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POSITIVITY OF OPERATOR-MATRICES OF HUA-TYPE

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This paper is dedicated to Professor Josip E. Pečarić

Submitted by F. Kittaneh

Abstract. Let A_j $(j = 1, 2, \dots, n)$ be strict contractions on a Hilbert space. We study an $n \times n$ operator-matrix:

$$\mathbf{H}_n(A_1, A_2, \dots, A_n) = [(I - A_j^* A_i)^{-1}]_{i,j=1}^n.$$

For the case $n = 2$, Hua [Inequalities involving determinants, Acta Math. Sinica, 5 (1955), 463–470 (in Chinese)] proved positivity, i.e., positive semi-definiteness of $\mathbf{H}_2(A_1, A_2)$. This is, however, not always true for $n = 3$. First we generalize a known condition which guarantees positivity of $\mathbf{H}_n$. Our main result is that positivity of $\mathbf{H}_n$ is preserved under the operator Möbius map of the open unit disc $\mathcal{D}$ of strict contractions.

1. Introduction and preliminaries

Let A_j $(j = 1, 2, \dots, n)$ be *strict contractions*, that is, $\|A_j\| < 1$, on a Hilbert space $\mathcal{H}$. Since all $I - A_j^* A_i$ and $I - A_i A_j^*$ are invertible, let us consider an $n \times n$ operator-matrix

$$\mathbf{H}_n(A_1, A_2, \dots, A_n) = [(I - A_j^* A_i)^{-1}]_{i,j=1}^n,$$

and its cousin

$$\mathbf{G}_n(A_1, A_2, \dots, A_n) = [(I - A_i A_j^*)^{-1}]_{i,j=1}^n.$$

Here $\mathbf{X} = [X_{i,j}]_{i,j=1}^n$ means that $X_{i,j}$ is the (i, j)-operator entry of $\mathbf{X}$. (Notice that Xu et al. [7] used $\mathbf{H}_n(A_1, A_2, \dots, A_n)$ for our $\mathbf{G}_n(A_1^*, A_2^*, \dots, A_n^*)$.)

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In this paper our interest is in *positivity*, i.e., positive semi-definiteness, of the operator-matrix $\mathbf{H}_n$ (and also that of $\mathbf{G}_n$). We will use the notation $\mathbf{X} \geq \mathbf{Y}$ to mean that both $\mathbf{X}, \mathbf{Y}$ are selfadjoint and $\mathbf{X} - \mathbf{Y}$ is positive. In particular $\mathbf{X} \geq 0$ means that $\mathbf{X}$ is positive. Here let us use $\mathbf{X} > 0$ to denote its positive definiteness, that is, $\mathbf{X}$ is positive and invertible.

For an operator-matrix $\mathbf{X} = [X_{i,j}]_{i,j=1}^n$ with invertible $X_{n,n}$, the *Schur complement* of the (n,n)-operator entry $X_{n,n}$ in $\mathbf{X}$, denoted by $\mathbf{X}/(n)$ in this paper, is the $(n-1) \times (n-1)$ operator-matrix defined by

$$\mathbf{X}/(n) = [X_{i,j} - X_{i,n}X_{n,n}^{-1}X_{n,j}]_{i,j=1}^{n-1}. \tag{1.1}$$

In this case, $\mathbf{X}$ is invertible if and only if $\mathbf{X}/(n)$ is invertible. Further the following relation holds (see [2, Section 7.7])

$$(\mathbf{X}/(n))^{-1} = \text{the top } (n-1) \times (n-1) \text{ operator-submatrix of } \mathbf{X}^{-1}. \tag{1.2}$$

For our purpose the following *Schur criteria* are quite useful. For selfadjoint $\mathbf{X}$ with invertible $X_{n,n}$ the positivity of $\mathbf{X}$ is equivalent to that $X_{n,n} \geq 0$ and $\mathbf{X}/(n) \geq 0$. Further $\mathbf{X} > 0$ if and only if $X_{n,n} > 0$ and $\mathbf{X}/(n) > 0$.

