A 1-NORM BOUND FOR INVERSES OF TRIANGULAR MATRICES WITH MONOTONE ENTRIES

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Abstract. This paper provides some new bounds for 1−norms of positive triangular matrices with monotonic column entries. The main theorem refines a recent inequality of Vecchio and Mallik in the case of constant diagonal. The results are shown to be in a sense best possible under the given constraints. En route some partial order inequalities are obtained.

1. Introduction

This paper provides some new bounds for 1−norms of positive triangular matrices with monotonic column entries. The main theorem refines a recent inequality of Vecchio and Mallik [11] in the case of constant diagonal. We refer the reader to Vecchio [10] and Vecchio and Mallik [11] (and the reference therein) for discussion of applications particularly those to stability analysis of linear methods for solving Volterra integral equations. Other references on the topic include [3]–[7] and [9].

The matrices of interest here are $n \times n$ truncations of infinite lower triangular (real) matrices, i.e.

$$A_n = \begin{bmatrix} a_{1,1} & & & & & \\ a_{2,1} & a_{2,2} & & & & \\ a_{3,1} & a_{3,2} & a_{3,3} & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ a_{n,1} & \cdots & & \cdot & a_{n,n-1} & a_{n,n} \end{bmatrix}. \tag{1.1}$$

The following result was proven in [11].

Theorem 1.1. *Assume that*

(i) $a_{i,j} \geq a > 0$, $j = 1, \ldots, i$, $i = 1, \ldots, n$,
(ii) $a_{i,i} \geq a_{i+1,i} \geq \cdots \geq a_{n,i}$, $i = 1, \ldots, n$,

and let

$$a_{min} = \min_{i=1,\ldots,n} \{a_{i,i}\}, \tag{1.2}$$

and $B_n = [b_{i,j}]$ be the inverse of the lower triangular matrix A_n. Then

$$\|B_n\|_1 \leq \frac{1}{a_{min}} + \frac{2}{a}. \tag{1.3}$$

The result in (1.3) was first proven in the case of triangular Toeplitz matrices in [10] and improved to the following in [2].

Theorem 1.2. *Suppose that the sequence $\{a_i\}_{i \geq 0}$ satisfies*

$$a_0 \geq a_1 \geq a_2 \geq \cdots a_n \geq a > 0, \tag{1.4}$$

for some constant a and all n and

$$C_n = \begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ \cdot & \cdot & \cdot & \cdot & \\ a_n & \cdots & \cdot & a_1 & a_0 \end{bmatrix}. \tag{1.5}$$

Then

$$\|C_n^{-1}\|_1 \leq \frac{2}{a}\left(1 - \rho(a, a_0)^{\lceil \frac{n}{2} \rceil}\right) \tag{1.6}$$

where ρ is the inverse ratio defined via

$$\rho(x, y) = 1 - x/y, \tag{1.7}$$

and, in particular

$$\|C_n^{-1}\|_1 \leq \frac{2}{a}, \tag{1.8}$$

independent of a_0 and n.

Here, we extend Theorem 1.2 (to non-Toeplitz matrices) and refine Theorem 1.1 in the case of constant diagonal. In particular we will prove the following.

Theorem 1.3. *Assume that the hypotheses of Theorem 1.1 are satisfied and in addition that*

$$a_{1,1} \le a_{2,2} \le \cdots \le a_{n,n}. \tag{1.9}$$

Then

$$\|B_n\|_1 \le \frac{2}{a}\left(\frac{a_{n,n}}{a_{1,1}}\right)\left(1 - \frac{\rho(a, a_{n,n})^{\lceil \frac{n}{2} \rceil} + \rho(a, a_{n,n})^{\lfloor \frac{n}{2} \rfloor}}{2}\right). \tag{1.10}$$

In particular, if

$$a_{1,1} = a_{2,2} = \cdots = a_{n,n} = a^*, \tag{1.11}$$

then

$$\|B_n\|_1 \le \frac{2}{a}\left(1 - \frac{\rho(a, a^*)^{\lceil \frac{n}{2} \rceil} + \rho(a, a^*)^{\lfloor \frac{n}{2} \rfloor}}{2}\right), \tag{1.12}$$

and hence

$$\|B_n\|_1 < \frac{2}{a}, \tag{1.13}$$

independent of a^*.

