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SPECTRAL RADIUS PRESERVERS OF PRODUCTS OF MONNEGATIVE MATRICES*

SEAN CLARK[1], CHI-KWONG LI[2]* AND LEIBA RODMAN[3]

This paper is dedicated to Professor Josip E. Pečarić

Submitted by M. Brešar

Abstract. A characterization of nonlinear spectral radius preserving maps is obtained for the usual and triple products of nonnegative matrices.

1. Introduction

Preserver problems concern the characterization of maps on matrices or operators leaving invariant a certain function, a certain subset or a certain relation. Earlier studies focused on *linear* maps with these properties. The literature on this subject is extensive; see, for example, [5, 13] and monographs [9, 10, 11]. Recently, researchers have studied preservers under mild assumptions. For instance, for a given function ν on a matrix set $\mathbf{M}$ with a binary operator $A \circ B$, researchers study maps $f : \mathbf{M} \to \mathbf{M}$ satisfying $\nu(f(A) \circ f(B)) = \nu(A \circ B)$ for all $A, B \in \mathbf{M}$; [2, 3, 14] is a small selection of recent work on the topic. There has been interest in studying such problem when $\nu(A)$ is the spectrum, the peripheral spectrum, the numerical radius, the spectral norm, etc. Moreover, these problems have also been considered for in more general contexts such as functions on operator algebras or uniform algebras [9, 7, 12]; the latter two papers served as a motivation for our study of peripheral spectrum preservers.

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* Corresponding author.

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It is worth noting that even without the linearity assumption, the preservers often end up to be linear and have certain “standard” or “expected” form. Even though the statements of results look similar to those of linear preservers, researchers often have to develop new techniques to solve the preserver problems with mild assumption. In some cases, one may get unexpected preservers, which lead to better understanding and insight to the subjects under consideration.

The purpose of this paper is to characterize preservers of the spectral radius, spectrum and peripheral spectrum of the product or the Jordan triple product on entrywise nonnegative matrices. We note that the literature on preservers in the context of entrywise nonnegative matrices is meager; see [8].

The following notation will be used throughout the paper:
M_n^+ stands for the set of $n \times n$ real matrices with nonnegative entries.
We assume $n \geq 2$ throughout.
$E_{ij} \in M_n^+$ the matrix unit: 1 in the (i,j)th position and zeros everywhere else.
$r(A)$ the spectral radius of a matrix A.
$\sigma(A)$ the spectrum (the set of eigenvalues) of a matrix A.
$\sigma_p(A) = \sigma(A) \cap \{\lambda \in \mathbb{C} : |\lambda| = r(A)\}$ the peripheral spectrum of A.
$\mathrm{tr}\,(A)$ the trace of A.
$p_A(t) = \det(tI - A)$ the characteristic polynomial of A.
A^t the transpose of A.
$\mathrm{diag}\,(x_1, \dots, x_n)$ diagonal matrix with $x_1, \dots, x_n$ on the diagonal (in that order)
$\mathcal{P} \subseteq M_n^+$ the group of permutation matrices.
$\mathcal{PD} \subseteq M_n^+$ the group of matrices of the form PD where P is a permutation matrix and D is a diagonal matrix with positive entries on the diagonal. Note that $A \in M_n^+$ has the property that A is invertible and $A^{-1} \in M_n^+$ if and only if $A \in \mathcal{PD}$ (see, e.g., [6] for a proof).

Here is our main theorem.

Theorem 1.1. *For $A, B \in M_n^+$, let $A * B$ denote the usual product $A * B = AB$ or the Jordan triple product $A * B = ABA$. Then the following statements* (1) - (4) *are equivalent for a surjective function $f : M_n^+ \longrightarrow M_n^+$:*

(1)
$$r(A * B) = r(f(A) * f(B)), \qquad (A, B \in M_n^+). \tag{1.1}$$

(2)
$$\sigma_p(A * B) = \sigma_p(f(A) * f(B)), \qquad (A, B \in M_n^+).$$

(3)
$$\sigma(A * B) = \sigma(f(A) * f(B)), \qquad (A, B \in M_n^+).$$

(4) *There exists a matrix $Q \in \mathcal{PD}$ such that either*
$$f(A) = Q^{-1} A Q, \qquad (A \in M_n^+),$$
or
$$f(A) = Q^{-1} A^t Q, \qquad (A \in M_n^+).$$

Note that in Theorem 1.1 the function f is not assumed to be linear or multiplicative *a priori.*

The result of Theorem 1.1 for $A*B = A+B$ was obtained in [6] without the surjective assumption. It would be interesting to remove the surjective assumption in Theorem 1.1. We are not able to do that at present.

Since for $A \in M_n^+$ we always have $r(A) \in \sigma_p(A)$, the implications (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1) are trivial. Also, (4) $\Longrightarrow$ (3) is easy to verify. It remains to show that (1) implies (4). We first present some preliminary and auxiliary results in Section 2. In particular, we prove a function $f : M_n^+ \to M_n^+$ having some special properties on matrix units will have the nice form described in Theorem 1.1 (4). Then we show that a function $f : M_n^+ \to M_n^+$ satisfying Theorem 1.1 (1) will possess the special properties on matrix units, and hence f has the form in Theorem 1.1 (4). This is done in Sections 3 and 4 for the usual product and Jordan triple product, respectively.

2. Preliminaries

In this section we present some known results and easy observations that will be often used, sometimes without explicit reference, throughout the paper. We list several well-known properties of nonnegative matrices and their spectral radii (see, for example, [4, Theorem 8.4.5] or [1]).

The following two observations are useful when considering the triple product. Let $\sqrt{E_{ij}} = \begin{cases} E_{ik} + E_{kj} : k \neq i, j & \text{if } i \neq j \\ E_{ii} & \text{if } i = j \end{cases}$ for which a trivial calculation shows $\sqrt{E_{ij}}^2 = E_{ij}$. Clearly our choice of the specific k in the above definition does not matter so long as it respects our constraint in each case. Note that this construction requires $n \geq 3$, so $n = 2$ will be covered separately.