Let us return to $\mathbf{H}_n(A_1, A_2, \ldots, A_n)$ and $\mathbf{G}_n(A_1, A_2, \ldots, A_n)$. In the case $n = 2$, for simplicity, let us write $A = A_1$ and $A_2 = B$. Hua [4] showed $\mathbf{H}_2(A, B) \geq 0$. Since $(I - B^*B)^{-1} > 0$, by the Schur criteria the Hua's positivity result is equivalent to the following inequality:

$$(I - A^*A)^{-1} - (I - B^*A)^{-1}(I - B^*B)(I - A^*B)^{-1} \geq 0. \tag{1.3}$$

With help of the identity (1.2), Xu et al. [7] gave a simple proof for the following identity due to Hua [4] which guarantees the positivity (1.3):

$$\begin{aligned}&(I - A^*A)^{-1} - (I - B^*A)^{-1}(I - B^*B)(I - A^*B)^{-1} \\ &= (I - B^*A)^{-1}(A - B)^*(I - AA^*)^{-1}(A - B)(I - A^*B)^{-1}.\end{aligned}$$

In [1] we proved also

$$(I - AA^*)^{-1} - (I - AB^*)^{-1}(I - BB^*)(I - BA^*)^{-1} \geq 0, \tag{1.4}$$

consequently $\mathbf{G}_2(A, B) \geq 0$. In this connection, let us point out that the following relation exists behind the inequality (1.4):

$$\begin{aligned}&(I - AA^*)^{-1} - (I - AB^*)^{-1}(I - BB^*)(I - BA^*)^{-1} \\ &= (I - AB^*)^{-1}\{A(A - B)^*(I - AA^*)^{-1}(A - B)A^* \\ &+(A - B)(A - B)^*\}(I - BA^*)^{-1}.\end{aligned}$$

What happens when $n \geq 3$? In [1] we showed that $\mathbf{H}_3(A_1, A_2, A_3) \geq 0$ is not always true, while Xu et al. [7] has shown that the situation is the same for $\mathbf{G}_3(A_1, A_2, A_3)$. Let us start with a relation between $\mathbf{H}_n(A_1, A_2, \ldots, A_n)$ and $\mathbf{G}_n(A_1, A_2, \ldots, A_n)$.

$$\begin{aligned}\mathbf{G}_n(A_1, A_2, \ldots, A_n) = [\overbrace{I, I, \ldots, I}^{n}]^*[\overbrace{I, I, \ldots, I}^{n}] + \operatorname{diag}(A_1, A_2, \ldots, A_n) \\ \times\mathbf{H}_n(A_1, A_2, \ldots, A_n) \cdot \operatorname{diag}(A_1, A_2, \ldots, A_n)^*.\end{aligned} \tag{1.5}$$

In fact, since $A(I - BA)^{-1} = (I - AB)^{-1}A$ for any strict contractions A, B,

$$I + A_i(I - A_j^*A_i)^{-1}A_j^* = I + (I - A_iA_j^*)^{-1}A_iA_j^* = (I - A_iA_j^*)^{-1}.$$

Since $[\overbrace{I, I, \ldots, I}^{n}]^*[\overbrace{I, I, \ldots, I}^{n}] \geq 0$, we can conclude from (1.5) the following.

Theorem 1.1. $\boldsymbol{H}_n(A_1, A_2, \ldots, A_n) \geq 0$ *implies* $\boldsymbol{G}_n(A_1, A_2, \ldots, A_n) \geq 0$.

Remark 1.2. The idea of the proof of Theorem 1.1 is implicit in Xu et al. [7].

However, $\mathbf{G}_n(A_1, A_2, \ldots, A_n) \geq 0$ does not imply $\mathbf{H}_n(A_1, A_2, \ldots, A_n) \geq 0$.

Example 1.3. When $\mathcal{H}$ is of 2-dimension, every operator is represented by a 2×2 matrix. Take $0 < \lambda < 1$ and let

$$A_1 = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ A_2 = \lambda \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } A_3 = 0.$$

Then $\mathbf{G}_3(A_1, A_2, A_3) \geq 0$ but $\mathbf{H}_3(A_1, A_2, A_3) \ngeq 0$.

In fact, simple computation will show that, with $\alpha \equiv \lambda^2$,

$$\mathbf{G}_3(A_1, A_2, A_3)/(3) = \frac{\alpha}{1-\alpha} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geq 0.$$

hence $\mathbf{G}_3(A_1, A_2, A_3) \geq 0$ by the Schur criteria. On the other hand

$$\mathbf{H}_3(A_1, A_2, A_3)/(3) = \frac{\alpha}{1-\alpha} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1-\alpha & 0 \\ 0 & 1-\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is not positive semi-definite, because it has a 2×2 principal submatrix $\begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}$, which is not positive semi-definite. Therefore $\mathbf{H}_3(A_1, A_2, A_3) \ngeq 0$ by the Schur criteria.

In [1] we showed that if A_j $(j = 1, 2, \ldots, n)$ are commuting normal operators, then $\mathbf{H}_n(A_1, A_2, \ldots, A_n) \geq 0$ and also $\mathbf{G}_n(A_1, A_2, \ldots, A_n) \geq 0$. In the next section we give a generalization of this result.