Note that triangular matrices satisfying (1.11) arise in the study of linear groups (see for instance [8]) and are particularly important in the theory of matrix decompositions.

The inequality in (1.12) is in a sense best possible. In particular, for $0 < a < a^*$, set

$$\mathcal{A}_n(a, a^*) = \{A = [a_{i,j}]_{n \times n} \mid A \text{ satisfies } (1.1), (i), (ii) \text{ and } (1.11)\}. \tag{1.14}$$

We have the following theorem regarding optimality.

Theorem 1.4. *For $0 < a < a^*$,*

$$\sup_{A \in \mathcal{A}_n(a,a^*)} \|A^{-1}\|_1 = \frac{2}{a}\left(1 - \frac{\rho(a, a^*)^{\lceil \frac{n}{2} \rceil} + \rho(a, a^*)^{\lfloor \frac{n}{2} \rfloor}}{2}\right). \tag{1.15}$$

Proof. We need to show that the bound in (1.12) is attained. To that end, suppose $a_{i,j} = a^* > 0$ for $i - j \in \{0, 1\}$ and $a_{i,j} \equiv a$ otherwise. It is easy to verify in this case, that for $1 \le j \le i \le n$,

$$b_{i,j} = (-1)^{i-j}\frac{1}{a^*}\left(1 - \frac{a}{a^*}\right)^{\lfloor \frac{i-j}{2} \rfloor}, \tag{1.16}$$

and hence,

$$
\begin{aligned}
\|A_n^{-1}\|_1 &= \sum_{i=1}^{n} |b_{i,1}| \\
&= \sum_{i=1}^{n} \frac{1}{a^*} \left(1 - \frac{a}{a^*}\right)^{\left\lfloor \frac{i-1}{2} \right\rfloor} = \sum_{i=0}^{n-1} \frac{1}{a^*} \left(1 - \frac{a}{a^*}\right)^{\left\lfloor \frac{i}{2} \right\rfloor} \\
&= \frac{1}{a^*} \left(\sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} \left(1 - \frac{a}{a^*}\right)^i + \sum_{i=0}^{\left\lceil \frac{n}{2} \right\rceil - 1} \left(1 - \frac{a}{a^*}\right)^i \right) \\
&= \frac{2}{a} \left(1 - \frac{\rho(a,a^*)^{\left\lceil \frac{n}{2} \right\rceil} + \rho(a,a^*)^{\left\lfloor \frac{n}{2} \right\rfloor}}{2} \right).
\end{aligned}
\tag{1.17}
$$

$\square$

Note also that

$$
\sup_{A \in \bigcup_{n \geq 1} \mathcal{A}_n(a,a^*)} \|A^{-1}\|_1 = 2/a.
\tag{1.18}
$$

The reader is referred to [1] for some discussion of bounds for inverses of matrices of the form in (1.1) when the condition of monotonicity within columns is replaced with that within rows.

2. Preliminaries and notation

In order to prove Theorem 1.3, consider the partial order on the set $\mathcal{V}_{b,a}$ of (arbitrary length) tuples $(a_1, a_2, \ldots, a_k)$ with

$$
b \geq a_1 > a_2 > \cdots > a_k = a
\tag{2.1}
$$

defined via

$$
\mathbf{v} \prec \mathbf{z} \text{ if } \mathbf{z} \text{ is a } suffix \text{ of } \mathbf{v}
\tag{2.2}
$$

where $\mathbf{z} = (z_1, z_2, \ldots, z_k)$ is a suffix of $\mathbf{v} = (v_1, v_2, \ldots, v_m)$ if $m > k$ and $\mathbf{v} = (v_1, \ldots, v_{m-k}, z_1, z_2, \ldots, z_k)$. For convenience, if $\mathbf{w} = (w_1, w_2, \ldots, w_r)$ we will write the $r+k-$tuple $(w_1, w_2, \ldots, w_r, z_1, z_2, \ldots, z_k)$ as $(\mathbf{w}; \mathbf{z})$. In addition, denote the length of $\mathbf{v}$ by $l(\mathbf{v}) = k$. The value v_1 will be referred to as the *initial* value of $\mathbf{v}$.