Since $r((BA)B) = r(B(BA)) = r(B^2A) = r(AB^2)$, we will use the following three equivalent conditions for the triple product interchangeably:

$$\begin{aligned} r(BAB) &= r(f(B)f(A)f(B)) \\ r(B^2A) &= r(f(B)^2 f(A)) \\ r(AB^2) &= r(f(A)f(B)^2). \end{aligned}$$

Lemma 2.1. *Let $f : M_n^+ \longrightarrow M_n^+$ be a map that satisfies* (1.1) *and is surjective. Assume further $n \geq 3$ if $A * B$ is the Jordan triple product. Then f is bijective.*

Proof. Since we assume surjectivity, we will prove injectivity. Suppose $A, B \in M_n^+$ satisfy $f(A) = f(B)$. For any (i, j) pair with $1 \leq i, j \leq n$, since AE_{ij} has all columns zero except for the jth column, and the jth column of AE_{ij} is just the ith column of A, we have $r(A\sqrt{E_{ij}}^2) = r(AE_{ij}) = a_{ji}$. Similarly $r(B\sqrt{E_{ij}}^2) = r(BE_{ij}) = b_{ji}$. Then by our spectral radius conditions,

$$a_{ji} = r(AE_{ij}) = r(f(A)f(E_{ij})) = r(f(B)f(E_{ij})) = r(BE_{ij}) = b_{ji},$$

or

$$a_{ji} = r(f(A)f(\sqrt{E_{ij}}^2)) = r(f(B)f(\sqrt{E_{ij}})^2) = r(B\sqrt{E_{ij}}^2) = b_{ji}.$$

Thus, $A = B$. $\square$

Remark 2.2. Since f is a bijection, it is simple to observe that its inverse f^{-1} fulfills (1.1) if f does, i.e.,

$$r(f(A) * f(B)) = r(A * B) = r(f^{-1}(f(A)) * f^{-1}(f(B))).$$

The following observations will be used throughout our discussion.

Lemma 2.3. *Assume that the function $f : M_n^+ \to M_n^+$ satisfies condition* (1) *in Theorem* 1.1. *Then:*

(a) *For any $A \in M_n^+$ we have $r(A) = r(f(A))$.*

(b) *$A \in M_n^+$ is nilpotent if and only if $f(A)$ is nilpotent.*

(c) *If $A \in M_n^+$ is nilpotent, i.e., $r(A) = 0$, then all diagonal elements of A and $f(A)$ are zeros.*

(d) *If in addition the range of f contains a matrix with positive entries, then A is nonzero if and only if $f(A)$ is nonzero.*

Proof. Condition (a) follows from setting $A = B$ in (1.1). Condition (b) follows trivially from (a).

Condition (c) follows from nilpotency and nonnegativity. A nilpotent matrix has all zero eigenvalues, so the sum of all eigenvalues, and equivalently the trace, is zero. Since the trace is the sum of the diagonal entries, all of which are nonnegative, we finally obtain that the diagonal entries must all be zero. By (b), we get the conclusion on $f(A)$.

For condition (d), let $A \in M_n^+$, and let $X \in M_n^+$ be the matrix with all entries equal to $1/n$. Then $X^2 = X$, and $\mathrm{tr}\,(A * X) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}/n$. Hence, if $A \neq 0$, then $\mathrm{tr}\,(A * X) \neq 0$ and hence $0 \neq r(A * X) = r(f(A) * f(X))$. It follows that $f(A) \neq 0$. Similarly, if $f(A) \neq 0$ and if $f(Y) = Z$, where Z is a matrix with all entries positive in the range of f, then $\mathrm{tr}\,(f(A) * Z) \neq 0$. So $0 \neq r(f(A) * Z) = r(A * Y)$ and hence $A \neq 0$. $\square$

Theorem 2.4. *Let $f : M_n^+ \to M_n^+$, $n \geq 3$, be such that*

$$r(A) = r(f(A)) \quad (A \in M_n^+), \tag{2.1}$$

and let $\mathcal{N} = \{(i,j) : 1 \leq i,j \leq n\}$. Assume that there exist a permutation τ on the set $\mathcal{N}$ and a collection of positive numbers $\{\gamma_{ij} : (i,j) \in \mathcal{N}\}$ that satisfies

(1) $\gamma_{ij} = 1/\gamma_{ji}$,

(2) $\tau(i,j) = (p,q) \Rightarrow \tau(j,i) = (q,p)$,

and f has the property that

$$f(\sum_{(i,j)\in S} E_{ij}) = \sum_{(i,j)\in S} \gamma_{ij} E_{\tau(i,j)}$$

for all $S \subseteq \mathcal{N}$, $S \neq \emptyset$. Then there exists a matrix $Q \in \mathcal{PD}$ such that either

$$f(E_{ij}) = Q^{-1} E_{ij} Q, \qquad ((i,j) \in \mathcal{N}), \tag{2.2}$$

or

$$f(E_{ij}) = Q^{-1} E_{ij}^t Q, \qquad ((i,j) \in \mathcal{N}). \tag{2.3}$$

Proof. Let $F_{ij} = f(E_{ij})$ for $1 \le i, j \le n$. We adjust our map f via $X \mapsto P^t f(X)P$ for a suitable permutation matrix P so that $F_{jj} = E_{jj}$ for $j = 1, \dots, n$. We proceed in 4 steps:

Step 1. We show that $F_{ij} = \gamma_{ij} E_{ij}$ or $F_{ij} = E_{ji}/\gamma_{ij}$. We may assume $i \ne j$. Let

$F_{ij} = \gamma_{ij} E_{pq}$, and assume $p \ne i, j$. Consider $A = E_{ij} + E_{ji} + E_{pp}$. Then clearly $r(A) = 1$. But $f(A) = \gamma_{ij} E_{pq} + E_{qp}/\gamma_{ij} + E_{pp}$, and $p_{f(A)}(t) = t^n - t^{n-1} - t^{n-2} = t^{n-2}(t^2 - t - 1)$, so $r(f(A)) = \frac{1+\sqrt{5}}{2} > 1 = r(A)$, a contradiction. Similarly, assume $q \ne i, j$, and we reach the same contradiction. In view of (2.1), we cannot have $p = q$. Therefore $F_{ij} = \gamma_{ij} E_{ij}$ or $F_{ij} = E_{ji}/\gamma_{ij}$.