Our main result of this paper is that positivity of $\mathbf{H}_n$ is preserved under an operator Möbius map of the open unit disc $\mathcal{D}$ of strict contractions.

2. Main Results

Theorem 2.1. *Let* A_j $(j = 1, 2, \ldots, n)$ *be strict contractions. If the products* $A_j^* A_i$ $(i, j = 1, 2, \ldots, n)$ *are commuting normal operators,* $\boldsymbol{H}_n(A_1, A_2, \ldots, A_n) \geq 0$.

Proof. Our idea of the proof is parallel to that of Xu et al. [7]. The assumption means that there is a commutative unital *-subalgebra $\mathcal{C} \subset B(\mathcal{H})$ such that $A_j^* A_i \in \mathcal{C}$ $(i, j = 1, 2, \ldots, n)$. Then by the Gelfand theorem (see [6, Theorem 4.4]) there is a *-isomorphism π of $\mathcal{C}$ to the commutative C*-algebra $C(\Omega)$ of continuus functions on a compact set Ω. Here the adjoint f^* of a function $f \in C(\Omega)$ is determined by

$$f^*(\omega) = \overline{f(\omega)} \quad (\omega \in \Omega). \tag{2.1}$$

Therefore we can write $f^* = \overline{f}$. Notice further that positivity of a $C(\Omega)$-matrix $[f_{i,j}]_{i,j=1}^n$ is equivalent to saying that for every $\omega \in \Omega$ the numerical matrix $[f_{i,j}(\omega)]_{i,j=1}^n$ is positive semi-definite in the usual sense.

Now let

$$f_{i,j} \equiv \pi(A_j^* A_i) \ \ (i,j = 1,2,\dots,n)$$

Then by (2.1)

$$f_{j,i} = \pi(A_i^* A_j) = \pi(A_j^* A_i)^* = \overline{f_{i,j}}.$$

Then since

$$[A_i^* A_j]_{i,j=1}^n = [A_1, A_2, ..., A_n]^* \cdot [A_1, A_2, ..., A_n] \geq 0$$

it follows that $[f_{j,i}]_{i,j=1}^n \geq 0$. Therefore for any $\omega \in \Omega$

$$[f_{i,j}(\omega)]_{i,j=1}^n = [\overline{f_{j,i}(\omega)}]_{i,j=1}^n \geq 0.$$

Recall the positivity theorem for *Schur product* (or Hadamard product) (see [3, Theorem 5.2.1]) that for two numerical $n \times n$ matrices

$$[\alpha_{i,j}]_{i,j=1}^n \geq 0 \ \text{ and } \ [\beta_{i,j}]_{i,j=1}^n \geq 0 \implies [\alpha_{i,j}\beta_{i,j}]_{i,j=1}^n \geq 0. \tag{2.2}$$

Then since

$$[(I - A_j^* A_i)^{-1}]_{i,j=1}^n = \sum_{k=0}^{\infty} [(A_j^* A_i)^k]_{i,j=1}^n,$$

and

$$[\pi((A_j^* A_i)^k)]_{i,j=1}^n = [f_{i,j}^k]_{i,j=1}^n,$$

it follows from the Schur product theorem (2.2) that

$$[(A_j^* A_i)^k]_{i,j=1}^n \geq 0 \ \ (k = 1,2,\dots),$$

consequently $[(I - A_j^* A_i)^{-1}]_{i,j=1}^n \geq 0$. □

In a similar way we can prove

Theorem 2.2. *Let A_j $(j = 1,2,\dots,n)$ be strict contractions. If the products $A_i A_j^*$ $(i,j = 1,\dots,n)$ are commuting normal operators, $\boldsymbol{G}_n(A_1, A_2, \dots, A_n) \geq 0$.*

Remark 2.3. Positivity of $\mathbf{G}_3(A_1, A_2, A_3)$ in Example 1.3 follows from Theorem 2.2.

