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STRUCTURE OF LOCALLY IDEMPOTENT ALGEBRAS

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This paper is dedicated to Professor Themistocles M. Rassias.

Submitted by M. Joita

Abstract. It is shown that every locally idempotent (locally m-pseudoconvex) Hausdorff algebra A with pseudoconvex von Neumann bornology is a regular (respectively, bornological) inductive limit of metrizable locally m-(k_B-convex) subalgebras A_B of A. In the case where A, in addition, is sequentially $\mathcal{B}_A$-complete (sequentially advertibly complete), then every subalgebra A_B is a locally m-(k_B-convex) Fréchet algebra (respectively, an advertibly complete metrizable locally m-(k_B-convex) algebra) for some $k_B \in (0,1]$. Moreover, for a commutative unital locally m-pseudoconvex Hausdorff algebra A over $\mathbb{C}$ with pseudoconvex von Neumann bornology, which at the same time is sequentially $\mathcal{B}_A$-complete and advertibly complete, the statements (a)–(j) of Proposition 3.2 are equivalent.

1. Introduction

1. Let $\mathbb{K}$ be the field $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers. A topological algebra A over $\mathbb{K}$ with separately continuous multiplication (in short a topological algebra) is *locally pseudoconvex* if it has a base $\mathcal{L}$ of neighbourhoods of zero, consisting of *balanced* and *pseudoconvex sets* that is, of sets O which satisfy the condition $\mu O \subset O$ for $|\mu| \leqslant 1$ and define a number $k_O \in (0,1]$ such that

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$O+O \subset 2^{\frac{1}{k_O}}O$. In particular, when $\inf\{k_O : O \in \mathcal{L}\} = 0$, then A is a *degenerated locally pseudoconvex* algebra and when $\inf\{k_O : O \in \mathcal{L}\} = k > 0$, A is a *locally k-convex* algebra. Moreover, A is a *locally convex* algebra if $k = 1$.

A topological algebra A is a *locally idempotent algebra* if it has a base of *idempotent* neighbourhoods of zero, that is, of neighbourhoods O such that $OO \subset O$. This class of topological algebras has been introduced in [29], p. 31. A topological algebra A is *locally m-pseudoconvex* (*locally m-(k-convex)*) if, at the same time, it is locally idempotent and locally pseudoconvex (respectively, locally idempotent and locally k-convex). In this case A has a base of neighbourhoods of zero which consists of idempotent and absolutely pseudoconvex[1] (respectively, idempotent and absolutely k-convex) sets. A locally m-(k-convex) algebra is *locally m-convex* if $k = 1$. Locally m-convex algebras (see, for example, [21], [23], [29] and [30]) and locally m-pseudoconvex algebra (see [1]–[8]) have been well studied, locally idempotent algebras (without any additional requirements) have been studied only in [24].

2. For any topological algebra A, $U \subset A$ and $k > 0$ let

$$\Gamma_k(U) = \Big\{ \sum_{v=1}^{n} \alpha_v u_v : n \in \mathbb{N}, u_v \in U, \alpha_v \in \mathbb{K} \text{ with } \sum_{v=1}^{n} |\alpha_v|^k \leqslant 1 \Big\}.$$

The *von Neumann bornology* $\mathcal{B}_A$ of a topological algebra A is the collection of all bounded subsets in A. If for every $B \in \mathcal{B}_A$ there exists a number $k_B \in (0,1]$ such that $\Gamma_{k_B}(B) \in \mathcal{B}_A$, then $\mathcal{B}_A$ is *pseudoconvex* (see, [17], p. 101, or [20], p. A1058). In particular, when the number k_B does not depend on B (that is, when $k_B = k$ for all $B \in \mathcal{B}_A$), then $\mathcal{B}_A$ is *k-convex* (see [31]), and when $k = 1$, then $\mathcal{B}_A$ is *convex*. It is known that the von Neumann bornology on any locally k-convex space is k-convex (see [31], Proposition 1.2.15) and there exists a non-convex space with convex von Neumann bornology (see [31], Example 1.2.7). Moreover (see [20], Theorems 1 and 2, [22] and [17], p. 102–103), the von Neumann bornology $\mathcal{B}_A$ on a locally pseudoconvex space A is pseudoconvex if $\mathcal{B}_A$ has a countable base, and every metrizable linear space is locally k-convex for some $k \in (0,1]$ if $\mathcal{B}_A$ is pseudoconvex.

3. A net $(x_\lambda)_{\lambda\in\Lambda}$ in a topological linear space X is said to *converge in the sense of Mackey* (sometimes, to *converge bornologically*) to an element $x_0 \in X$ if there exist a balanced set $B \in \mathcal{B}_A$ and for every $\varepsilon > 0$ an index $\lambda_\varepsilon \in \Lambda$ such that $x_\lambda - x_0 \in \varepsilon B$ whenever $\lambda > \lambda_\varepsilon$. It is easy to see that every net which converges in the sense of Mackey (shortly, is *Mackey convergent*) converges also in the topological sense. The converse is false in general (see [18], p. 122, or [31], Proposition 1.2.4), but it is true in case when X is a metrizable topological linear space (see, [18], p. 27).

A map f from X into another topological linear space Y is *Mackey continuous* at $x_0 \in X$ (see, for example, [17], p. 10) if for each net $(x_\lambda)_{\lambda\in\Lambda}$, which converges to x_0 in X in the sense of Mackey, the net $(f(x_\lambda))_{\lambda\in\Lambda}$ converges to $f(x_0)$ in Y

[1]A subset $U \subset A$ is *absolutely k-convex* if $\lambda u + \mu v \in U$ for all $u, v \in U$ and $\lambda, \mu \in \mathbb{K}$ with $|\lambda|^k + |\mu|^k \leqslant 1$ and is *absolutely pseudoconvex* if it is absolutely k-convex for some $k \in (0,1]$, which depends on the set U.

in the sense of Mackey. Moreover, a map f from X into Y is called *Mackey continuous* if f is Mackey continuous at every point of X, and f is *bounded* if $f(B) \in \mathcal{B}_Y$ for each $B \in \mathcal{B}_X$.

A net $(x_\lambda)_{\lambda\in\Lambda}$ in a topological linear space X is called a *Mackey–Cauchy net* if there exist a balanced set $B \in \mathcal{B}_X$ and for every $\varepsilon > 0$ a number $\lambda_\varepsilon \in \Lambda$ such that $x_\lambda - x_\mu \in \varepsilon B$ whenever $\lambda > \mu > \lambda_\varepsilon$. It is easy to see that every Mackey–Cauchy net is a Cauchy net in the sense of topology. The converse statement is false in general (see [18], p. 122) but it is true in case of metrizable topological linear spaces (see [18], p. 27, or [31], Proposition 1.2.5). We say that a topological linear space X is *sequentially* $\mathcal{B}_X$*-complete* if every Mackey–Cauchy sequence in X converges in the sense of topology. Consequently, every sequentially complete (as well as complete) topological linear space X is sequentially $\mathcal{B}_X$-complete space.