Step 2. We show that $F_{1j} = \gamma_{1j} E_{1j}$ and $F_{j1} = E_{j1}/\gamma_{1j}$, after possible replacement of f by a map of the form $X \mapsto f(X)^t$.

We may assume that $F_{12} = \gamma_{12} E_{12}$ and $F_{21} = E_{21}/\gamma_{21}$; otherwise, use the map $X \mapsto f(X)^t$. Now consider $A = E_{12} + E_{2j} + E_{j1}$ for $3 \le j \le n$. Then $r(A) = 1 = r(f(A))$. But

$$f(A) = F_{12} + F_{2j} + F_{j1} = \gamma_{12} E_{12} + F_{2j} + F_{j1},$$

and if $F_{2j} = \gamma_{j2} E_{j2}$ or $F_{j1} = \gamma_{1j} E_{1j}$, then $f(A)$ is nilpotent, and $r(f(A)) = 0$, a contradiction. So $F_{j1} = E_{j1}/\gamma_{1j}$ which gives us $F_{1j} = \gamma_{1j} E_{1j}$.

Step 3. We show that $F_{ij} = \gamma_{ij} E_{ij}$ and $F_{ji} = E_{ji}/\gamma_{ij}$.

We only need to consider the case $1 < i < j \le n$. So let $A = E_{1i} + E_{ij} + E_{j1}$, and we have

$$f(A) = F_{1i} + F_{ij} + F_{j1} = \gamma_{1i} E_{1i} + F_{ij} + E_{j1}/\gamma_{1j}.$$

But again, if $F_{ij} = E_{ji}/\gamma_{ij}$ then $f(A)$ is nilpotent. But then $r(A) = 1 > 0 = r(f(A))$, a contradiction. Therefore, $f(E_{ij}) = \gamma_{ij} E_{ij}$ and $f(E_{ji}) = E_{ji}/\gamma_{ij}$.

Step 4. We show $\gamma_{ij} = \gamma_{1j}/\gamma_{1i}$.

Consider the same matrix $A = E_{1i} + E_{ij} + E_{j1}$, with $r(A) = 1 = r(f(A))$. But $f(A) = \gamma_{1i} E_{1i} + \gamma_{ij} E_{ij} + E_{j1}/\gamma_{1j}$, so

$$p_{f(A)}(t) = t^n - \gamma_{1i}\gamma_{ij}/\gamma_{1j} t^{n-3} = t^{n-3}(t^3 - \gamma_{1i}\gamma_{ij}/\gamma_{1j}).$$

Then

$$\sigma(f(A)) = \{0, \sqrt[3]{\gamma_{1i}\gamma_{ij}/\gamma_{1j}}, \omega \sqrt[3]{\gamma_{1i}\gamma_{ij}/\gamma_{1j}}, \omega^2 \sqrt[3]{\gamma_{1i}\gamma_{ij}/\gamma_{1j}}\}$$

(zero is present only if $n > 3$), where ω is the primitive cubic root of 1. Since $r(f(A)) = 1$, we have $\sqrt[3]{\gamma_{1i}\gamma_{ij}/\gamma_{1j}} = 1$, so $\gamma_{1i}\gamma_{ij}/\gamma_{1j} = 1$. Our conclusion follows.

Now replace f by the map $X \mapsto Df(X)D^{-1}$ with $D = \operatorname{diag}(1, \gamma_{12}, \dots, \gamma_{1n})$. Then we have $f(E_{ij}) = F_{ij} = E_{ij}$ for all $1 \le i, j \le n$. Therefore, reversing our modifications, for $Q = PD \in \mathcal{PD}$, then f must be of the form (2.2) or of the form (2.3). $\square$

3. The usual product

This section concerns the proof of "(1) ⇒ (4)" of Theorem 1.1 for the usual product $A * B = AB$.

For the rest of this section, we always assume that f is a bijective (cf. Lemma 2.1) map on M_n^+ that satisfies

$$r(AB) = r(f(A)f(B)), \qquad (A, B \in M_n^+). \tag{3.1}$$

Let us now define a set of matrices useful for our proof.

Definition 3.1. For every $A \in M_n^+$ define $\mathcal{F}(A) = \{X \in M_n^+ : r(AX) > 0\}$.

The next few results examine and exploit the relationships between these sets and the bijectivity of our function to extract relationships between a matrix and its image.

Lemma 3.2. *Suppose $A = [a_{ij}] \in M_n^+$, $B = [b_{ij}] \in M_n^+$.*

(a) *If there is (i, j) pair such that $b_{ij} > 0 = a_{ij}$ then $\mathcal{F}(B) \setminus \mathcal{F}(A)$ is non-empty.*

(b) *The inclusion $\mathcal{F}(A) \subseteq \mathcal{F}(B)$ holds if and only if $b_{ij} > 0$ whenever $a_{ij} > 0$, i.e., there is $\gamma > 0$ such that $\gamma B - A \in M_n^+$.*

(c) *The following two conditions are equivalent:*

(c.1) *$\mathcal{F}(A) = \mathcal{F}(B)$.*

(c.2) *$a_{ij} = 0$ if and only if $b_{ij} = 0$.*

Proof. (a) Let $a_{ij} = 0$, and $b_{ij} > 0$. Let $X = E_{ji}$. Then

$$r(BX) = b_{ij} > 0 = a_{ij} = r(AX).$$

So $X \in \mathcal{F}(B)$ and $X \notin \mathcal{F}(A)$. The result follows.

(b) The necessity follows from (a): if $a_{ij} > 0 = b_{ij}$, then $\exists X \in \mathcal{F}(A) \setminus \mathcal{F}(B)$.

To prove the sufficiency, assume that $b_{ij} > 0$ whenever $a_{ij} > 0$. Then there is $\gamma > 0$ such that $\gamma B - A \in M_n^+$. So $\gamma B \geq A$ (entrywise inequality), and then $\gamma BX \geq AX$ and $r(\gamma BX) \geq r(AX)$ for all $X \in M_n^+$. (We use here the well known monotonicity property of the spectral radius, see, e.g., [4, Theorem 8.1.18] or [1].) Thus, for any $X \in \mathcal{F}(A)$,

$$r(\gamma BX) = \gamma r(BX) \geq r(AX) > 0.$$

So $r(BX) > 0$, thus $X \in \mathcal{F}(B)$. It follows that $\mathcal{F}(A) \subseteq \mathcal{F}(B)$.