In the linear systems theory (see [8, Chapter 10]), for a time-invariant linear system with a state-space realization matrix $\begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}$ it is common to consider the operator-valued function, called the *transfer function*, defined as

$$\zeta \longmapsto B_{2,2} + B_{2,1}(\zeta I - B_{1,1})^{-1} B_{1,2}$$

for complex numbers ζ for which $\zeta I - B_{1,1}$ are invertible. In operator theory, however, it is more convenient to consider a linear-fractional transformation $\Theta(\zeta)$ defined as

$$\Theta(\zeta) = B_{2,2} + \zeta B_{2,1}(I - \zeta B_{1,1})^{-1} B_{1,2}.$$

(See [5, Chapter 6])

Extending the variable from a number ζ to an operator Z, let us define a map $\Theta(Z)$ as

$$\Theta(Z) = B_{2,2} + B_{2,1} Z (I - B_{1,1} Z)^{-1} B_{1,2}. \tag{2.3}$$

For a contraction B, define its *defect operator* D_B as

$$D_B = (I - B^* B)^{1/2}. \tag{2.4}$$

The following relations are immediate from definition (2.4)

$$B D_B = D_{B^*} B, \quad \text{and} \quad B^* D_{B^*} = D_B B^*, \tag{2.5}$$

and for any strict contraction Z the operators $I - B^* Z$ and $I - Z B^*$ are invertible and the following relation holds

$$Z(I - B^* Z)^{-1} = (I - Z B^*)^{-1} Z. \tag{2.6}$$

Lemma 2.4. *When B is a strict contraction, the operator-matrix $\begin{bmatrix} B^* & D_B \\ -D_{B^*} & B \end{bmatrix}$ is unitary, and the map*

$$\Theta(Z) = B - D_{B^*} Z (I - B^* Z)^{-1} D_B = B - D_{B^*}(I - Z B^*)^{-1} Z D_B$$

satisfies the following relations that for any strict contractios Z, W

$$I - \Theta(Z)^* \Theta(W) = D_B (I - Z^* B)^{-1} (I - Z^* W)(I - B^* W)^{-1} D_B.$$

Proof. The proof of unitarity is immediate from (2.5) and omitted. Now since

$$\begin{aligned} \Theta(Z)^*\Theta(W) &= B^*B - D_B(I - Z^*B)^{-1} Z^* D_{B^*} B - B^* D_{B^*} W (I - B^* W)^{-1} D_B \\ &\quad + D_B (I - Z^* B)^{-1} Z^* (I - BB^*) W (I - B^* W)^{-1} D_B, \end{aligned}$$

by (2.5) and (2.6) we can see

$$\begin{aligned} I - \Theta(Z)^*\Theta(W) &= D_B\{I + (I - Z^*B)^{-1} Z^* B + B^* W (I - B^* W)^{-1} \\ &\quad - (I - Z^* B)^{-1}(I - BB^*) W (I - B^* W)^{-1}\} D_B \\ &= D_B (I - Z^* B)^{-1}\{(I - Z^* B)(I - B^* W) + Z^* B (I - B^* W) \\ &\quad + (I - Z^* B) B^* W - Z^*(I - BB^*) W\}(I - B^* W)^{-1} D_B \\ &= D_B (I - Z^* B)^{-1}(I - Z^* W)(I - B^* W)^{-1} D_B. \end{aligned}$$

∎

Given a complex number β with $|\beta| < 1$, the Möbius transformation at β

$$M_\beta(\zeta) \equiv \frac{\beta - \zeta}{1 - \overline{\beta}\zeta}$$

is a conformal map of the open unit disc of the complex plane, which maps 0 to β and β to 0, and is involutive, that is, $M_\beta\left(M_\beta(\zeta)\right) = \zeta$.

The following is an analogy for the case of the open unit disc $\mathcal{D}$ of strict contractions.

Proposition 2.5. *For a strict contraction B, the Möbius map $\Theta_B(\cdot)$ at B, defined by*

$$\Theta_B(Z) \equiv D_{B^*}^{-1}(B - Z)(I - B^*Z)^{-1}D_B,$$

is an involutive map of the open unit disc $\mathcal{D}$, that is,

$$\Theta_B(\Theta_B(Z)) = Z \quad (Z \in \mathcal{D}).$$

It is clear from the definition that $\Theta_B(Z)$ is *holomorphic* with respect to the operator variable Z. Since $\Theta(\cdot)$ is involutive, its inverse is also holomorphic. Therefore $\Theta_B(\cdot)$ becomes a *biholomorphic* map of the open unit disc $\mathcal{D}$ of strict contractions, and is considered as a natural generalization of the Möbius transformation on the open unit disc of the complex plane.

