3.1) holds. The proof of this theorem is completed. □

Theorem 3.2. *Let φ be a holomorphic self-map of B and $g \in H(B)$. Then the following statements are equivalent.*

(1) *$T_{g,\varphi} : \mathcal{LB} \to \mathcal{LB}$ is compact;*
(2) *$T_{g,\varphi} : \mathcal{LB}_0 \to \mathcal{LB}$ is compact;*
(3) *$g \in \mathcal{LB}$ and*

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)\ln\frac{e}{1-|z|^2}\ln\ln\frac{4}{1-|\varphi(z)|^2}|\Re g(z)| = 0. \qquad (3.3)$$

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3). Suppose that $T_{g,\varphi} : \mathcal{LB}_0 \to \mathcal{LB}$ is compact. Then it is obvious that $T_{g,\varphi} : \mathcal{LB}_0 \to \mathcal{LB}$ is bounded. Taking the function $f(z) = 1$, and employing the boundedness of $T_{g,\varphi} : \mathcal{LB}_0 \to \mathcal{LB}$ we obtain that $g \in \mathcal{LB}$. Let $(\varphi(z_k))_{k\in\mathbb{N}}$ be a sequence in B such that $\lim_{k\to\infty}|\varphi(z_k)| = 1$. Set

$$f_k(z) = \Big(\ln\ln\frac{4}{1-\langle z, \varphi(z_k)\rangle}\Big)^2\Big(\ln\ln\frac{4}{1-|\varphi(z_k)|^2}\Big)^{-1}.$$

Similarly to the proof of Theorem 3.1 we see that f_k is a bounded sequence in $\mathcal{LB}_0$. Moreover, $f_k \to 0$ uniformly on compact subsets of B as $k \to \infty$. By Lemma 2.1,

$$\lim_{k\to\infty} \|T_{g,\varphi} f_k\|_{\mathcal{LB}} = 0. \tag{3.4}$$

We have

$$\begin{aligned} \|T_{g,\varphi} f_k\|_{\mathcal{LB}} &= \sup_{z\in B}(1-|z|^2)\ln\frac{e}{1-|z|^2}|\Re(T_{g,\varphi} f_k)(z)| \\ &= \sup_{z\in B}(1-|z|^2)\ln\frac{e}{1-|z|^2}|f_k(\varphi(z)\Re g(z)| \\ &\geq (1-|z_k|^2)\ln\frac{e}{1-|z_k|^2}\ln\ln\frac{4}{1-|\varphi(z_k)|^2}|\Re g(z_k)|, \end{aligned}$$

which together with (3.4) imply

$$\lim_{k\to\infty}(1-|z_k|^2)\ln\frac{e}{1-|z_k|^2}\ln\ln\frac{4}{1-|\varphi(z_k)|^2}|\Re g(z_k)| = 0.$$

This proves that (3.3) holds.

(3) $\Rightarrow$ (1). Suppose that $g \in \mathcal{LB}$ and (3.3) holds. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{LB}$ with $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathcal{LB}} \leq L$ and suppose $f_k \to 0$ uniformly on compact subsets of B as $k \to \infty$. By Lemma 2.1 we only to show that

$$\|T_{g,\varphi} f_k\|_{\mathcal{LB}} \to 0 \quad \text{as } k \to \infty.$$

By (3.3) we have that if given $\varepsilon > 0$, there is a constant $\delta(0 < \delta < 1)$, such that when $\delta < |\varphi(z)| < 1$ we have

$$(1-|z|^2)\ln\frac{e}{1-|z|^2}\ln\ln\frac{4}{1-|\varphi(z)|^2}|\Re g(z)| < \varepsilon. \tag{3.5}$$

Let $G = \{w \in B : |w| \leq \delta\}$. (3.5) along with the fact that $g \in \mathcal{LB}$ shows that

$$\begin{aligned} &\|T_{g,\varphi} f_k\|_{\mathcal{LB}} = \sup_{z\in B}(1-|z|^2)\ln\frac{e}{1-|z|^2}|\Re(T_{g,\varphi} f_k)(z)| \\ = \ & \sup_{z\in B}(1-|z|^2)\ln\frac{e}{1-|z|^2}|\Re g(z) f_k(\varphi(z))| \\ \leq \ & \Big(\sup_{\{z\in B:\, |\varphi(z)|\leq\delta\}} + \sup_{\{z\in B:\, \delta\leq|\varphi(z)|<1\}}\Big)(1-|z|^2)\ln\frac{e}{1-|z|^2}|\Re g(z)||f_k(\varphi(z))| \\ \leq \ & \|g\|_{\mathcal{LB}}\sup_{w\in G}|f_k(w)| \\ & +C\|f_k\|_{\mathcal{LB}}\sup_{\{z\in B:\, \delta\leq|\varphi(z)|<1\}}(1-|z|^2)\ln\frac{e}{1-|z|^2}\ln\ln\frac{4}{1-|\varphi(z)|^2}|\Re g(z)| \\ \leq \ & \|g\|_{\mathcal{LB}}\sup_{w\in G}|f_k(w)| + CL\varepsilon. \end{aligned}$$