4. For any topological algebra A (over $\mathbb{K}$) let $m(A)$ denote the set of all closed regular two-sided ideals in A (which are maximal as left or right ideals) and let $\hom A$ denote the set of all nontrivial continuous linear and multiplicative maps from A onto $\mathbb{K}$. A topological algebra A is a *Gelfand–Mazur algebra* (see, for example, [1]-[8] and [21]) if A/M is topologically isomorphic to $\mathbb{K}$ for each $M \in m(A)$. It is easy to see that every Gelfand–Mazur algebra A with non-empty set $m(A)$ is exactly such topological algebra for which there is a bijection $\varphi \to \ker \varphi$ between $\hom A$ and $m(A)$. Therefore, only in case of Gelfand–Mazur algebras it is possible to use the Gelfand theory, well-known for commutative (complex) Banach algebras.

5. A topological algebra A is *simplicial* (see [3], p. 15) if every closed regular left (right or two-sided) ideal of A is contained in some closed maximal left (respectively, right or two-sided) ideal of A. It is known (see[2] [6], Corollary 6) that every commutative unital locally m-pseudoconvex Hausdorff algebra is simplicial.

6. It is known that every locally m-convex Hausdorff algebra is a bornological inductive limit (with continuous canonical injections) of metrizable locally m-convex subalgebras of A (see [9], Proposition on p. 943, or [10], Theorem II.4.3) and every complete locally m-convex algebra is a bornological inductive limit (with continuous canonical injections) of locally m-convex Fréchet subalgebras of A (see [9], p. 941, or [10], Theorem II.4.2). Later on this result was generalized to the case of a sequentially $\mathcal{B}_A$-complete locally m-convex Hausdorff algebra A (see [26], Theorem 2.1) and to the case of an advertibly complete locally m-convex Hausdorff algebra A (see [12], Theorem 6.2, or [15], Theorem 3.14). All these results hold in case of locally m-(k-convex) algebras as well, but not in general in the case of degenerated locally m-pseudoconvex algebras.

In this paper these results are generalized to the case of locally idempotent Hausdorff algebras A with pseudoconvex von Neumann bornology. It is shown (as an application) that for every commutative unital locally m-pseudoconvex Hausdorff algebra A over $\mathbb{C}$ with pseudoconvex von Neumann bornology, which at the same time is sequentially $\mathcal{B}_A$-complete and advertibly complete, the statements (a)–(j) of Proposition 3.2 are equivalent.

[2] For complete algebras see [4], Proposition 2, or [13], Corollary 7.1.14, and for locally m-convex algebras see, for example, [14], pp. 321–322.

2. Main result

The following structural result for locally idempotent algebras holds.

Theorem 2.1. 1) *Let A be a locally idempotent Hausdorff algebra with pseudoconvex von Neumann bornology $\mathcal{B}_A$. Then every basis β_A of $\mathcal{B}_A$ defines an inductive system $\{A_B : B \in \beta_A\}$ of metrizable locally m-(k_B-convex) subalgebras A_B of A with $k_B \in (0,1]$ such that A is a regular inductive limit*[3] *of this system.*

2) *Let A be a locally m-pseudoconvex Hausdorff algebra with pseudoconvex von Neumann bornology*[4] *$\mathcal{B}_A$. Then every basis β_A of $\mathcal{B}_A$ defines an inductive system $\{A_B : B \in \beta_A\}$ of metrizable locally m-(k_B-convex) subalgebras A_B of A with $k_B \in (0,1]$ such that A is a bornological inductive limit of this system with continuous canonical injections from A_B into A.*

In case, when A, in addition, is sequentially $\mathcal{B}_A$-complete, then every subalgebra A_B in the inductive system $\{A_B : B \in \beta_A\}$ is a locally m-(k_B-convex) Fréchet algebra, and when A is sequentially advertibly complete, then every A_B in the inductive system $\{A_B : B \in \beta_A\}$ is an advertibly complete metrizable locally m-(k_B-convex) algebra for each $B \in \beta_A$.

Proof. 1) Let A be a locally idempotent Hausdorff algebra such that the von Neumann bornology $\mathcal{B}_A$ of A is pseudoconvex, β_A a basis of $\mathcal{B}_A$ and $\mathfrak{L}_A$ a base of idempotent balanced neighbourhoods of zero in A. Then every $B \in \beta_A$ defines a number $k_B \in (0,1]$ such that $\Gamma_{k_B}(B) \in \mathcal{B}_A$. For each $n \in \mathbb{N}$ and $B \in \beta_A$ let

$$\mathfrak{L}_n^B = \{O \in \mathfrak{L}_A : \Gamma_{k_B}(B) \subset nO\}.$$

If for fixed $B \in \beta_A$ some of the sets $\mathfrak{L}_n^B$ are empty, then we omit such sets $\mathfrak{L}_n^B$, receiving in this way a sequence of numbers (v_n) (which depends on B) and a sequence of sets $(\mathfrak{L}_{v_n}^B)$, in which all members $\mathfrak{L}_{v_n}^B$ are non-empty. Further, we put

$$\mathfrak{O}_n^B = \bigcap\{O : O \in \mathfrak{L}_{v_n}^B\}.$$

As every set $\mathfrak{O}_n^B$ is non-empty and idempotent in A, then

$$C_n^B(k_B) = \mathrm{cl}_A(\Gamma_{k_B}(\mathfrak{O}_n^B))$$

is a closed, idempotent (see [19], p. 103, and [23], Lemma 1.3) and absolutely k_B-convex subset of A for each $n \in \mathbb{N}$ and $B \in \beta_A$. Therefore, there is a countable set of k_B-homogeneous submultiplicative seminorms p_n^B on

$$A_B = \{a \in A : C_n^B(k_B) \text{ absorbs } a \text{ for each } n \in \mathbb{N}\},$$

defined by

$$p_n^B(a) = \inf\{|\mu|^{k_B} : a \in \mu C_n^B(k_B)\}$$

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[3]An iductive limit A of A_i with $i \in I$ is a *regular inductive limit* (see, for example, [19], p. 83), if $\mathcal{B}_A \subset \bigcup\{\mathcal{B}_{A_i} : i \in I\}$, and A is a *bornological inductive limit* (see, for example, [18], p. 34), if $\mathcal{B}_A = \bigcup\{\mathcal{B}_{A_i} : i \in I\}$.

