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THE HYERS–ULAM STABILITY FOR TWO FUNCTIONAL EQUATIONS IN A SINGLE VARIABLE

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Abstract. We apply the Luxemburg–Jung fixed point theorem in generalized metric spaces to study the Hyers–Ulam stability for two functional equations in a single variable.

1. Introduction and preliminaries

According to [8], the study of stability problems for functional equations originated from a talk of S. Ulam before the Mathematics Club of the University of Wisconsin in 1940, when he proposed the following problem:

Let E and $E^{'}$ be Banach spaces. Does there exist for each $\varepsilon > 0$ a $\delta > 0$ such that, to each function f from E into $E^{'}$ such that $\|f(x+y)-f(x)-f(y)\| \leq \delta$ for all $x, y \in E$ there corresponds a linear transformation $l(x)$ of E into $E^{'}$ satisfying the inequality $\|f(x)-l(x)\| \leq \varepsilon$ for all x in E?

A year later, D.H. Hyers answered this question in the affirmative. He designed as a $\delta-$linear transformation between two Banach spaces E and E' any mapping $f : E \to E'$ such that

$$\|f(x+y)-f(x)-f(y)\| < \delta(x, y \in E)$$

and proved the following theorem, which says that the Cauchy functional equation is "stable in the sense of Hyers–Ulam":

Theorem. (cf. [8, Theorem 1]) *Let E and E' be Banach spaces and let $f(x)$ be a δ-linear transformation of E into E'. Then the limit $l(x) = \lim_{n\to\infty} f(2^n x)/2^n$ exists for each $x \in E$, $l(x)$ is a linear transformation, and $\|f(x)-l(x)\| \leq \delta$*

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for all $x \in E$. *Moreover* $l(x)$ *is the only linear transformation satisfying this inequality.*

Subsequently, the result of Hyers has been generalized by considering unbounded Cauchy differences (T. Aoki [2], for additive mappings and Th.M. Rassias [19], for linear mappings). The paper of Th.M. Rassias [19] has provided a great influence in the development of the theory of stability of functional equations, see e.g., [20, 7, 9, 16, 15].

Baker ([3]) studied the stability of a nonlinear functional equation by using the Banach fixed point theorem. Recently, Radu ([18], see also [5]) pointed out that many theorems concerning the stability of functional equations are consequences of the fixed point alternative of Margolis and Diaz [14]. In 1996, G. Isac and Th.M. Rassias [11] were the first mathematicians to introduce applications of stability theory of functional equations for the proof of new fixed point theorems. The reader is referred to the book [10] for an extensive account of both old and new developments of noinlinear methods with applications to fixed point theory.

In this note we apply a fixed point theorem of Jung ([12]) to study the Hyers–Ulam stability for two functional equations in a single variable. First, we extend a theorem of Baker [3] and Agarwal et al. [1] and then we obtain a stability result (in the sense of Ulam) for a functional equation discussed in [17].

2. Fixed points in generalized metric spaces

The notion of complete generalized metric space has been introduced by Luxemburg in [13], by allowing the value $+\infty$ for the distance mapping.

If (X, d) is a generalized metric space then the relation $\sim$ on X defined by $x \sim y$ if and only if $d(x, y) < +\infty$ is an equivalence relation on X, which determines a unique decomposition (called the canonical decomposition) of X into disjoint equivalence classes, $X = \cup\{X_\alpha, \alpha \in A\}$. If $d_\alpha = d \mid_{X_\alpha \times X_\alpha}$, then (X, d) is a complete generalized metric space if and only if (X_α, d_α) is a complete metric space for each $\alpha \in A$.

The fixed point theorems of the alternative on generalized metric spaces can be obtained from the corresponding fixed point theorems on appropriate metric spaces. Namely, see [12, Theorem 3.1] , if (X, d) is a generalized metric space, $X = \cup\{X_\alpha, \alpha \in A\}$ is its canonical decomposition and $T : X \to X$ is a mapping such that

$$d(T(x), T(y)) < +\infty \text{ whenever } d(x, y) < +\infty,$$

then T has a fixed point if and only if $T_\alpha = T \mid_{X_\alpha} : X_\alpha \to X_\alpha$ has a fixed point for some $\alpha \in A$.

Definition 2.1. A mapping $\varphi : [0, \infty] \to [0, \infty]$ is called *a generalized strict comparison function* if it is nondecreasing, $\varphi(\infty) = \infty$, $lim_{n\to\infty}\varphi^n(t) = 0$ for all $0 < t < \infty$ and $t - \varphi(t) \to \infty$ as $t \to \infty$. Let (X, d) be a generalized metric space and φ be a generalized strict comparison function. A mapping $f : X \to X$ is called a *strict φ-contraction* if

$$d(f(x), f(y)) \le \varphi(d(x, y))$$

for all $x, y \in X$.