For a triangular double sequence $\{d_{i,j}\}_{j<i<n}$ satisfying $0 \leq d_{i,j} < 1$ for $j < i < n$ and

$$
\sum_{i=j+1}^{n} d_{i,j} \leq x < 1, \; j = 1, 2, \ldots, n-1
\tag{2.3}
$$

define the function D via

$$D(\mathbf{v}) = d_{v_k, v_{k-1}} \cdot d_{v_{k-1}, v_{k-2}} \cdots d_{v_3, v_2} \cdot d_{v_2, v_1} \qquad (2.4)$$

for $\mathbf{v} = (v_1, v_2, \ldots, v_k)$.

Note that it follows directly from the definition of D, the inequality in (2.3) and the non-negativity of $\{d_{i,j}\}$ that $D(\mathbf{v}) < D(\mathbf{z})$ for $\mathbf{v} \prec \mathbf{z}$.

Lemma 2.1. *Consider a set of tuples $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$. If $\mathbf{v}_i \prec \mathbf{z}$ for $1 \leq i \leq k$ and $\mathbf{v}_i \not\prec \mathbf{v}_j$ for $i \neq j$ (i.e. $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ forms an antichain that is bounded above by $\mathbf{z}$) then*

$$D(\mathbf{z}) \geq D(\mathbf{v}_1) + D(\mathbf{v}_2) + \cdots + D(\mathbf{v}_k). \qquad (2.5)$$

Proof. Let $\mathbf{z}_2$ be the least upper bound for $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, i.e. $\mathbf{z}_2 = \min\{\mathbf{w} \preceq \mathbf{z} : \mathbf{v}_i \preceq \mathbf{w}, 1 < i < k\}$. Clearly, $\mathbf{z}_2 \preceq \mathbf{z}$. We will show that

$$\sum_{i=1}^{k} D(\mathbf{v}_i) \leq D(\mathbf{z}_2). \qquad (2.6)$$

The result is immediate for $k = 1$. Hence suppose (2.6) holds for $1 \leq k < K$.

Now, suppose that there exists a $\mathbf{z}_3 \prec \mathbf{z}_2$ and a set $S \subset \{1, 2, \ldots, K\}$ such that $2 \leq \|S\| \leq K - 1$, $\mathbf{v}_i \prec \mathbf{z}_3$ if $i \in S$, and $\mathbf{v}_i \not\prec \mathbf{z}_3$, if $i \in S^c$. then by induction, we have

$$\sum_{i=1}^{K} D(\mathbf{v}_i) \leq D(\mathbf{z}_3) + \sum_{i \in S^c} D(\mathbf{v}_i). \qquad (2.7)$$

Considering the set $\{\mathbf{z}_3\} \cup \{\mathbf{v}_i : i \in S^c\}$ and applying induction again we have the inequality in (2.6).

Otherwise $\mathbf{v}_i$ is of the form $\mathbf{v}_i = (\mathbf{w}_i; (t_i); \mathbf{z}_3)$, $i = 1, 2, \ldots, k$, where $t_l \neq t_j$ for $l \neq j$ and $\mathbf{z}_3 = (z_{3,1}, \ldots, z_{3,l(\mathbf{z}_3)}) \preceq \mathbf{z}_2$. In this case, by (2.3),

$$\sum_{i=1}^{K} D(\mathbf{v}_i) \leq \sum_{i=1}^{K} D((t_i); \mathbf{z}_3) = \sum_{i=1}^{K} d_{t_i, z_{3,1}} D(\mathbf{z}_3) = D(\mathbf{z}_3) \sum_{i=1}^{K} d_{t_i, z_{3,1}} \leq D(\mathbf{z}_3)$$

$$(2.8)$$

and the proof is complete. □

The following lemma will be crucial.