(c) Note that $\mathcal{F}(A) = \mathcal{F}(B)$ if and only if $\mathcal{F}(A) \subseteq \mathcal{F}(B) \subseteq \mathcal{F}(A)$. By (b), this is equivalent to any one of the following conditions:

(i) $b_{ij} > 0$ if and only if $a_{ij} > 0$. (ii) $b_{ij} = 0$ if and only if $a_{ij} = 0$. □

Corollary 3.3. *A matrix $X \in M_n^+$ has exactly k nonzero entries if and only if there is a sequence of matrices $X_1, \ldots, X_k, \ldots X_{n^2}$ in M_n^+ with $X_k = X$ such that $\mathcal{F}(X_i)$ is proper non-empty subset of $\mathcal{F}(X_{i+1})$ for $i = 1, \ldots, n^2 - 1$.*

Proof. If $X_k = X$ has exactly k nonzero entries, we can replace zero entries with nonzero entries one at a time to get $X_{k+1}, \ldots, X_{n^2}$. Similarly, we can replace nonzero entries with zeros one at a time to get the required $X_{k-1}, X_{k-2}, \ldots, X_1$.

Observe that since we only replaced $k-1$ nonzero entries with zeros, $X_1 \neq 0$. So this construction yields the desired sequence.

Conversely, if $X = X_1, X_2, \ldots, X_{n^2}$ have the described property, then $X_1 \neq 0$ because $\mathcal{F}(X_1)$ is non-empty. Moreover, by Lemma 3.2, X_{i+1} has at least one more nonzero entry than X_i. It follows that X_i must have exactly i nonzero entries for each i, so the result follows. $\square$

Note that for $A \in M_n^+$, we have

$$\begin{aligned} \mathcal{F}(f(A)) &= \{X \in M_n^+ : r(f(A)X) > 0\} = \{X \in M_n^+ : r(Af^{-1}(X)) > 0\} \\ &= \{f(Y) \in M_n^+ : r(AY) > 0\} = f(\mathcal{F}(A)). \end{aligned}$$

Thus, we have the following.

Lemma 3.4. *If $A \in M_n^+$, then $\mathcal{F}(f(A)) = f(\mathcal{F}(A))$.*

Corollary 3.5. *A matrix $X \in M_n^+$ has exactly k nonzero entries if and only if $f(X)$ has exactly k nonzero entries.*

Proof. Let $X \in M_n^+$ such that X has exactly k nonzero entries. By Corollary 3.3 there exist $X_1, \ldots, X_{n^2}$ in M_n^+ with $X_k = X$ such that $\mathcal{F}(X_i)$ is proper non-empty subset of $\mathcal{F}(X_{i+1})$ for $i = 1, \ldots, n^2-1$. By Lemma 3.4, we have

$$\mathcal{F}(f(X_i)) = f(\mathcal{F}(X_i)) \subseteq f(\mathcal{F}(X_{i+1})) = \mathcal{F}(f(X_{i+1})),$$

and the inclusion is strict in view of bijectivity of f. Thus, $f(X_1), \ldots, f(X_{n^2})$ is a sequence satisfying Corollary 3.3. So, $f(X_k) = f(X)$ has k nonzero entries. Applying the above proof to f^{-1} in place of f (see Remark 2.2) we see that $f^{-1}(X)$ has k nonzero entries. $\square$

This concludes our direct involvement with our (3.1) sets. Now we will use our obtained results to characterize the image of another set of useful matrices: the matrix units.

Lemma 3.6. *Let $f(E_{ij}) = F_{ij}$ for $1 \leq i, j \leq n$. Then:*

(a) *For $i \neq j$, we have $F_{ij} = \gamma_{ij} E_{pq}$ for some $1 \leq p, q \leq n$, $p \neq q$, where $\gamma_{ij} > 0$.*

(b) *$\{F_{11}, \ldots, F_{nn}\} = \{E_{11}, \ldots, E_{nn}\}$.*

(c) *If $i \neq j$, then*

$$F_{ij} = \gamma_{ij} E_{pq} \qquad \Longrightarrow \qquad F_{ji} = \gamma_{ij}^{-1} E_{qp};$$

thus $\gamma_{ji} = \gamma_{ij}^{-1}$.

(d) *There is a permutation τ of $\{(i,j) : 1 \leq i, j \leq n\}$ with the properties that*

$$\tau(i,j) = (p,q) \qquad \Longrightarrow \qquad \tau(j,i) = (q,p) \tag{3.2}$$

and $F_{ij} = \gamma_{ij} E_{\tau(i,j)}$ for all pairs (i,j), $1 \leq i, j \leq n$ (we take $\gamma_{ii} = 1$ for $i = 1, 2, \ldots, n$).

(e) *For $f([a_{kl}]) = [x_{kl}]$, we have $\gamma_{ij} a_{ij} = x_{\tau(i,j)}$.*

Proof. For (a) let $i \neq j$. From Corollary 3.5, F_{ij} has exactly one nonzero entry. But $r(F_{ij}) = r(E_{ij}) = 0$, so this nonzero entry is not on the diagonal, thus $F_{ij} = \gamma_{ij} E_{pq}$ for some positive γ_{ij}, $p \neq q$.

For (b), by Corollary 3.5 F_{ij} has exactly one nonzero entry for all i, j. For $i = j$, since $r(F_{ii}) = r(E_{ii}) = 1$, this nonzero entry must be on the diagonal, and it must be 1. So $F_{ii} = E_{pp}$ for some p. Furthermore, since $r(F_{ii}F_{kk}) = r(E_{ii}E_{kk}) = 0$ for $i \neq k$, no two F_{ii}'s can have the same nonzero position, so we get the desired result.

For (c), let $i \neq j$. So $F_{ij} = \gamma_{ij} E_{pq}$ for some (p, q), $p \neq q$. Since $r(F_{ij}F_{ji}) = r(E_{ij}E_{ji}) = 1$, then the nonzero entry of F_{ji} must be in the transposed position to the nonzero entry of F_{ij} to get a nonzero entry on the diagonal. Furthermore, these entries therefore must be inverse to each other. Thus $F_{ji} = E_{qp}/\gamma_{ij}$.