Theorem 2.2. *Let (X, d) be a complete generalized metric space and $T : X \to X$ be a strict φ-contraction such that $d(x_0, T(x_0)) < +\infty$ for some $x_0 \in X$. Then T has a unique fixed point in the set $X_{\alpha_0} := \{y \in X, d(x_0, y) < \infty\}$ and the sequence $(T^n(x))_{n\in N}$ converges to the fixed point x^* for every $x \in Y$. Moreover, $d(x_0, T(x_0)) \leq \delta$ implies $d(x^*, x_0) \leq \delta_\varphi := sup\{t > 0, t - \varphi(t) \leq \delta\}$.*

Proof. Let $X = \cup\{X_\alpha, \alpha \in A\}$ be the canonical decomposition of X. Since $d(x_0, T(x_0)) < +\infty$, both x_0 and $T(x_0)$ belong to the class X_{α_0}. On the other hand, it is easy to show that $\varphi(t) < t$ for all $t \in (0, \infty)$. Thus, for every $y \in X_{\alpha_0}$,

$$d(x_0, T(y)) \leq d(x_0, T(x_0)) + d(T(x_0), T(y))$$
$$\leq d(x_0, T(x_0)) + \varphi(d(x_0, y)) \leq d(x_0, T(x_0)) + d(x_0, y) < \infty$$

that is, X_{α_0} is an invariant subset for T. This means that the restriction $T_{\alpha_0} = T\mid_{X_{\alpha_0}}$ is a strict φ-contraction on the metric space (X_{α_0}, d) and now the conclusion follows from a well known fixed point result in metrical fixed point theory (see e.g., [21, Theorem 7.1.1] or [4, section 2.5]). □

3. Hyers–Ulam stability of the nonlinear functional equation $f(x) = F(x, f(\eta(x)))$

The Hyers–Ulam stability for the nonlinear functional equation

$$f(x) = F(x, f(\eta(x)))$$

where $\eta : S \to S$ and $F : S \times X \to X$ are given mappings is discussed in [3] and [1] (for the generalized stability of this equation see [6] and [5]). In the next theorem we slightly improve [3, Theorem 2] and from [1, Theorem 13], by considering comparison functions.

Theorem 3.1. *Let S be a nonempty set and (X, d) be a complete metric space. Let $\eta : S \to S, F : S \times X \to X$. Suppose that*

$$d(F(x, u), F(x, v)) \leq \varphi(d(u, v)) \qquad (x \in S, u, v \in X)\,,$$

where $\varphi : [0, \infty] \to [0, \infty]$ is a generalized strict comparison function and let $f : S \to X, \delta > 0$ be such that

$$d(f(x), F(x, f(\eta(x)))) \leq \delta \qquad (x \in S)\,.$$

Then there is a unique mapping $f_s : S \to X$ such that

$$f_s(x) = F(x, f_s(\eta(x))) \qquad (x \in S)$$

and

$$d(f(x), f_s(x)) \leq \delta_\varphi \qquad (x \in S)$$

where $\delta_\varphi := sup\{t : t - \varphi(t) \leq \delta\}$.

Proof. Consider the set Y of all mappings a from S to X. According to [3, Theorem 2], the formula $\rho(a, b) = sup\{d(a(x), b(x)), x \in S\}$ defines a (generalized) complete metric on Y. Next, let us define the mapping T from Y to Y as follows: for every $a \in Y$ and $x \in S$, $T(a)(x) = F(x, a(\eta(x)))$. Then, for all $a, b \in Y$ and $x \in S$,

$$d(T(a)(x), T(b)(x)) = d(F(x, a(\eta(x))), F(x, b(\eta(x))))$$

$$\leq \varphi(d(a(\eta(x)), b(\eta(x))) \leq \varphi(\rho(a, b)).$$

Therefore,

$$\rho(T(a), T(b)) \leq \varphi(\rho(a, b)) \qquad (a, b \in Y)$$

that is, T is a strict φ-contraction on Y.

As $d(f(x), F(x, f(\eta(x)))) \leq \delta$ $(x \in S)$ means that $\rho(f, T(f)) \leq \delta$, from Theorem 2.2 it follows that there is a unique f_s in Y such that $f_s = T(f_s)$ and $d(f(x), f_s(x)) \leq sup\{t : t - \varphi(t) \leq \delta\}$ $(x \in S)$. □

4. The Hyers–Ulam stability of the equation $\mu \circ f \circ \eta = f$

Let X be a nonempty set, (Y, d) be a metric space and $\eta : X \to X$, $\mu : Y \to Y$ be two given functions. In the following we deal with the Hyers–Ulam stability problem for the functional equation $\mu \circ f \circ \eta = f$, where $f : X \to Y$ is an unknown mapping. The Hyers–Ulam–Rassias stability of this equation has been studied in [17] and [5].