[4]For example, when A is a locally m-(k-convex) Hausdorff algebra for some $k \in (0,1]$, because in this case the von Newmann bornology $\mathcal{B}_A$ is k-convex (see [31], Proposition 1.2.15).

for each $a \in A_B$. It is not difficult to verify that $B \subset A_B$ for each $B \in \beta_A$ (because $B \subset v_n C_n^B(k_B)$ for each $n \in \mathbb{N}$), A_B is a subalgebra of A,

$$A = \bigcup_{B \in \beta_A} A_B \tag{2.1}$$

and

$$\mathfrak{L}_A = \bigcup_{n \in \mathbb{N}} \mathfrak{L}_{v_n}^B \tag{2.2}$$

for each fixed $B \in \beta_A$. Moreover, every $U \in \mathcal{B}_A$ defines a set $B_0 \in \beta_A$ such that $U \subset B_0 \subset \Gamma_{k_{B_0}}(B_0)$. Since

$$\frac{1}{v_n} U \subset \mathfrak{O}_n^{B_0} \subset \Gamma_{k_{B_0}}(\mathfrak{O}_n^{B_0}) \subset C_n^{B_0}(k_{B_0})$$

for each $n \in \mathbb{N}$, then $C_n^{B_0}(k_{B_0})$ absorbs U for each $n \in \mathbb{N}$. Hence $U \subset A_{B_0}$ and $p_n^{B_0}(u) \leqslant |v_n|^{k_{B_0}}$ for each $u \in U$ and each fixed $n \in \mathbb{N}$. It means that U is bounded in A_{B_0}. Consequently, every bounded subset of A is bounded in some subalgebra A_B of A, where $B \in \beta_A$.

Let now $n \in \mathbb{N}$ be fixed and $B, B' \in \beta_A$. We define the ordering on β_A by inclusion: we say that $B \prec B'$ if and only if $B \subset B'$. Since β_A is a basis of $\mathcal{B}_A$, then for any $B, B' \in \beta_A$ there exists a $B'' \in \beta_A$ such that $B \cup B' \subset B''$ (see, for example, [18], p. 18). Hence, $(\beta_A, \prec)$ is a directed set. Now for any $B, B' \in \beta_A$ with $B \prec B'$ it is true that[5] $\mathfrak{L}_{v_n}^{B'} \subset \mathfrak{L}_{v_n}^{B}$, $\mathfrak{O}_n^B \subset \mathfrak{O}_n^{B'}$, $C_n^B(k_B) \subset C_n^{B'}(k_{B'})$, $A_B \subset A_{B'}$ and

$$p_n^{B'}(a)^{k_B} \leqslant p_n^B(a)^{k_{B'}} \tag{2.3}$$

for each $n \in \mathbb{N}$ and $a \in A_B$.

For each pair $B, B' \in \beta_A$ with $B \prec B'$, let $i_{B'B}$ denote the canonical injection of A_B into $A_{B'}$ and for each $B \in \beta_A$ let i_B denote the canonical injection of A_B into A. Then

$$p_n^{B'}(i_{B'B}(a))^{k_B} \leqslant p_n^B(a)^{k_{B'}}$$

for each $n \in \mathbb{N}$ and $a \in A_B$ by the equality (2.3). Taking this into account, $\{A_B, i_{B'B}; \beta_A\}$ is an inductive system (with continuous canonical injections $i_{B'B}$) of metrizable locally m-(k_B-convex) algebras A_B and A is, by (2.1), a regular inductive limit of this system (with not necessarily continuous canonical injections i_B).

2) Let A be a locally m-pseudoconvex Hausdorff algebra with pseudoconvex von Neumann bornology $\mathcal{B}_A$. Then the injection i_B from A_B into A is continuous for each $B \in \beta_A$. To show this, let $B \in \beta_A$ and O be an arbitrary neighbourhood of zero in A. Since A is locally m-pseudoconvex, then there are a number $k \in (0, 1]$ and a closed absolutely k-convex idempotent neighbourhood O_0 of zero in A such that $O_0 \subset O$. Moreover, there exists a number $k_B \in (0, 1]$ such that $\Gamma_{k_B}(B) \in \mathcal{B}_A$, because $\mathcal{B}_A$ is pseudoconvex. Similarly as above (see the footnote[5]), we can

[5]Without loss of generality, we can assume that $k_{B'} \leqslant k_B$, otherwise in the role of k_B we can take the number $k_{B'}$ since $\Gamma_{k_{B'}}(B) \subset \Gamma_{k_B}(B)$ if $k_B \leqslant k_{B'}$ (in this case $\Gamma_{k_{B'}}(B) \in \mathcal{B}_A$). Thus, if $k_{B'} \leqslant k_B$, then $\Gamma_{k_B}(U) \subset \Gamma_{k_{B'}}(U)$ for any $U \subset A$.

assume that $k \leqslant k_B$. Now O_0 defines a number $n_0 \in \mathbb{N}$ such that $O_0 \in \mathfrak{L}^B_{v_{n_0}}$ by (2.2). Hence $\mathfrak{O}^B_{n_0} \subset O_0$. Therefore, from

$$O^B_{n_0} \subset C^B_{n_0}(k_B) = \mathrm{cl}_A(\Gamma_{k_B}(\mathfrak{O}^B_{n_o})) \subset \mathrm{cl}_A(\Gamma_k(\mathfrak{O}^B_{n_0})) \subset \mathrm{cl}_A \Gamma_k(O_0) = O_0 \subset O$$

follows that $i_B(O^B_{n_0}) \subset O$, where $O^B_{n_0} = \{a \in A_B : p^B_{n_0}(a) < 1\}$ is a neighbourhood of zero in A_B for each fixed $B \in \beta_A$. Hence, i_B is continuous.

Next, let U be a bounded subset in A_B. Then for any $n \in \mathbb{N}$ there is a positive number M_n such that $p^B_n(u) \leqslant M_n^{k_B}$ for all $u \in U$. Hence O defines $n \in \mathbb{N}$ such that

$$U \subset M_n C^B_n(k_B) = M_n \mathrm{cl}_A(\Gamma_k(\mathfrak{O}^B_n)) \subset M_n \mathrm{cl}_A \Gamma_k(O_0) = M_n O_0 \subset M_n O\,.$$

That is, $U \in \mathcal{B}_A$. Consequently, every locally m-pseudoconvex Hausdorff algebra A with pseudoconvex von Neumann bornology $\mathcal{B}_A$ is a bornological inductive limit of metrizable m-(k_B-convex) subalgebras A_B with continuous canonical injections from A_B into A.