Since for $(p, q) \notin \{(i, j), (j, i)\}$ we have

$$r(F_{ij}F_{pq}) = r(E_{ij}E_{pq}) = 0 = r(E_{ji}E_{pq}) = r(F_{ji}F_{pq}),$$

it is clear that no other F_{pq} shares its nonzero position with F_{ij} or F_{ji}. Then our τ can be defined by $\tau(i, j) = (p, q)$ if $F_{ij} = \gamma_{ij} E_{pq}$, and is bijective by our above discussion, so it is the permutation required by (d). Property (3.2) holds in view of (c).

Finally for (e), note that

$$\begin{aligned} a_{ij} &= r(AE_{ji}) = r([x_{ij}]f(E_{ji})) = r\left([x_{ij}]\gamma_{ji}E_{\tau(j,i)}\right) \\ &= r\left([x_{ij}]\gamma_{ij}^{-1}E^t_{\tau(i,j)}\right) = \gamma_{ij}^{-1}x_{\tau(i,j)} \end{aligned}$$

by (d). It follows that $x_{\tau(i,j)} = \gamma_{ij} a_{ij}$. □

Corollary 3.7. *Let $\mathcal{N} = \{(i, j) : 1 \leq i, j \leq n\}$, and let $S \subseteq \mathcal{N}$, $S \neq \emptyset$. Then*

$$f\Big(\sum_{(i,j)\in S} E_{ij}\Big) = \sum_{(i,j)\in S} F_{ij}.$$

Proof. Let $A = \sum_{(i,j)\in S} E_{ij} = [a_{ij}]$, and $f(A) = [x_{ij}]$.

First observe that $a_{ij} = 1$ if $(i, j) \in S$ and $a_{ij} = 0$ otherwise. From (e) we have that $\gamma_{ij} a_{ij} = x_{\tau(i,j)}$. Then $x_{\tau(i,j)} = \gamma_{ij}$ if $(i, j) \in S$ and $x_{\tau(i,j)} = 0$ otherwise. Representing $f(A)$ as a combination of matrix units, we get

$$f(A) = \sum_{(i,j)\in\mathcal{N}} x_{ij}E_{ij} = \sum_{\tau(i,j)\in S} \gamma_{ij}E_{\tau(i,j)} = \sum_{(i,j)\in S} F_{ij}.$$

□

We are now ready to present the proof of Theorem 1.1 for the usual product.

Proof. Recall that the bijective map $f : M_n^+ \longrightarrow M_n^+$ has the property (3.1). If $n = 2$, then Theorem 1.1 follows easily from Lemma 3.6 and Corollary 3.7. Thus suppose $n > 2$. By Lemma 3.6 and Corollary 3.7, the hypotheses of Theorem 2.4 are satisfied. Thus, either (2.2) or (2.3) holds. For $f(E_{ij}) = Q^{-1}E_{ij}Q$, define

$\hat{f}_1(A) = Qf(A)Q^{-1}$, and for $f(E_{ij}) = Q^{-1}E_{ij}^tQ$, define $\hat{f}_2(A) = \hat{f}_1(A)^t$. Then $\hat{f}_1, \hat{f}_2 : M_n^+ \longrightarrow M_n^+$ since $Q, f(A), Q^{-1} \in M_n^+$. Also, for all $A, B \in M_n^+$,

$$r(AB) = r(f(A)f(B)) = r(Qf(A)(Q^{-1}Q)f(B)Q^{-1}) = r(\hat{f}_1(A)\hat{f}_1(B)),$$

and

$$r(AB) = r(\hat{f}_1(A)\hat{f}_1(B)) = r(\hat{f}_1(B)^t\hat{f}_1(A)^t) = r(\hat{f}_2(B)\hat{f}_2(A)) = r(\hat{f}_2(A)\hat{f}_2(B)),$$

thus we may use our previous machinery on these functions.

Trivially, $\hat{f}_1(E_{ij}) = \hat{f}_2(E_{ij}) = E_{ij}$. Applying Lemma 3.6 to $\hat{f}_k$, we have $\gamma_{ij} = 1$ and τ the identity permutation. Then for $A = [a_{ij}] \in M_n^+$ and $\hat{f}_k(A) = [x_{ij}]$, we apply part (e) of that lemma to get $x_{ij} = x_{\tau(i,j)} = \gamma_{ij}a_{ij} = a_{ij}$, and so $\hat{f}_k(A) = A$. Thus, $f(A) = Q^{-1}\hat{f}_1(A)Q = Q^{-1}AQ$ or $f(A) = Q^{-1}\hat{f}_2(A)^tQ = Q^{-1}A^tQ$. □

4. The Triple Product

This section concerns the proof of "(1) ⇒ (4)" of Theorem 1.1 for the Jordan triple product $A * B = ABA$. For the rest of this section, we always assume that the surjective map f on M_n^+ satisfies

$$r(ABA) = r(f(A)f(B)f(A)), \qquad (A, B \in M_n^+). \tag{4.1}$$

Note that by Lemma 2.1 f is automatically bijective if $n > 2$.

We first treat the special case when $n = 2$.

Lemma 4.1. *Suppose $n = 2$ and $A * B = ABA$. Then the implication "(1) ⇒ (4)" of Theorem 1.1 holds.*

Proof. We divide the proof into several assertions. We will use the observation that $A \in M_2^+$ is a non-zero nilpotent if and only if it has exactly one non-zero entry at the off-diagonal position.

Assertion 1 $\{f(E_{11}), f(E_{22})\} = \{E_{11}, E_{22}\}$.

To see this, let $f(E_{ii}) = [x_{pq}]$. Observe that since all diagonal entries of a nilpotent matrix are 0, $r(E_{ii}^2N) = r(E_{ii}N) = 0$ for any nilpotent matrix N. So for N such that $f(N) = E_{12}$, we have $r(f(E_{ii})^2E_{12}) = r(E_{ii}^2N) = 0$. But

$$f(E_{ii})^2E_{12} = \begin{pmatrix} 0 & x_{11}^2 + x_{12}x_{21} \\ 0 & x_{21}(x_{11} + x_{22}) \end{pmatrix},$$

so $x_{21}(x_{11} + x_{22}) = 0$. By similar argument, we see that $x_{12}(x_{11} + x_{22}) = 0$.