Proof. First let us show the map $\Theta_B(\cdot)$ is nothing but the linear-fractinal transformation $\Theta(\cdot)$ of the unitary operator-matrix $\begin{bmatrix} B^* & D_B \\ -D_{B^*} & B \end{bmatrix}$. In fact, by definition and (2.5)

$$\begin{aligned} \Theta(Z) &= B - D_{B^*}Z(I - B^*Z)^{-1}D_B \\ &= D_{B^*}^{-1}\left\{D_{B^*}BD_B^{-1}(I - B^*Z) - (I - BB^*)Z\right\}(I - B^*Z)^{-1}D_B \\ &= D_{B^*}^{-1}(B - Z)(I - B^*Z)^{-1}D_B = \Theta_B(Z). \end{aligned}$$

Next $\Theta_B(\cdot)$ maps the open unit disc $\mathcal{D}$ to itself. In fact, by Lemma 2.4

$$I - \Theta_B(Z)^*\Theta_B(Z) = D_B(I - Z^*B)^{-1}(I - Z^*Z)(I - B^*Z)^{-1}D_B > 0 \quad (Z \in \mathcal{D}).$$

Finally the involutivity follows from the following two relations:

$$B - \Theta(Z) = D_{B^*}Z(I - B^*Z)^{-1}D_B$$

and

$$\begin{aligned} I - B^*\Theta(Z) &= I - B^*B + B^*D_{B^*}Z(I - B^*Z)^{-1}D_B \\ &= D_B^2 + D_BB^*Z(I - B^*Z)^{-1}D_B \\ &= D_B\{I + B^*Z(I - B^*Z)^{-1}\}D_B = D_B(I - B^*Z)^{-1}D_B. \end{aligned}$$

□

Corollary 2.6. *If an operator-matrix $[B_{i,j}]_{i,j=1}^2$ with $\|B_{2,2}\| < 1$ is unitary, then the map*

$$\Theta(Z) \equiv B_{2,2} + B_{2,1}Z(I - B_{1,1}Z)^{-1}B_{1,2}$$

is a biholomorphic map of the open unit disc $\mathcal{D}$ of strict contractions.

Proof. Let $B = B_{2,2}$. Then it is easy to see from unitarity that there are unitary U, V such that

$$B_{1,1} = UB^*V,\ B_{1,2} = UD_B \ \text{ and } \ B_{2,1} = -D_B^*V.$$

Then we have

$$\Theta(Z) = \Theta_B(VZU)\ (Z \in \mathcal{D}),$$

where $\Theta_B(\cdot)$ is the Möbius map at B. Finally since $Z \longmapsto VZU$ is a biholomorphic map of $\mathcal{D}$, the assertion follows from Proposition 2.5. □

The following is the main result of this paper.

Theorem 2.7. *Let B be a strict contraction, and $\Theta_B(\cdot)$ the Möbius map at B on the open unit disc $\mathcal{D}$ of strict contractions. Then for any $A_i \in \mathcal{D}$ $(i = 1, 2, \ldots, n)$*

$$\boldsymbol{H}_n(A_1, A_2, \ldots, A_n) \geq 0 \ \ \textit{implies} \ \ \boldsymbol{H}_n\left(\Theta_B(A_1), \Theta_B(A_2), \ldots, \Theta_B(A_n)\right) \geq 0.$$

Proof. Since by Lemma 2.4

$$(I - \Theta_B(A_j)^*\Theta_B(A_i))^{-1} = D_B^{-1}(I - B^*A_i)(I - A_j^*A_i)^{-1}(I - A_j^*B)D_B^{-1},$$

we have

$$\mathbf{H}_n\left(\Theta_B(A_1), \Theta_B(A_2), \ldots, \Theta_B(A_n)\right) = \mathbf{D} \cdot \mathbf{H}_n(A_1, A_2, \ldots, A_n) \cdot \mathbf{D}^*$$

where

$$\mathbf{D} = \operatorname{diag}\left(D_B^{-1}(I - B^*A_1), D_B^{-1}(I - B^*A_2), \ldots, D_B^{-1}(I - B^*A_n)\right).$$

This identity proves the assertion. □

Remark 2.8. It is not clear whether or not

$$\mathbf{G}_n(A_1, A_2, \ldots, A_n) \geq 0 \ \ \text{implies} \ \ \mathbf{G}_n\left(\Theta_B(A_1), \Theta_B(A_2), \ldots, \Theta_B(A_n)\right) \geq 0.$$

Remark 2.9. In Introduction we stated that $\mathbf{H}_2(A, B) \geq 0$ is valid for any strict contraction A, B. Let us show that this result is included in the combination of Theorem 2.2 and Theorem 2.7. In fact, consider the Möbius map $\Theta_B(\cdot)$ at B. Then by Proposition 2.5 $A = \Theta_B(\tilde{A})$ where $\tilde{A} = \Theta_B(A)$ and $B = \Theta_B(0)$ and by Theorem 2.2 $\mathbf{H}_2(\tilde{A}, 0) \geq 0$. Then apply Theorem 2.7 to see $\mathbf{H}_2(A, B) \geq 0$.

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