Observe that G is a compact subset of B, then it gives that

$$\lim_{k\to\infty}\sup_{w\in G}|f_k(w)| = 0.$$

By letting $k \to \infty$ in the last inequality, we can deduce that

$$\limsup_{k\to\infty}\|T_{g,\varphi} f_k\|_{\mathcal{LB}} \leq CL\varepsilon.$$

Since $\varepsilon > 0$ is an arbitrary positive number it follows that the last limit is equal to zero. Therefore, $T_{g,\varphi} : \mathcal{LB} \to \mathcal{LB}$ is compact. The proof is finished. $\square$

Theorem 3.3. *Let φ be a holomorphic self-map of B and $g \in H(B)$. Then $T_{g,\varphi} : \mathcal{LB}_0 \to \mathcal{LB}_0$ is bounded if and only if $T_{g,\varphi} : \mathcal{LB}_0 \to \mathcal{LB}$ is bounded and $g \in \mathcal{LB}_0$.*

Proof. Suppose that $T_{g,\varphi} : \mathcal{LB}_0 \to \mathcal{LB}_0$ is bounded. It is obvious that $T_{g,\varphi} : \mathcal{LB}_0 \to \mathcal{LB}$ is bounded. Taking the function $f(z) = 1$, and employing the boundedness of $T_{g,\varphi} : \mathcal{LB}_0 \to \mathcal{LB}_0$ we see that $g \in \mathcal{LB}_0$.

Conversely, assume that $T_{g,\varphi} : \mathcal{LB}_0 \to \mathcal{LB}$ is bounded and $g \in \mathcal{LB}_0$. Then, for each polynomial $p(z)$, we have that

$$(1-|z|^2)\ln\frac{e}{1-|z|^2}|\Re(T_{g,\varphi}p)(z)| \le (1-|z|^2)\ln\frac{e}{1-|z|^2}|\Re g(z)|\|p\|_\infty,$$

from which it follows that $T_{g,\varphi}p \in \mathcal{LB}_0$. Since the set of all polynomials is dense in $\mathcal{LB}_0$, we have that for every $f \in \mathcal{LB}_0$ there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that $\|f - p_k\|_{\mathcal{LB}} \to 0$, as $k \to \infty$. Hence

$$\|T_{g,\varphi}f - T_{g,\varphi}p_k\|_{\mathcal{LB}} \le \|T_{g,\varphi}\|\|f - p_k\|_{\mathcal{LB}} \to 0$$

as $k \to \infty$, since the operator $T_{g,\varphi} : \mathcal{LB}_0 \to \mathcal{LB}$ is bounded. Since $\mathcal{LB}_0$ is closed subset of $\mathcal{LB}$, we obtain

$$T_{g,\varphi}(\mathcal{LB}_0) \subset \mathcal{LB}_0.$$

Therefore $T_{g,\varphi} : \mathcal{LB}_0 \to \mathcal{LB}_0$ is bounded. $\square$

Theorem 3.4. *Let φ be a holomorphic self-map of B and $g \in H(B)$. Assume that $T_{g,\varphi} : \mathcal{LB} \to \mathcal{LB}_0$ is bounded. Then the following statements are equivalent*

(1) *$T_{g,\varphi} : \mathcal{LB} \to \mathcal{LB}_0$ is compact;*
(2) *$T_{g,\varphi} : \mathcal{LB}_0 \to \mathcal{LB}_0$ is compact;*
(3)

$$\lim_{|z|\to 1}(1-|z|^2)\ln\frac{e}{1-|z|^2}\ln\ln\frac{4}{1-|\varphi(z)|^2}|\Re g(z)| = 0. \tag{3.6}$$

Proof. (1) $\Rightarrow$ (2). It is obvious.