Lemma 2.2. *For fixed $s \geq 1$, set $S_{i,s} = 0$ for $i < s$, $S_{s,s} = 1$ and for $s+1 \leq m \leq n$, inductively,*

$$S_{m,s} = \sum_{i=s}^{m-1} d_{m,i} S_{i,s}. \qquad (2.9)$$

Then, for $Q \subseteq \{s+1, \ldots, n\}$, we have

$$\sum_{i \in Q} S_{i,s} \leq x_s + x_s^2 + \cdots + x_s^{\|Q\|}, \tag{2.10}$$

where $x_s = \max_{t=s,\ldots,n-1} \sum_{i=t+1}^{n} d_{i,t}$.

Proof. Note that it follows from straightforward induction that for $m > s$,

$$S_{m,s} = \sum_{\mathbf{v}=(m,\ldots,s) \in \mathcal{V}_{m,s}} D(\mathbf{v}). \tag{2.11}$$

Note that in (2.11), the summation is over all tuples $v = (v_1, v_2, \ldots, v_{l(\mathbf{v})})$ with

$$m = v_1 > v_2 > \cdots > v_{l(\mathbf{v})} = s. \tag{2.12}$$

Now, define

$$L_{m,s}^{k} = \sum_{\substack{\mathbf{v} \in \mathcal{V}_{m,s} \\ l(\mathbf{v})=k+1}} D(\mathbf{v}) \tag{2.13}$$

We will show inductively that

$$L_{m,s}^{k} \leq x_s^k. \tag{2.14}$$

First note that by (2.3) and the definition of x_s,

$$L_{m,s}^{1} = d_{s+1,s} + d_{s+2,s} + d_{s+3,s} + \cdots + d_{m,s} \leq x_s. \tag{2.15}$$

Thus assume that (2.14) is true for $k < K$. Then, since $x_1 \geq x_2 \geq \cdots \geq x_n$,

$$L_{m,s}^{K} = \sum_{i=s+1}^{m} d_{i,s} L_{m,i}^{K-1} \leq \sum_{i=s+1}^{m} d_{i,s} x_i^{K-1} \leq x_s^{K-1} \sum_{i=s+1}^{m} d_{i,s} \leq x_s^{K}. \tag{2.16}$$

Now, define the sets

$$\mathcal{R}_1 = \{\mathbf{v} \in \mathcal{V}_{m,s} | 2 \leq l(\mathbf{v}) \leq \|Q\| + 1\} \tag{2.17}$$

and

$$\mathcal{R}_2 = \{\mathbf{v} \in \mathcal{V}_{m,s} | \mathbf{v} = (i, \ldots, s), \ i \in Q\} \tag{2.18}$$

and consider the quantity

$$H_Q = \sum_{k=1}^{\|Q\|} L_{m,s}^{k} - \sum_{i \in Q} S_{i,s} \tag{2.19}$$

$$= \sum_{\mathbf{v} \in \mathcal{R}_1} D(\mathbf{v}) - \sum_{\mathbf{v} \in \mathcal{R}_2} D(\mathbf{v}). \tag{2.20}$$

We will prove that for all sets Q, $H_Q \geq 0$. The result will then follow from (2.20) and the inequality in (2.14).