Let now, in addition, A be sequentially $\mathcal{B}_A$-complete, $B \in \beta_A$, (a_m) a Cauchy sequence in A_B,

$$V_B = \{a_k - a_l : k, l \in \mathbb{N}\}$$

and

$$O^B_{n\nu} = \{a \in A_B : p^B_n(a) < \nu\}$$

for each $n \in \mathbb{N}$ and $\nu > 0$. Then V_B is bounded in A_B, $O^B_{n\nu}$ is a neighbourhood of zero in A_B and $O^B_{n\nu} = \nu^{\frac{1}{k_B}} O^B_{n1}$ for each $n \in \mathbb{N}$ and $\nu > 0$. Hence, for each $n \in \mathbb{N}$ there exists a number $\mu_n > 0$ such that $V_B \subset \mu_n O^B_{n1}$. Now, let $\epsilon > 0$, (α_n) a sequence of positive numbers, which converges to 0, $\lambda_n = \frac{\mu_n}{\alpha_n}$ for each $n \in \mathbb{N}$ and

$$U = \bigcap_{n \,\in\, \mathbb{N}} \lambda_n O^B_{n1}.$$

Then U is a bounded and balanced subset in A_B, $\frac{\lambda_n}{\mu_n} = \frac{1}{\alpha_n}$ tends to ∞, if $n \to \infty$, and there is a number $s \in \mathbb{N}$ such that $\frac{\lambda_n}{\mu_n} \geqslant \frac{1}{\epsilon}$ for each $n > s$. Hence $\mu_n \leqslant \epsilon\lambda_n$ and $V_B \subset \mu_n O^B_{n1} \subset \epsilon\lambda_n O^B_{n1}$ for each $n > s$. Since

$$W_B = \bigcap_{n \,\leqslant\, s} \epsilon\lambda_n O^B_{n1}$$

is a neighborhood of zero in A_B, then there exists $l \in \mathbb{N}$ and $\alpha > 0$ such that $O^B_{l\alpha} \subset W_B$. Thus

$$V_B \cap O^B_{l\alpha} \subset \Big(\bigcap_{n>s} \epsilon\lambda_n O^B_{n1}\Big) \bigcap \Big(\bigcap_{n\leqslant s} \epsilon\lambda_n O^B_{n1}\Big) = \bigcap_{n\in\mathbb{N}} \epsilon\lambda_n O^B_{n1} = \epsilon U. \tag{2.4}$$

As (a_m) is a Cauchy sequence in A_B, then there is a number $r \in \mathbb{N}$ such that $a_s - a_t \in O^B_{l\alpha}$, whenever $s > t > r$. Taking this into account, it is clear by (2.4), that $a_s - a_t \in \epsilon U$, whenever $s > t > r$. Consequently, (a_m) is a Mackey–Cauchy sequence in A_B. Since, the canonical injection i_B of A_B into A is continuous, then U is bounded in A in the present case and (a_m) is a Cauchy-Mackey sequence also in A. Hence, (a_m) converges in A say, to a_0.

As (a_m) is a bounded sequence in A_B, then for each fixed $n \in \mathbb{N}$ there exists a number $M_n > 0$ such that

$$p_n^B(a_m) < M_n^{k_B}$$

for all $m \in \mathbb{N}$. Hence, $a_m \in M_n C_n^B(k_B)$ for each fixed $n \in \mathbb{N}$ and all $m \in \mathbb{N}$. It is easy to see that $M_n C_n^B(k_B)$ is a closed and balanced subset of A. Therefore

$$a_0 \in M_n C_n^B(k_B) = \mu\Big(\frac{M_n}{\mu}\Big) C_n^B(k_B) \subset \mu C_n^B(k_B),$$

whenever $|\mu| \geqslant M_n$. Consequently, $C_n^B(k_B)$ absorbs a_0 for each $n \in \mathbb{N}$. Hence, $a_0 \in A_B$. Since (a_n) is a Cauchy sequence in A_B, then for each $\epsilon > 0$ there exist $\delta \in (0, \epsilon)$ and $r_\delta \in \mathbb{N}$ such that $p_n^B(a_s - a_t) < \delta$, whenever $s > t > r_\delta$. Taking this into account, $p_n^B(a_0 - a_t) \leq \delta < \epsilon$ for each $t > r_\delta$, because p_n^B is continuous on A_B. Consequently, (a_n) converges to a_0 in A_B. It means that every A_B is a locally m-(k-convex) Fréchet algebra.

Let now A be a sequentially advertibly complete locally m-pseudoconvex Hausdorff algebra with pseudoconvex von Neumann bornology $\mathcal{B}_A$, β_A a basis of $\mathcal{B}_A$ and let $B \in \beta_A$. Then the canonical injection i_B from A_B into A is continuous (as it has been shown above). Therefore the topology τ_{A_B} on A_B, defined by the system of seminorms $\{p_n^B : n \in \mathbb{N}\}$, is stronger than the topology $\tau|_{A_B}$ on A_B, induced by the topology of A. If (a_n) is a Cauchy sequence in A_B which is advertibly convergent, then there exists an element $a \in A_B$ such that sequences $(a \circ a_n)$ and $(a_n \circ a)$ converge to θ_A in the topology τ_{A_B}. Since τ_{A_B} is stronger than $\tau|_{A_B}$, then (a_n) is a Cauchy sequence in A which advertibly converges in the topology of A as well. Hence, (a_n) converges in A, because A is sequentially advertibly complete.

Let a_0 be the limit of (a_n) in A. It is easy to see that a_0 is the quasi-inverse of a in A. Since every Cauchy sequence is bounded, then, similarily as above, $C_n^B(k_B)$ absorbs a_0 for all $n \in \mathbb{N}$. Thus, $a_0 \in A_B$. Since $(a_n) = (a_0 \circ (a \circ a_n))$ converges to $a_0 \circ \theta_A = a_0$, then A_B is an advertibly complete metrizable locally m-(k_B-convex) algebra with $k_B \in (0, 1]$ for each $B \in \mathcal{B}$. □

3. Applications

1. Let A be a topological algebra over $\mathbb{C}$, Qinv$A$ the set of all quasi-invertible elements (if A is a unital algebra, let Inv$A$ be the set of all invertible elements) in A and let $a \in A$. The set

$$\mathrm{sp}_A(a) = \{\lambda \in \mathbb{C} \setminus \{0\} : \frac{a}{\lambda} \notin \mathrm{Qinv}A\} \cup \{0\}$$