Assume that $x_{11} + x_{22} = 0$. Then

$$f(E_{ii})^2 = \operatorname{diag}(x_{12}x_{21}, x_{12}x_{21}) = x_{12}x_{21}I.$$

But for $i \neq j$,

$$0 = r(E_{ii}E_{jj}) = r(f(E_{ii})^2f(E_{jj})) = r(x_{12}x_{21}f(E_{jj})) = x_{12}x_{21}r(f(E_{jj})) = x_{12}x_{21}.$$

This is impossible, for then $r(f(E_{ii})) = 0 \neq 1 = r(E_{ii})$ which is a contradiction. So we must have $x_{21} = x_{12} = 0$.

Thus, $f(E_{11}) = \operatorname{diag}(x_{11}, x_{22})$ and $f(E_{22}) = \operatorname{diag}(y_{11}, y_{22})$ for some nonnegative numbers $x_{11}, x_{22}, y_{11}, y_{22}$. But then

$$f(E_{11})^2 f(E_{22}) = \begin{pmatrix} x_{11}^2 y_{11} & 0 \\ 0 & x_{22}^2 y_{22} \end{pmatrix}.$$

Since $0 = r(E_{ii}E_{jj}) = r(f(E_{ii})^2 f(E_{jj}))$, we have $x_{11}^2 y_{11} + x_{22}^2 y_{22} = 0$. Now, since $x_{11} + x_{22} > 0$ and $y_{11} + y_{22} > 0$, $f(E_{11})$ and $f(E_{22})$ must have exactly one nonzero entry on the diagonal in different positions, and that nonzero entry must be 1. This completes the proof of Assertion 1.

Replacing f by the map $X \mapsto P^t f(X) P$ for a suitable permutation matrix P, we may assume that $f(E_{ii}) = E_{ii}$. Additionally, up to transposition, $f(E_{12}) = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$ for $\gamma \geq 0$. Observe that since E_{ij} is nonzero, we have $\gamma > 0$. We will assume this is the case since if it is not we can instead consider the map $X \to f(X)^{\mathrm{t}}$.

After these modifications, we can proceed to prove the following.

Assertion 2 *Let $A = [a_{ij}]$ and $f(A) = [f_{ij}]$. Then $f_{ii} = a_{ii}$.*

To see this, simply consider $f_{ii} = r(E_{ii}^2 f(A)) = r(E_{ii}^2 A) = a_{ii}$.

Assertion 3
Let $X = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. Then

$$f(X) = E_{11} + \gamma E_{12} \qquad \text{and} \qquad f(Y) = \gamma^{-1} E_{21} + E_{22}. \tag{4.2}$$

Let $f(X) = [x_{ij}]$, $f(Y) = [y_{ij}]$. Then $x_{11} = 1 = y_{22}$, $x_{22} = 0 = y_{11}$ by the previous assertion. But then

$$f(X)^2 = \begin{pmatrix} 1 + x_{12}x_{21} & x_{12} \\ x_{21} & x_{12}x_{21} \end{pmatrix}, \qquad f(Y)^2 = \begin{pmatrix} y_{12}y_{21} & y_{12} \\ y_{21} & 1 + y_{12}y_{21} \end{pmatrix},$$

which gives us

$$0 = r(X^2 E_{22}) = r(f(X)^2 E_{22}) = x_{12}x_{21}$$

and

$$0 = r(Y^2 E_{11}) = r(f(Y)^2 E_{11}) = y_{12}y_{21}.$$

Now

$$0 = r(X^2 E_{12}) = r(f(X)^2 \gamma E_{12}) = \gamma x_{21},$$

so $x_{21} = 0$. Similarly,

$$1 = r(Y^2 E_{12}) = r(f(Y)^2 \gamma E_{12}) = \gamma y_{21},$$

so $y_{21} = \gamma^{-1}$ and $y_{12} = 0$.

Finally, observe $X^2 Y = X$, so

$$1 = r(X^2 Y) = r(f(X)^2 f(Y)) = r(f(X)f(Y)).$$

But $f(X)f(Y) = \begin{pmatrix} \gamma^{-1}x_{12} & x_{12} \\ 0 & 0 \end{pmatrix}$, therefore, $x_{12} = \gamma$, giving us the desired result (4.2).

We can now modify our function by $X \to Df(X)D^{-1}$ where $D = \operatorname{diag}(\gamma^{-1}, 1)$ so $f(X) = X$ and $f(Y) = Y$. With this additional modification, we can complete the proof of our lemma by proving the following.

Assertion 4 *$f(A) = A$ for all $A \in M_2^+$.*

Let $A = [a_{ij}]$ and $f(A) = [x_{ij}]$. Then by Assertion 2 $x_{ii} = a_{ii}$. Furthermore,

$$X^2A = XA = \begin{pmatrix} a_{11} + a_{21} & a_{12} + a_{22} \\ 0 & 0 \end{pmatrix}$$

and

$$f(X)^2 f(A) = f(X)f(A) = \begin{pmatrix} a_{11} + x_{21} & x_{12} + a_{22} \\ 0 & 0 \end{pmatrix},$$

so

$$a_{11} + x_{21} = r(X^2 f(A)) = r(X^2 A) = a_{11} + a_{21},$$

thus $a_{21} = x_{21}$.

Repeating the same calculation with Y yields $a_{12} = x_{12}$. Thus $a_{ij} = x_{ij}$, so $f(A) = A$. $\square$

Now, we turn to the case when $n > 2$.

Remark 4.2. It is clear from our consideration of $n = 2$ that the arguments in Section 3 are not directly extendable to the triple product. However, we shall adapt the same approach and modify it as needed to obtain our new result for $n \geq 3$. For those results having similar proofs exhibited in Section 3, we shall restate the results but often suppress the proof.