We define the following scheme for matching elements $\mathbf{z}$ in $\mathcal{R}_1$ with (possibly empty) subsets $\mathcal{S}(\mathbf{z})$ of $\mathcal{R}_2$ such that $D(\mathbf{z}) \geq \sum_{\mathbf{v} \in S(\mathbf{z})} D(\mathbf{v})$ and $\{\mathcal{S}(\mathbf{z}) | \mathbf{z} \in \mathcal{R}_1\}$ is a partition of R_2. In particular for $2 \leq t \leq n$, set

$$\mathcal{J}_t = \{\mathbf{v} \in \mathcal{R}_1 : l(v) = t\}, \tag{2.21}$$

and recursively in $t \geq 2$, for $\mathbf{z} \in \mathcal{J}_t$ let

$$S(\mathbf{z}) = \{\mathbf{v} \in \mathcal{R}_2 | \mathbf{v} \text{ is a maximal element in the set } \mathcal{W}(\mathbf{z})\}. \tag{2.22}$$

where

$$\mathcal{W}(\mathbf{z}) = \{\mathbf{w} \in \mathcal{R}_2 | \mathbf{w} \preceq \mathbf{z} \text{ and } \mathbf{w} \notin \bigcup_{\substack{\mathbf{v} \succ \mathbf{z} \\ \mathbf{v} \in \mathcal{R}_1}} \mathcal{S}(\mathbf{v})\}. \tag{2.23}$$

Here, again, the maximality in (2.22) is with respect to the given partial order on $\mathcal{V}_{m,s}$.

Now, fix $\mathbf{z} \in \mathcal{J}_t$ for some $2 \leq t \leq n$ and suppose $\{\mathbf{v}_1, \mathbf{v}_2\} \subset \mathcal{S}(\mathbf{z})$ with $\mathbf{v}_1 \neq \mathbf{v}_2$. The fact that $\mathbf{v}_1 \nprec \mathbf{v}_2$ and $\mathbf{v}_2 \nprec \mathbf{v}_1$ follows from the maximality in (2.22). We then have that Lemma 2.1 is applicable and

$$D(\mathbf{z}) \geq \sum_{\mathbf{v} \in S(\mathbf{z})} D(\mathbf{v}), \tag{2.24}$$

as required. In addition, by the definition of $\mathcal{W}$ we have that the sets $\mathcal{S}(\mathbf{z})$, $\mathbf{z} \in \mathcal{R}_1$ are pairwise disjoint. To see that $\mathcal{R}_2 \subset \bigcup_{\mathbf{z} \in \mathcal{R}_1} \mathcal{S}(\mathbf{z})$, first suppose $\mathbf{v} \in \mathcal{R}_2$. Let $K_{\mathbf{v}}$ be a maximal chain in $\mathcal{V}_{m,s}$ such that $\mathbf{v} \in K_{\mathbf{v}}$ and set $\mathcal{T}_1 = K_{\mathbf{v}} \cap \mathcal{R}_1 = \{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_r\}$ and $\mathcal{T}_2 = K_{\mathbf{v}} \cap \mathcal{R}_2 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q\}$, where $\mathbf{v}_1 \succ \mathbf{v}_2 \succ \cdots \succ \mathbf{v}_q$ and $\mathbf{z}_1 \succ \mathbf{z}_2 \succ \cdots \succ \mathbf{z}_r$. Note that $\|\mathcal{T}_1\| = \|Q\|$ and $\|\mathcal{T}_2\| \leq \|Q\|$ (since the only possible initial values for tuples are those in Q) and by (2.22), $\mathbf{v}_i \in \mathcal{S}(\mathbf{z}_i)$ for $1 \leq i \leq r$ and in particular $\mathbf{v} \in \bigcup_{\mathbf{z} \in \mathcal{R}_1} \mathcal{S}(\mathbf{z})$. Since $\bigcup_{\mathbf{z} \in \mathcal{R}_1} \mathcal{S}(\mathbf{z}) \subset \mathcal{R}_2$ by (2.22), the result is proven. $\square$

3. Proof of the main theorem

In this section we prove Theorem 1.3.

First note that the lower triangular matrix $B_n = [b_{i,j}] = A_n^{-1}$ satisfies $b_{s,s} = 1/a_{s,s}$ and

$$b_{m,s} = \sum_{j=s}^{m-1} -\alpha_{m,j} b_{j,s}, \tag{3.1}$$

for $1 \leq s < m \leq n$, where $\alpha_{m,j} = (a_{m,j}/a_{m,m})$ for $1 \leq j \leq m \leq n$ (see for instance [1]).