(if A has a unit e_A, then $\mathrm{sp}_A(a) = \{\lambda \in \mathbb{C} : a - \lambda e_A \notin \mathrm{Inv}A\}$) is the *spectrum* of a and

$$\mathrm{r}_A(a) = \sup\{|\lambda| : \lambda \in \mathrm{sp}_A(a)\}$$

the *spectral radius* of a. If $\hom A$ is not empty, then

$$\{\varphi(a) : \varphi \in \hom A\} \subset \mathrm{sp}_A(a)$$

for each $a \in A$. In particular, when

$$\mathrm{sp}_A(a) = \{\varphi(a) : \varphi \in \hom A\} \cup S,$$

where $S = \{0\}$ if $a \notin \bigcup\{\ker\varphi : \varphi \in \hom A\}$ and $S = \emptyset$ otherwise, we will say that A is a *topological algebra with functional spectrum.*

2. For any topological algebra A let τ_M denote the *Mackey closure topology* on A, that is,

$$\tau_M = \{O \subset A : \forall\, a \in O \text{ and } \forall \text{ balanced } B \in \mathcal{B}_A\ \exists\, \lambda > 0 \text{ such that } a + \lambda B \subset O\}.$$

Then every element of τ_M is a *Mackey open* subset and every element U, for which $A \setminus U \in \tau_M$, is a *Mackey closed* subset in A. It is easy to show (see, for example, [18], p. 37 and p. 120) that a subset $O \subset A$ is Mackey open if and only if for every $a \in O$ and for every net $(a_\lambda)_{\lambda\in\Lambda}$ in A, which converges to a in the sense of Mackey, there is an index $\lambda_0 \in \Lambda$ such that $a_\lambda \in O$ for all $\lambda \succ \lambda_0$ and O is Mackey closed if and only if for every net $(a_\lambda)_{\lambda\in\Lambda}$ in O, which converges to a_0 in the sense of Mackey, element $a_0 \in O$. A topological algebra A is called a *Q-algebra* (*Mackey Q-algebra*) if the set Qinv$A$ (if A is a unital algebra, then the set Inv$A$) is open (respectively, is Mackey open) in A. It is easy to see that every Q-algebra is a Mackey Q-algebra. Nevertheles, there are Mackey Q-algebras (see [16], Example 3.9) which are not Q-algebras.

Lemma 3.1. *Let A be a topological algebra. Then A is a Mackey Q-algebra if and only if* Qinv$A$ *has a non-empty interior in the Mackey closure topology.*

Proof. Let S denote the interior of Qinv$A$ in the Mackey closure topology. If A is a Mackey Q-algebra, then $\theta_A \in S$. Assume now that S is not empty. For every fixed $b \in A$ let $l_b(a) = b \circ a$ and $r_b(a) = a \circ b$ for each $a \in A$. It is easy to see that the maps l_b and r_b are Mackey continuous on A. If now $a \in$ Qinv$A$ and $s \in S$, then[6] $l_{s\circ a_q^{-1}}(a) = r_{a_q^{-1}\circ s}(a) = s \in S$. To show that

$$W = l^{-1}_{s\circ a_q^{-1}}(S) \cap r^{-1}_{a_q^{-1}\circ s}(S) \subset \mathrm{Qinv}A,$$

let $w \in W$ an arbitrary element. Then

$$l_{s\circ a_q^{-1}}(w), r_{a_q^{-1}\circ s}(w) \in S \subset \mathrm{Qinv}A.$$

Hence, there exist $x, y \in A$ such that

$$x \circ l_{s\circ a_q^{-1}}(w) = l_{s\circ a_q^{-1}}(w) \circ x = \theta_A$$

and

$$y \circ r_{a_q^{-1}\circ s}(w) = l_{s\circ a_q^{-1}}(w) \circ y = \theta_A.$$

Therefore

$$[x \circ (s \circ a_q^{-1})] \circ w = x \circ [(s \circ a_q^{-1}) \circ w] = \theta_A$$

and

$$w \circ [(a_q^{-1} \circ s) \circ y] = [w \circ (a_q^{-1} \circ s)] \circ y = \theta_A.$$

Now $x \circ (s \circ a_q^{-1}) = (a_q^{-1} \circ s) \circ y$ and $w \in$ Qinv$A$.

To show that W is Mackey open, let $w_0 \in W$ and $(w_\alpha)_{\alpha\in\mathcal{A}}$ be a net in A which Mackey converges to w_0. Since $l_{s\circ a_q^{-1}}$ and $r_{a_q^{-1}\circ s}$ are Mackey continuous maps, then $(l_{s\circ a_q^{-1}}(w_\alpha))_{\alpha\in\mathcal{A}}$ converges to $l_{s\circ a_q^{-1}}(w_0) \in S$ and $(r_{a_q^{-1}\circ s}(w_\alpha))_{\alpha\in\mathcal{A}}$ converges

[6]Here and later on a_q^{-1} denotes the quasi-inverse of $a \in A$.

to $r_{a_q^{-1}\circ s}(w_0) \in S$ in the sense of Mackey. Therefore, there exist $\alpha_1, \alpha_2 \in \mathcal{A}$ such that $l_{s\circ a_q^{-1}}(w_\alpha) \in S$, whenever $\alpha \succ \alpha_1$ and $r_{a_q^{-1}\circ s}(w_\alpha) \in S$, whenever $\alpha \succ \alpha_2$. Let $\alpha_0 \in \Lambda$ be such that $\alpha_0 \succ \alpha_1$ and $\alpha_0 \succ \alpha_2$. Then $w_\alpha \in W$, whenever $\alpha \succ \alpha_0$. Consequently, W is a Mackey open neighbourhood of a, because of which Qinv$A$ is a Mackey open set in A. □

Proposition 3.2. *Let A be a topological Hausdorff algebra over $\mathbb{C}$ with pseudoconvex von Neumann bornology $\mathcal{B}_A$. If* $\hom A$ *is not empty and, in addition, A satisfies the following conditions:*

(α) *A is sequentially $\mathcal{B}_A$-complete;*
(β) *if $a \in A$ and $\mathrm{r}_A(a) < 1$, then the set $\{a^n : n \in \mathbb{N}\}$ is bounded in A;*
(γ) *if $a \in A$ and $\varphi(a) \neq 1$ for each $\varphi \in \hom A$, then*[7] *$a \in$ Qinv$A$;*
(δ) *A is representable in the form of a regular inductive limit of barrelled subalgebras A_i of A with $i \in I$ such that the canonical injections $\iota_i : A_i \to A$ are continuous,*

then the following statements are equivalent:

(a) *every $a \in A$ is bounded*[8]*;*
(b) $\mathrm{sp}_A(a)$ *is bounded for each $a \in A$;*
(c) $\mathrm{sp}_A(a)$ *is compact for each $a \in A$;*
(d) r_A *is a bounded map from A into $[0, \infty)$;*
(e) r_A *is Mackey continuous at θ_A;*
(f) r_A *is a Mackey continuous map;*
(g) *the set $\{a \in A : \mathrm{r}_A(a) < 1\}$ is Mackey open;*
(h) *the interior of* Qinv$A$ *in the Mackey closure topology on A is not empty;*
(i) *A is a Mackey Q-algebra;*
(j) Hom$A$ *is an equibounded*[9] *set.*

Proof. (a) ⇒ (b) It is known (see [7], Theorem 4.2) that $\mathrm{r}_A(a) < \infty$ if A is sequentially $\mathcal{B}_A$-complete and every element in A is bounded. Therefore from the statement (a) follows (b).