For every $A \in M_n^+$ define

$$\widetilde{\mathcal{F}}(A) = \{X \in M_n^+ : r(AX^2) > 0\}.$$

Lemma 4.3. *Suppose $A = [a_{ij}], B = [b_{ij}] \in M_n^+$.*

(a) *If there is (i, j) pair such that $b_{ij} > 0 = a_{ij}$ then $\widetilde{\mathcal{F}}(B) \setminus \widetilde{\mathcal{F}}(A) \neq \emptyset$.*
(b) *The inclusion $\widetilde{\mathcal{F}}(A) \subseteq \widetilde{\mathcal{F}}(B)$ holds if and only if $b_{ij} > 0$ whenever $a_{ij} > 0$, i.e., there is $\gamma > 0$ such that $\gamma B - A \in M_n^+$.*
(c) *The following two conditions are equivalent:*
 (c.1) *$\widetilde{\mathcal{F}}(A) = \widetilde{\mathcal{F}}(B)$.*
 (c.2) *$a_{ij} = 0$ if and only if $b_{ij} = 0$.*

Proof. (a) Let $a_{ij} = 0$, and $b_{ij} > 0$. Let $X = \sqrt{E_{ji}}$. Then $r(BX^2) = b_{ij} > 0 = a_{ij} = r(AX^2)$, so $X \in \widetilde{\mathcal{F}}(B)$ and $X \notin \widetilde{\mathcal{F}}(A)$. The result follows.

(b) The necessity follows from (a): if $a_{ij} > 0 = b_{ij}$, then there exists $X \in \widetilde{\mathcal{F}}(A) \setminus \widetilde{\mathcal{F}}(B)$.

To prove the sufficiency, assume that $b_{ij} > 0$ whenever $a_{ij} > 0$. Then there is $\gamma > 0$ such that $\gamma B - A \in M_n^+$. So $\gamma B \geq A$, so $r(\gamma BX) \geq r(AX)$ for all $X \in M_n^+$. Thus, for any $X \in \widetilde{\mathcal{F}}(A)$,

$$r(\gamma BX^2) = \gamma r(BX^2) \geq r(AX^2) > 0,$$

thus $X \in \widetilde{\mathcal{F}}(B)$. It follows that $\widetilde{\mathcal{F}}(A) \subseteq \widetilde{\mathcal{F}}(B)$.

Finally, (c) follows easily from (b). $\square$

Corollary 4.4. *A matrix $X \in M_n^+$ has exactly k nonzero entries if and only if there is a sequence of matrices $X_1, X_2, \dots X = X_k, \dots, X_{n^2}$ in M_n^+ such that $\widetilde{\mathcal{F}}(X_i)$ is proper non-empty subset of $\widetilde{\mathcal{F}}(X_{i+1})$ for $i = 1, \dots, n^2 - 1$.*

Proof. Similar to that of Corollary 3.3. $\square$

Note that for $A \in M_n^+$, we have

$$\begin{aligned}\widetilde{\mathcal{F}}(f(A)) &= \{X \in M_n^+ : r(f(A)X^2) > 0\} = \{X \in M_n^+ : r(A(f^{-1}(X))^2) > 0\} \\ &= \{f(Y) \in M_n^+ : r(AY^2) > 0\} = f(\widetilde{\mathcal{F}}(A)).\end{aligned}$$

So, we have the following.

Lemma 4.5. *If $A \in M_n^+$, then $\widetilde{\mathcal{F}}(f(A)) = f(\widetilde{\mathcal{F}}(A))$.*

Corollary 4.6. *A matrix $X \in M_n^+$ has exactly k nonzero entries if and only if $f(X)$ has exactly k nonzero entries.*

Proof. Similar to that of Corollary 3.5. $\square$

Lemma 4.7. *Let $f(E_{ij}) = F_{ij}$ for $1 \le i, j \le n$.*
(a) *If $i \ne j$, then $F_{ij} = \gamma_{ij} E_{pq}$ for some $1 \le p, q \le n$, with $p \ne q$ and $\gamma_{ij} > 0$.*
(b) *We have $\{F_{11}, \dots, F_{nn}\} = \{E_{11}, \dots, E_{nn}\}$.*
(c) *There is a permutation τ of $\{(i,j) : 1 \le i, j \le n\}$ such that $F_{ij} = \gamma_{ij} E_{\tau(i,j)}$ and $F_{ji} = \gamma_{ji} E^t_{\tau(i,j)}$, for all pairs (i,j); moreover, τ satisfies the property:*

$$\tau(i,j) = (p,q) \qquad \Longrightarrow \qquad \tau(j,i) = (q,p).$$

(d) *For $f([a_{ij}]) = [x_{ij}]$, we have $\gamma_{ij} a_{ij} = x_{\tau(i,j)}$.*

Proof. For (a), let $i \ne j$. By Corollary 4.6, F_{ij} has exactly one nonzero entry. But $r(F_{ij}) = r(E_{ij}) = 0$, so this nonzero entry is not on the diagonal, thus $F_{ij} = \gamma_{ij} E_{pq}$ for some $\gamma_{ij} > 0$, $p \ne q$.

For (b), again by Corollary 4.6 F_{ii} has exactly one nonzero entry, and since $r(F_{ii}) = r(E_{ii}) = 1$, this nonzero entry must be on the diagonal, and it must be equal 1. So $F_{ii} = E_{pp}$ for some p. Furthermore, since $r(F_{ii}F_{kk}) = r(E_{ii}E_{kk}) = 0$ for $i \ne k$, no two F_{ii}'s can have the same nonzero position, so we get the desired result.

For (c), let $i \ne j$. By Lemma 4.3 (a) we have that $\widetilde{\mathcal{F}}(E_{ij}) \not\subseteq \widetilde{\mathcal{F}}(E_{ji})$ and vice versa, so by Lemma 4.5 we have $\widetilde{\mathcal{F}}(F_{ij}) \not\subseteq \widetilde{\mathcal{F}}(F_{ji})$. It is clear then that F_{ij} and F_{ji} do not have the same nonzero position. However, consider $A = E_{ij} + E_{ji}$. Then by our lemmas, we know that $\widetilde{\mathcal{F}}(E_{ij}) \subseteq \widetilde{\mathcal{F}}(A)$ and $\widetilde{\mathcal{F}}(E_{ji}) \subseteq \widetilde{\mathcal{F}}(A)$, so it follows that $\widetilde{\mathcal{F}}(F_{ij}) \subseteq \widetilde{\mathcal{F}}(f(A))$ and $\widetilde{\mathcal{F}}(F_{ji}) \subseteq \widetilde{\mathcal{F}}(f(A))$. By Corollary 4.6, $f(A)$ must have exactly two nonzero entries. But by Lemma 4.3 (b), the nonzero positions of F_{ij} and F_{ji} must lie in the nonzero positions of $f(A)$, so each must occupy a distinct nonzero position of $f(A)$. (It is not possible for F_{ij} and F_{ji} to have the same nonzero position; indeed, if they did, then we would have $\widetilde{\mathcal{F}}(F_{ij}) = \widetilde{\mathcal{F}}(F_{ji})$ and