Define $h_{i,j} = a_{j,j} b_{i,j}$ for $1 \leq j \leq i \leq n$, so that $h_{s,s} = 1$ and for $1 \leq s < m \leq n$,

$$h_{m,s} = \sum_{j=s}^{m-1} -\alpha_{m,j} h_{j,s}. \tag{3.2}$$

We have the following lemma (contrast with Equation (2.3) in [11]).

Lemma 3.1. *Suppose that* $[a_{i,j}]$ *satisfies the hypotheses of Theorem 1.3. Then*

$$h_{i,j} = S_{i,j} - S_{i,j+1}, \tag{3.3}$$

for $1 \le j \le i \le n$, *where* $\{S_{i,j}\}$ *is as in (2.9) for the nonnegative double sequence* $\{d_{i,j}\}$ *defined via*

$$d_{m,j} = \alpha_{m-1,j} - \alpha_{m,j}, \tag{3.4}$$

for $1 \le j < m \le n$. *In addition, (2.3) is satisfied with*

$$x_s = \max_{t=s,\dots,n-1} \sum_{i=t+1}^{n} d_{i,t} \le 1 - \frac{a}{a_{n,n}} = x. \tag{3.5}$$

Proof. First, note that by (3.4), (ii) and (1.9)

$$d_{m,j} = \alpha_{m-1,j} - \alpha_{m,j} = \frac{a_{m-1,j}}{a_{m-1,m-1}} - \frac{a_{m,j}}{a_{m,m}} \ge 0, \tag{3.6}$$

and

$$\sum_{m=j+1}^{n} d_{m,j} = \alpha_{j,j} - \alpha_{n,j} = 1 - \frac{a_{n,j}}{a_{n,n}} \le 1 - \frac{a}{a_{n,n}} < 1. \tag{3.7}$$

In addition, for $s+2 \le m \le n$, $s = 1, 2 \dots, n$,

$$
\begin{aligned}
h_{m,s} - h_{m-1,s} &= \sum_{j=s}^{m-1} -\alpha_{m,j} h_{j,s} + \sum_{j=s}^{m-2} \alpha_{m-1,j} h_{j,s} \\
&= \sum_{j=s}^{m-2} (\alpha_{m-1,j} - \alpha_{m,j}) h_{j,s} - \alpha_{m,m-1} h_{m,m-1},
\end{aligned}
\tag{3.8}
$$

and hence since $d_{m,m-1} = \alpha_{m-1,m-1} - \alpha_{m,m-1} = 1 - \alpha_{m,m-1}$,

$$h_{m,s} = \sum_{j=s}^{m-2} d_{m,j} h_{j,s} + (1 - \alpha_{m,m-1}) h_{m,m-1} = \sum_{j=s}^{m-1} d_{m,j} h_{j,s}. \tag{3.9}$$

In addition,

$$
\begin{aligned}
S_{m,s} - S_{m,s+1} &= \sum_{i=s}^{m-1} d_{m,i} S_{i,s} - \sum_{i=s+1}^{m-1} d_{m,i} S_{i,s+1} \\
&= d_{m,s} S_{s,s} + \sum_{i=s+1}^{m-1} d_{m,i} (S_{i,s} - S_{i,s+1}) \\
&= d_{m,s} (S_{s,s} - S_{s,s+1}) + \sum_{i=s+1}^{m-1} d_{m,i} (S_{i,s} - S_{i,s+1}) \\
&= \sum_{i=s}^{m-1} d_{m,i} (S_{i,s} - S_{i,s+1}),
\end{aligned}
\tag{3.10}
$$

since $S_{s,s+1} = 0$.