(2) $\Rightarrow$ (3). Suppose that $T_{g,\varphi} : \mathcal{LB}_0 \to \mathcal{LB}_0$ is compact. Then it is clear that $T_{g,\varphi} : \mathcal{LB}_0 \to \mathcal{LB}_0$ is bounded. Taking $f(z) \equiv 1$ we obtain

$$g \in \mathcal{LB}_0. \tag{3.7}$$

By the compactness of $T_{g,\varphi} : \mathcal{LB}_0 \to \mathcal{LB}_0$ we see that $T_{g,\varphi} : \mathcal{LB}_0 \to \mathcal{LB}$ is compact. From Theorem 3.2 we have

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)\ln\frac{e}{1-|z|^2}\ln\ln\frac{4}{1-|\varphi(z)|^2}|\Re g(z)| = 0. \tag{3.8}$$

In terms of (3.8), for every $\varepsilon > 0$, there exists an $r \in (0,1)$, such that when $r < |\varphi(z)| < 1$,

$$(1-|z|^2)\ln\frac{e}{1-|z|^2}\ln\ln\frac{4}{1-|\varphi(z)|^2}|\Re g(z)| < \varepsilon \tag{3.9}$$

According to (3.7), there exists a $\delta \in (0,1)$, such that when $\delta < |z| < 1$,

$$(1-|z|^2)\ln\frac{e}{1-|z|^2}|\Re g(z)| \leq \frac{\varepsilon}{\ln\ln\frac{4}{1-r^2}}. \tag{3.10}$$

Therefore, if $\delta < |z| < 1$ and $r < |\varphi(z)| < 1$, by (3.9) we have

$$(1-|z|^2)\ln\frac{e}{1-|z|^2}\ln\ln\frac{4}{1-|\varphi(z)|^2}|\Re g(z)| < \varepsilon. \tag{3.11}$$

If $\delta < |z| < 1$ and $|\varphi(z)| \leq r$, by (3.10) we obtain

$$\begin{aligned} &(1-|z|^2)\ln\frac{e}{1-|z|^2}\ln\ln\frac{4}{1-|\varphi(z)|^2}|\Re g(z)| \\ < \ & \ln\ln\frac{4}{1-r^2}(1-|z|^2)\ln\frac{e}{1-|z|^2}|\Re g(z)| < \varepsilon. \end{aligned} \tag{3.12}$$

Combining (3.11) with (3.12), we obtain (3.6), as desired.

(3) $\Rightarrow$ (1). From the proof of Theorem 3.1 we have

$$\begin{aligned} &(1-|z|^2)\ln\frac{e}{1-|z|^2}|\Re(T_{g,\varphi}f)(z)| \\ \leq \ & C(1-|z|^2)\ln\frac{e}{1-|z|^2}\ln\ln\frac{4}{1-|\varphi(z)|^2}|\Re g(z)|\|f\|_{\mathcal{LB}}. \end{aligned} \tag{3.13}$$

Taking the supremum in (3.13) over all $f \in \mathcal{LB}$ such that $\|f\|_{\mathcal{LB}} \leq 1$, by letting $|z| \to 1$, we arrive at

$$\lim_{|z|\to 1}\sup_{\|f\|_{\mathcal{LB}}\leq 1}(1-|z|^2)\ln\frac{e}{1-|z|^2}|\Re(T_{g,\varphi}f)(z)| = 0.$$

Combining this with Lemma 2.2 we see that $T_{g,\varphi} : \mathcal{LB} \to \mathcal{LB}_0$ is compact. The proof is completed. □

Let $\varphi(z) = z$. From Theorems 3.1, 3.2, 3.3, 3.4, we immediately get the following results.

Corollary 3.5. *Assume that $g \in H(B)$. Then the following statements are equivalent.*

(1) $T_g : \mathcal{LB} \to \mathcal{LB}$ *is bounded;*
(2) $T_g : \mathcal{LB}_0 \to \mathcal{LB}$ *is bounded;*
(3)

$$\sup_{z\in B}(1-|z|^2)\ln\frac{e}{1-|z|^2}\ln\ln\frac{4}{1-|z|^2}|\Re g(z)| < \infty.$$

Corollary 3.6. *Assume that $g \in H(B)$. Then $T_g : \mathcal{LB}_0 \to \mathcal{LB}_0$ is bounded if and only if $T_g : \mathcal{LB}_0 \to \mathcal{LB}$ is bounded and $g \in \mathcal{LB}_0$.*

Corollary 3.7. *Assume that $g \in H(B)$. Then the following statements are equivalent.*

(1) $T_g : \mathcal{LB} \to \mathcal{LB}$ *is compact;*
(2) $T_g : \mathcal{LB}_0 \to \mathcal{LB}$ *is compact;*
(3) $T_g : \mathcal{LB}_0 \to \mathcal{LB}_0$ *is compact;*

(4) $T_g : \mathcal{LB} \to \mathcal{LB}_0$ *is compact;*
(5)

$$\lim_{|z|\to 1}(1-|z|^2)\ln\frac{e}{1-|z|^2}\ln\ln\frac{4}{1-|z|^2}|\Re g(z)| = 0.$$

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