$$f(\widetilde{\mathcal{F}}(E_{ij})) = \widetilde{\mathcal{F}}(f(E_{ij})) = \widetilde{\mathcal{F}}(F_{ij}) = \widetilde{\mathcal{F}}(F_{ji}) = \widetilde{\mathcal{F}}(f(E_{ji})) = f(\widetilde{\mathcal{F}}(E_{ji})),$$

which gives a contradiction because f is bijective and $\widetilde{\mathcal{F}}(E_{ij}) \neq \widetilde{\mathcal{F}}(E_{ji})$.) Furthermore, since $r(f(A)) = r(A) = 1$, the nonzero positions of A must form a cycle, and so must be transposed of each other.

Thus if $F_{ij} = \gamma_{ij}E_{pq}$ then it must be the case that $F_{ji} = \gamma_{ji}E_{qp} = \gamma_{ji}E_{pq}^t$. By our previous considerations it is clear that each F_{ij} has a unique nonzero position (p,q) with respect to one another so we may define a bijection $\tau(i,j) = (p,q)$ accordingly, giving us the required permutation.

Finally for (d), we temporarily return to our square roots. By Corollary 4.6, for $i \neq j$ we have that $f(\sqrt{E_{ij}})$ has exactly two nonzero entries, so either $f(\sqrt{E_{ij}})^2 = 0$ or $f(\sqrt{E_{ij}})^2$ has exactly 1 nonzero entry (note that $f(\sqrt{E_{ij}})$ is nilpotent by Lemma 2.3). Since

$$r(F_{ji}f(\sqrt{E_{ij}})^2) = r(E_{ji}\sqrt{E_{ij}}^2) = 1,$$

it must be the latter case. Moreover, if $F_{ji} = \gamma_{ji}E_{\tau(j,i)}$, then it is clear that $f(\sqrt{E_{ij}})^2 = E_{\tau(i,j)}/\gamma_{ji}$. Now let $A = [a_{ij}]$ and $f(A) = [x_{ij}]$. Fix $i \neq j$, and consider $r(A\sqrt{E_{ji}}^2) = r(AE_{ji}) = a_{ij}$ and

$$r(A\sqrt{E_{ji}}^2) = r(f(A)f(\sqrt{E_{ji}})^2) = r([x_{ij}]E_{\tau(j,i)}/\gamma_{ij}) = x_{\tau(i,j)}/\gamma_{ij}.$$

Therefore, $x_{\tau(i,j)} = \gamma_{ij}a_{ij}$. In the case $i = j$, $\sqrt{E_{ii}} = E_{ii}$, so

$$a_{ii} = r(AE_{ii}) = r(AE_{ii}^2) = r([x_{ij}]F_{ii}) = r([x_{ij}]E_{\tau(i,i)}) = x_{\tau(i,i)}. \qquad \square$$

Corollary 4.8. *Let $\mathcal{N} = \{(i,j) : 1 \leq i,j \leq n\}$, and let $S \subseteq \mathcal{N}$, $S \neq \emptyset$. Then*

$$f(\sum_{(i,j)\in S} E_{ij}) = \sum_{(i,j)\in S} F_{ij}.$$

The proof is completely analogous to that of Corollary 3.7.

We note the following equalities:

$$\gamma_{ji} = 1/\gamma_{ij}, \qquad (i,j, \quad 1 \leq i,j \leq n).$$

Indeed, $f(E_{ij} + E_{ji}) = F_{ij} + F_{ji}$, so

$$\gamma_{ij}\gamma_{ji} = r(F_{ij} + F_{ji}) = r(E_{ij} + E_{ji}) = 1.$$

We are now ready to prove the implication "(1) ⇒ (4)" in Theorem 1.1 for the Jordan triple product.

Proof. Recall that $f : M_n^+ \longrightarrow M_n^+$ has the property (4.1). Using Corollary 4.8 and Theorem 2.4, we see that f must satisfy either (2.2) or (2.3). For $f(E_{ij}) = Q^{-1}E_{ij}Q$, define $\hat{f}_1(A) = Qf(A)Q^{-1}$, and for $f(E_{ij}) = Q^{-1}E_{ij}^tQ$, define $\hat{f}_2(A) = \hat{f}_1(A)^t$. Then $\hat{f}_1, \hat{f}_2 : M_n^+ \longrightarrow M_n^+$ since $Q, f(A), Q^{-1} \in M_n^+$. Also, for all $A, B \in M_n^+$,

$$\begin{aligned} r(AB^2) &= r(f(A)f(B)^2) = r(Qf(A)f(B)^2Q^{-1}) = r(Qf(A)(Q^{-1}Q)f(B)^2Q^{-1}) \\ &= r((Qf(A)Q^{-1})(Qf(B)Q^{-1})^2) = r(\hat{f}_1(A)\hat{f}_1(B)^2), \end{aligned}$$

and

$$\begin{aligned} r(AB^2) &= r(\hat{f}_1(A)\hat{f}_1(B)^2) = r((\hat{f}_1(B)^t)^2\hat{f}_1(A)^t) \\ &= r(\hat{f}_2(B)^2\hat{f}_2(A)) = r(\hat{f}_2(A)\hat{f}_2(B)^2). \end{aligned}$$

Trivially, $\hat{f}_1(E_{ij}) = \hat{f}_2(E_{ij}) = E_{ij}$. Applying Lemma 4.7 to $\hat{f}_k$ we have $\gamma_{ij} = 1$ and $\tau(i,j) = (i,j)$. Now the proof is completed exactly as in the case of the usual product. □

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[1,2,3] Department of Mathematics, College of William and Mary, Williamsburg, VA 23187-8795, USA.

E-mail address: siclar@wm.edu, ckli@math.wm.edu, lxrodm@math.wm.edu