(b) ⇒ (a) Let $a \in A$ and let $\mathrm{sp}_A(a)$ be bounded. Then there is a number $M > 0$ such that $\mathrm{r}_A(a) < M$ or $\mathrm{r}_A(\frac{a}{M}) < 1$. Therefore $\{(\frac{a}{M})^n : n \in \mathbb{N}\}$ is bounded in A by the assumption (β). It means that from the statement (b) follows (a).

(b) ⇒ (c) Suppose that there is an element $a \in A$ such that $\mathrm{sp}_A(a)$ is not closed in $\mathbb{C}$. Then there exists a complex number

$$\mu_a \in \mathrm{cl}_{\mathbb{C}}(\mathrm{sp}_A(a)) \setminus \mathrm{sp}_A(a)$$

[7] If $a \in A \setminus \bigcup\{\ker\varphi : \varphi \in \hom A\}$ and $\lambda \in \mathrm{sp}_A(a) \setminus \{0\}$, then $\frac{a}{\lambda} \notin$ Qinv$A$. Hence, by applying the statement (γ), there exists a map $\varphi \in \hom A$ such that $\lambda = \varphi(a)$. It means that $\mathrm{sp}_A(a) \setminus \{0\} \subset \{\varphi(a) : \varphi \in \hom A\}$. Otherwise $\mathrm{sp}_A(a) \subset \{\varphi(a) : \varphi \in \hom A\}$. Hence, from ($\gamma$) follows that A has functional spectrum.

[8] An $a \in A$ is *bounded* if there is a $\lambda \in \mathbb{C} \setminus \{0\}$ such that the set $\{(\frac{a}{\lambda})^n : n \in \mathbb{N}\}$ is bounded in A.

[9] Here and later on Hom$A$ denotes the set of nontrivial (not necessarily continuous) homomorphisms from A onto $\mathbb{C}$. A family $\mathcal{F}$ of maps f from a topological linear space X into another topological linear space Y is *equibounded* if the set $\bigcup\{f(B) : f \in \mathcal{F}\}$ is bounded in Y for each bounded set B of X.

such that $\frac{1}{\mu_a}a \in \operatorname{Qinv}A$ ($\mu_a \neq 0$ because $0 \in \operatorname{sp}_A(a)$). Since

$$\operatorname{sp}_A(a) = \{\varphi(a) : \varphi \in \hom A\} \cup S,$$

where $S = \{0\}$ if $a \notin \bigcup\{\ker\varphi : \varphi \in \hom A\}$ and $S = \emptyset$ otherwise, by the assumption (γ), then there is a sequence (φ_n) in $\hom A$ such that the sequence $(\varphi_n(a))$ converges to μ_a in $\mathbb{C}$. It is well known (see, for example, [27], Theorem 1.6.11) that

$$\operatorname{sp}_A(a_q^{-1}) = \Big\{\frac{\lambda}{\lambda - 1} : \lambda \in \operatorname{sp}_A(a)\Big\}.$$

Therefore

$$\operatorname{sp}_A\Big[\Big(\frac{a}{\mu_a}\Big)_q^{-1}\Big] = \Big\{\frac{\varphi(a)}{\varphi(a) - \mu_a} : \varphi \in \hom A\Big\}.$$

Thus,

$$\operatorname{sp}_A\Big[\Big(\frac{a}{\mu_a}\Big)_q^{-1}\Big]$$

is not bounded which is not possible. Hence, $\operatorname{sp}_A(a)$ is closed in $\mathbb{C}$ for each $a \in A$ and every bounded closed subset in $\mathbb{C}$ is compact.

(c) $\Rightarrow$ (b) is clear.

(b) $\Rightarrow$ (d) Since

$$\operatorname{r}_A(a) = \sup\{f_\varphi(a) : \varphi \in \hom A\} < \infty$$

for each $a \in A$ by the condition (b) and the assumption (γ), where the function f_φ, defined by $f_\varphi(a) = |\varphi(a)|$ for each $a \in A$ and each $\varphi \in \hom A$, is continuous (consequently, is lower semicontinuous too), then r_A is a lower semicontinuous function on A (see, for example, [28], p. 97). Therefore

$$O_\varepsilon = \{a \in A : \operatorname{r}_A(a) \leqslant \varepsilon\}$$

is closed set in A for each $\varepsilon > 0$.

Let $B_0 \in \mathcal{B}_A$. By the assumption (δ) there are barrelled subalgebras A_i with $i \in I$ in A such that A is a regular inductive limit of subalgebras A_i and the cannonical injections $\iota_i : A_i \to A$ are continuous. Therefore, there exists an index $i_0 \in I$ such that $B_0 \subset A_{i_0}$ and B_0 is bounded in A_{i_0}. Moreover, if $g_{i_0} = \operatorname{r}_A \circ \iota_{i_0}$, then

$$U_{i_0}^\varepsilon = \{b \in A_{i_0} : g_{i_0}(b) \leqslant \varepsilon\} = \iota_{i_0}^{-1}(O_\varepsilon)$$

is a barrel in A_{i_0} for each $\varepsilon > 0$. Hence, $U_{i_o}^\varepsilon$ is a neighbourhood of zero in A_{i_0} for each $\varepsilon > 0$, because every A_i is barrelled. Now $U_{i_0}^\varepsilon$ defines a number $\mu_\varepsilon > 0$ such that $B_0 \subset \mu_\varepsilon U_{i_0}^\varepsilon$. Since $g_{i_0}(A_{i_0}) \subset [0,\infty)$ by the contition (b) and $\{[0,\delta) : \delta > 0\}$ is a base of 0 in $[0,\infty)$, then for every neighbourhood O of zero in $[0,\infty)$ there is a number $\varepsilon > 0$ such that $[0,\varepsilon] \subset O$. Therefore,

$$\operatorname{r}_A(B_0) \subset \mu_\varepsilon g_{i_0}(U_{i_0}^\varepsilon) \subset \mu_\varepsilon[0,\varepsilon] \subset \mu_\varepsilon O.$$

Consequently, r_A is a bounded map.