Comparing (3.9) and (3.10) and noting that $h_{s,s} = 1 = S_{s,s} - S_{s,s+1}$ and $h_{s+1,s} = -\alpha_{s+1,s} = (1 - \alpha_{s+1,s}) - 1 = d_{s+1,s}S_{s,s} - 1 = S_{s+1,s} - S_{s+1,s+1}$, the result follows. $\square$

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. Employing Lemma 3.1 and the definition of $\{h_{i,j}\}$, we have

$$\|A_n^{-1}\|_1 = \|B_n\|_1 \quad = \quad \max_{1 \le j \le n} \sum_{i=j}^{n} |b_{i,j}|$$

$$= \quad \max_{1 \le j \le n} \sum_{i=j}^{n} \left| \frac{1}{a_{j,j}}(S_{i,j} - S_{i,j+1}) \right|. \tag{3.11}$$

Now, fix $1 \le j \le n$. We have, by the nonnegativity of $\{S_{i,j}\}$, that

$$\sum_{i=j}^{n} |S_{i,j} - S_{i,j+1}| \quad = \quad |S_{j,j} - S_{j,j+1}| + |S_{j+1,j} - S_{j+1,j+1}| +$$

$$\sum_{i \in Q_1} (S_{i,j} - S_{i,j+1}) + \sum_{i \in Q_1^c} (S_{i,j+1} - S_{i,j})$$

$$\le \quad |S_{j,j} - S_{j,j+1}| + |S_{j+1,j} - S_{j+1,j+1}| + \sum_{i \in Q_1} S_{i,j} + \sum_{i \in Q_1^c} S_{i,j+1},$$

$$\tag{3.12}$$

where $Q_1 = \{j + 2 \le i \le n | S_{i,j} > S_{i,j+1}\}$.

Noting that $S_{j,j} = 1$, $S_{j,j+1} = 0$, $S_{j+1,j} = d_{j+1,j} < 1$ and $S_{j+1,j+1} = 1$, we have from (3.12) that

$$\sum_{i=j}^{n} |S_{i,j} - S_{i,j+1}| \quad \le \quad 2 + \sum_{i \in Q_1} S_{i,j} + \sum_{i \in Q_1^c} S_{i,j+1}. \tag{3.13}$$

.

Letting $y = \|Q_1\| \le n - j - 1$, recalling $x_j \le 1 - a/a_{n,n} = x < 1$ and employing Lemma 2.2 gives

$$\sum_{i=j}^{n} |S_{i,j} - S_{i,j+1}| \quad \le \quad (1 + x + \cdots + x^y) + (1 + x + \cdots + x^{n-j-1-y})$$

$$\le \quad \frac{1 - x^{y+1}}{1 - x} + \frac{1 - x^{n-(y+1)}}{1 - x} \le \frac{2 - (x^{y+1} + x^{n-(y+1)})}{1 - x}$$

$$\tag{3.14}$$

.

By the convexity of the function f defined via $f(t) = x^t$, we have that $x^{y+1} + x^{n-(y+1)} \geq x^{\lfloor n/2 \rfloor} + x^{\lceil n/2 \rceil}$. Thus, returning to (3.11), we obtain

$$
\begin{aligned}
\|A_n^{-1}\|_1 &\leq \frac{2}{\min_i\{a_{i,i}\}} \frac{1 - \frac{x^{\lfloor n/2 \rfloor} + x^{\lceil n/2 \rceil}}{2}}{1 - x} \\
&= \frac{2}{a_{1,1}} \left(\frac{1 - \frac{(1-a/a_{n,n})^{\lceil \frac{n}{2} \rceil} + (1-a/a_{n,n})^{\lfloor \frac{n}{2} \rfloor}}{2}}{a/a_{n,n}} \right).
\end{aligned} \tag{3.15}
$$

In the case when $a_{i,i} = a^*$ for all i, (3.15) gives

$$
\|A_n^{-1}\|_1 \leq \frac{2}{a} \left(1 - \frac{(1-a/a^*)^{\lceil \frac{n}{2} \rceil} + (1-a/a^*)^{\lfloor \frac{n}{2} \rfloor}}{2} \right), \tag{3.16}
$$

as required. □

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