Theorem 4.1. *Let X be a nonempty set, (Y, d) be a complete metric space and $\eta : X \to X$, $\mu : Y \to Y$ be two given functions. Suppose that $f : X \to Y$ satisfies*

$$d((\mu \circ f \circ \eta)(x), f(x)) \leq \delta \qquad (x \in X),$$

where δ is a given positive real number. If $\varphi : [0, \infty] \to [0, \infty]$ is a generalized strict comparison function and

$$d(\mu(u), \mu(v)) \leq \varphi(d(u, v)) \qquad (u, v \in Y),$$

then there exists a unique mapping $c : X \to Y$, which satisfies both the equation

$$\mu \circ c \circ \eta = c$$

and the estimation

$$d(f(x), c(x)) \leq \delta_\varphi \qquad (x \in X).$$

Moreover,

$$c(x) = \lim_{n \to \infty} (\mu^n \circ f \circ \eta^n)(x) \qquad (x \in X).$$

Proof. Let $E := \{a : X \to Y\}$ and $\rho(a, b) = sup\{d(a(x), b(x)), x \in S\}$. For every $f \in E$, define $T(f) : X \to Y$ by $T(f) = \mu \circ f \circ \eta$.

From the definition of T it follows that if $a, b \in E$ then, for all $x \in X$,

$$d(T(a)(x), T(b)(x)) = d(\mu \circ a \circ \eta(x), \mu \circ b \circ \eta(x))$$

$$\leq \varphi(d(a(f(x)), b(f(x))) \leq \varphi(\rho(a, b)).$$

Therefore,

$$\rho(T(a), T(b)) \leq \varphi(\rho(a, b)) \qquad (a, b \in E).$$

As $d((\mu \circ f \circ \eta)(x), f(x)) \leq \delta \qquad (x \in X)$ means that $\rho(f, T(f)) \leq \delta$, we can use Theorem 2.2 to conclude the proof.

□

References

[1] R.P. Agarwal, B. Xu and W. Zhang, *Stability of functional equations in single variable*, J. Math. Anal. Appl. **288** (2003), 852–869.

[2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. M. Soc. Japan **2** (1950), 64–66.

[3] J.A. Baker, *The stability of certain functional equations*, Proc. AMS **112** (1991), 729–732.

[4] V. Berinde, *Iterative approximation of fixed points*, Efemeride (2002).

[5] L. Cădariu and V. Radu, *Fixed point method for the generalized stability of functional equations in single variable*, Fixed Point Theory and Applications, to appear.

[6] L. Cădariu, M.S. Moslehian and V. Radu, *An application of Banach's fixed point theorem to the stability of a general functional equation*, preprint.

[7] G.L. Forti, *Hyers–Ulam stability of functional equations in several variables*, Aequationes Math. **50** (1995), 143–190.

[8] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA **27** (1941), 222–224.

[9] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Basel, 1998.

[10] D.H. Hyers, G. Isac and Th.M. Rassias, *Topics in Nonlinear Analysis and Applications*, World Scientific Publishing Company, Singapore, New Jersey, London, 1997.

[11] G. Isac and Th.M. Rassias, *Stability of ψ-additive mappings: Applicaqtions to nonlinear analysis*, International Journal of Mathematics and Mathematical Sciences **19**(2)(1996), 219–228.

[12] C.F.K. Jung, *On generalized complete metric spaces*, Bull. A.M.S. **75**(1969), 113–116.

[13] W.A.J. Luxemburg, *On the convergence of successive approximations in the theory of ordinary differential equations*, II, Nederl. Akad. Wetensch. Proc. Ser. A 61 = Indag. Math. **20** (1958), 540–546.

[14] B. Margolis and J.B. Diaz, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.

[15] D. Miheţ and V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, Journal of Mathematical Analysis and Applications, **343** (2008), 567–572.

[16] A.K. Mirmostafee and M.S. Moslehian, *Fuzzy versions of HyersUlamRassias theorem*, Fuzzy Sets and Systems **159** (6) (2008), 720–729.

[17] A. Gilanyi, Z. Kaiser and Z. Pales, *Estimates to the stability of functional equations*, Aequationes Math. **73** (2007), 125–143.

[18] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory, Cluj–Napoca **IV**(1) (2003), 91–96.

[19] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[20] Th.M. Rassias, *On the Stability of Functional Equations and a Problem of Ulam*, Acta Applicandae Mathematicae **62** (2000), 23–130.

[21] I.A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj–Napoca, 2001.

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