(d) $\Rightarrow$ (e) Let $(a_\lambda)_{\lambda\in\Lambda}$ be a net in A which converges to θ_A in the sense of Mackey. Then there exist a balanced set $B \in \mathcal{B}_A$ and for any $\varepsilon > 0$ an index $\lambda_0 \in \Lambda$ such that $a_\lambda \in \varepsilon B$, whenever $\lambda \succ \lambda_0$. Since $\operatorname{r}_A(a_\lambda) \in \varepsilon\, \operatorname{r}_A(B)$, whenever $\lambda \succ \lambda_0$ and $\operatorname{r}_A(B)$ is bounded in $[0,\infty)$ by the statement (d), then $(\operatorname{r}_A(a_\lambda))_{\lambda\in\Lambda}$

converges to $\mathrm{r}_A(\theta_A) = 0$ in $[0,\infty)$ in the sense of Mackey. Therefore, r_A is Mackey continuous at θ_A.

(e) ⇒ (f) Let $(a_\lambda)_{\lambda\in\Lambda}$ be a net in A which converges to $a_0 \in A$ in the sense of Mackey. Then the net $(a_\lambda - a_0)_{\lambda\in\Lambda}$ converges to θ_A in A in the sense of Mackey. Therefore the net $(\mathrm{r}_A(a_\lambda - a_0))_{\lambda\in\Lambda}$ converges to 0 in $[0,\infty)$ (because from the convergence of net in the sense of Mackey follows the convergence of it in the sense of topology). Since r_A is subadditive by the assumption (γ), then

$$|\mathrm{r}_A(a) - \mathrm{r}_A(b)| \leqslant \mathrm{r}_A(a-b)$$

for all $a, b \in A$. Hence, the net $(\mathrm{r}_A(a_\lambda))_{\lambda\in\Lambda}$ converges to $\mathrm{r}_A(a_0)$ in the sense of topology, consequently, also in the sense of Mackey (because $[0,\infty)$ is a metric space).

(f) ⇒ (g) Let $U = A \setminus \{a \in A : \mathrm{r}_A(a) < 1\}$ and $(a_\lambda)_{\lambda\in\Lambda}$ a net in U which converges to $a_0 \in A$ in the sense of Mackey. Then $\mathrm{r}_A(a_\lambda) \geqslant 1$ for each $\lambda \in \Lambda$. Since the net $(\mathrm{r}_A(a_\lambda))_{\lambda\in\Lambda}$ converges to $\mathrm{r}_A(a_0)$ by the statement (f), then $\mathrm{r}_A(a_0) \geqslant 1$ or $a_0 \in U$. Hence, U is Mackey closed. Consequently, $\{a \in A : \mathrm{r}_A(a) < 1\}$ is Mackey open.

(g) ⇒ (h) The set $O = \{a \in A : \mathrm{r}_A(a) < 1\}$ is a neighbourhood of zero in A in the Mackey closure topology by the statement (g). If now $a \in O$, then $\varphi(a) < 1$ for each $\varphi \in \hom A$ because A has functional spectrum by the assumption (γ) and $O \subset \mathrm{Qinv}A$. Consequently, the interior of $\mathrm{Qinv}A$ in the Mackey closure topology is not empty.

(h) ⇒ (i) The statement (i) follows from (g) by Lemma 3.1.

(i) ⇒ (b) The set $\mathrm{Qinv}A$ is a neighbourhood of zero in the Mackey closure topology on A by the statement (i). Therefore for each $a \in A$ there is a number $\mu_a > 0$ such that $\frac{a}{\mu_a} \in \mathrm{Qinv}A$ or $\mu_a \neq \mathrm{sp}_A(a)$. Hence, $\mathrm{r}_A(a) < \mu_a$. It means that $\mathrm{sp}_A(a)$ is bounded for each $a \in A$.

(d) ⇒ (j) Since

$$\{\varphi(a) : \varphi \in \hom A\} \subset \{\varphi(a) : \varphi \in \mathrm{Hom}A\} \subset \mathrm{sp}_A(a)$$

for each $a \in A$ and A has functional spectrum by the assumption (γ), then

$$\mathrm{r}_A(a) = \sup\{|\varphi(a)| : \varphi \in \mathrm{Hom}A\}$$

for each $a \in A$. Hence,

$$\bigcup_{\varphi\in\,\mathrm{Hom}A} \varphi(B)$$

is bounded in $[0,\infty)$ for each $B \in \mathcal{B}_A$ by the statement (d). Hence, $\mathrm{Hom}A$ is a equibounded set.

(j) ⇒ (d) Let $\mathrm{Hom}A$ be an equibounded set. Then for each $B \in \mathcal{B}_A$ there exists a number $M_B > 0$ such that $|\varphi(a)| < M_B$ for all $a \in B$ and $\varphi \in \mathrm{Hom}A$. Therefore, $\mathrm{r}_A(B)$ is bounded. Hence, the statement (d) is true. □

Theorem 3.3. *Let A be a commutative unital locally m-pseudoconvex Hausdorff algebra over $\mathbb{C}$ with pseudoconvex von Neumann bornology. If, at the same time, A is sequentially $\mathcal{B}_A$-complete and advertibly complete, then all the statements (a)–(j) of Proposition* 3.2 *are equivalent.*

Proof. Let A be a commutative unital locally m-pseudoconvex Hausdorff algebra over $\mathbb{C}$. Then A is an advertive (see [3], Corollary 2) simplicial (see [6], Corollary 5; for complete case see [3], Proposition 2) Gelfand–Mazur algebra (see [2], Corollary 2, or [1], Lemma 1.11). Therefore (see [3], Proposition 8), hom A is not empty and A satisfies the condition (γ) of Proposition 3.2. Let $\{p_\lambda : \lambda \in \Lambda\}$ be a saturated family of k_λ-homogeneous seminorms (with $k_\lambda \in (0,1]$ for each $\lambda \in \Lambda$), which defines the topology of A. If $a \in A$ and $\mathrm{r}_A(a) < 1$, then there is a number ρ such that $\mathrm{r}_A(a) < \rho < 1$. Since A is advertibly complete, then

$$\mathrm{r}_A(a) = \sup_{\lambda\in\Lambda} \lim_{n\to\infty} \sqrt[k_\lambda n]{p_\lambda(a^n)}$$

for each $a \in A$ (see [3], Proposition 12). Therefore, for every $\lambda \in \Lambda$ there is a number $n_\lambda \in \mathbb{N}$ such that $p_\lambda(a^n) < \rho^{k_\lambda} < 1$, whenever $n > n_\lambda$. It means that $p_\lambda(a^n) < \infty$ for all $\lambda \in \Lambda$. Hence, the set $\{a^n : n \in \mathbb{N}\}$ is bounded in A. That is, A satisfies the condition (β) of Proposition 3.2. Since A satisfies also the condition (δ) of Proposition 3.2 by Theorem 2.1, then the statements (a)–(j) are equivalent by Proposition 3.2. □

Corollary 3.4. *Let A be a commutative unital locally m-(k-convex) Hausdorff algebra over $\mathbb{C}$ for some $k \in (0,1]$. If, at the same time, A is sequentially $\mathcal{B}_A$-complete and advertibly complete (in particular, A is complete), then all the statements (a)–(j) of Proposition 3.2 are equivalent.*

Remark 3.5. Corollary 3.4 in case $k = 1$ has been partly proved in many papers (see, for example, [12], Proposition 4.3, and [26], Proposition 4.1, for complete case see [25], Proposition 3.3; [11], Theorem on the p. 61 and others).

References

1. Mart Abel, *Structure of Gelfand–Mazur algebras,* Dissertationes Mathematicae Universitatis Tartuensis **31**, Tartu University Press, Tartu, 2003.
2. Mati Abel, *Gelfand–Mazur algebras,* Topological vector spaces, algebras and related areas, (Hamilton, ON, 1994), 116–129, Pitman Res. Notes Math. Ser. **316**, Longman Sci. Tech., Harlow, 1994.
3. Mati Abel, *Advertible topological algebras,* General topological algebras (Tartu, 1999), 14–24, Math. Stud. (Tartu), **1**, Est. Math. Soc., Tartu, 2001.
4. Mati Abel, *Descriptions of the topological radical in topological algebras,* General topological algebras (Tartu, 1999), 25–31, Math. Stud. (Tartu), **1**, Est. Math. Soc., Tartu, 2001.
5. Mati Abel, *Survey of results on Gelfand–Mazur algebras,* Non-normed topological algebras (Rabat, 2000), 14–25, E. N. S. Takaddoum Publ., Rabat, 2004.
6. Mati Abel, *Inductive limits of Gelfand–Mazur algebras,* Int. J. Pure Appl. Math. **16** (2004), no. 3, 363–378.
7. Mati Abel, *Topological algebras with pseudoconvexly bounded elements,* Topological algebras, their applications, and related topics, 21–33, Banach Center Publ., **67**, Polish Acad. Sci., Warsaw, 2005.
8. Mati Abel, A. Kokk, *Locally pseudoconvex Gelfand–Mazur algebras,* Eesti NSV Tead. Akad. Toimetised Füüs.-Mat. **37** (1988), no. 4, 377–386 (in Russian).
9. M. Akkar, *Sur la structure des algèbres topologiques localement multiplicativement convexes,* C. R. Acad. Sc. Paris Ser. A **279** (1974), 941–944.
10. M. Akkar, *Etude spectrale et structures d'algèbres topologiques et bornologiques complètes,* Thése Sci. Math., Univ. de Bordeaux, 1976.

11. M. Akkar, *Caractèrisation des algébres localement m-convexes dont l'ensemble des caractères est équiborné,* Colloq. Math. **68** (1995), no. 1, 59–65.
12. M. Akkar, A. Beddaa, M. Oudadess, *Sur une classe d'algèbres topologiques,* Bull. Belg. Math. Soc. Simon Stevin **3** (1996), no. 1, 13–24.
13. V.K. Balachandran, *Topological algebras,* North-Holland Mathematics Studies, **185**, North-Holland Publishing Co., Amsterdam (2000).
14. E. Beckenstein, L. Narici and Ch. Suffel, *Topological algebras,* North-Holland Mathematics Studies, **24**, North-Holland Publ. Co., Amsterdam-New York-Oxford (1977).
15. A. Beddaa, *Algèbres localement convexes advertiblement complètes et continuité automatique des morphismes,* These Docteur d'Etat Sciences Mathematiques. Univ. Mohamed V de Rabat, Rabat, 1997.
16. A. El Kinani, *Advertible complétude et structure de Q-algèbre*, Rend. Circ. Mat. Palermo (2) **50**, no. 3, (2001),427–442.
17. H. Hogbe-Nlend, *Théorie des bornologies et applications*, Lecture Notes in Mathematics **213**, Springer-Verlag, Berlin-New York, 1971.
18. H. Hogbe-Nlend, *Bornologies and Functional Analysis. Introductory course on the theory of duality topology-bornology and its use in functional analysis.*, North-Holland Mathematics Studies **26**, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.
19. H. Jarchow, *Locally Convex Spaces. Mathematische Leitfäden*, B.G. Teubner, Stuttgart, 1981.
20. J.-P. Ligaud, *Sur les rapports entre topologies et bornologies pseudoconvexes*, C. R. Acad. Sci. Paris. Sér. A–B **271** (1970), A1058–A1060.
21. A. Mallios, *Topologial Algebras. Selected Topics*, North-Holland Mathematics Studies **124**, North-Holland Publishing Co., Amsterdam, 1986.
22. R.C. Metzler, *A remark on bounded sets in linear topological spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **15** (1967), 317–318.
23. E.A. Michael, *Locally multiplicatively-convex topologial algebras,* Mem. Amer. Math. Soc., 1952.
24. V. Murali, *Locally idempotent algebras,* Math. Japon. **30** (1985), N. 5, 736–776.
25. M. Oudadess, *A note on m-convex and pseudo-Banach structures,* Rend. Circ. Mat. Palermo (2) **41** (1992), no. 1, 105–110.
26. M. Oudadess, *Functional boundedness of some M-complet m-convex algebras,* Bull. Greek Math. Soc. **39** (1997), 17–20.
27. C.E. Rickart, *General theory of Banach algebras,* D. van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1960.
28. Z. Semadeni, *Banach spaces of continuous functions,* Vol. I, PWN, Warszawa, 1971.
29. W. Żelazko, *Metric generalizations of Banach Algebras,* Rozprawy Mat. **XLVII**, PWN, Warszawa, 1965.
30. W. Żelazko, *Selected topics in topological algebras,* Lecture Notes Series **31**, Aarhus Universitet, Aarhus, 1971.
31. L. Waelbroeck, *Bornological quotients. With the collaboration of Guy Noël,* Académie Royale de Belgique, Classe des Sciences, Brussels, 2005.

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