EXISTENCE RESULTS FOR QUASI VARIATIONAL INEQUALITIES

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This paper is dedicated to Professor Dr. Themistocles M. Rassias.

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ABSTRACT. It is well known that the quasi variational inequalities are equivalent to the fixed point problems. We this equivalent alternative formulation to discuss the existence of a solution of quasi variational inequalities under some mild conditions. Since the quasi variational inequalities include variational inequalities, implicit complementarity problems and optimization problems as special cases, results proved in this paper continue to hold these problems. This shows that results proved in this paper can be viewed as an important and significant improvement and refinement of the previous results.

1. INTRODUCTION

Variational inequality theory, which was introduced by Stampacchia [29] in 1964, has emerged as a fascinating and interesting branch of mathematical and engineering sciences. The ideas and techniques of the variational inequalities are being applied in a variety of diverse areas of sciences and proved to be productive and innovative. It has been shown that this theory provides a simple, natural and unified framework for a general treatment of unrelated problems, see [1]–[36] and the references therein. These activities have motivated to generalize and extend the variational inequalities and related optimization problems in several directions using new and novel techniques. A useful and important generalization

of the variational inequalities is known as quasi variational inequality with wide range of applications in industry, finance, economics, engineering, network analysis, structural analysis, optimizations and operation research. In recent several iterative type methods including projection, Wiener-Hopf equations, updating the technique of solution, auxiliary principle and splitting methods for solving variational inequalities, see, for example, Noor [17]–[20], [23, 24] and the references therein for details. We would like to emphasize that the projection method and its variant forms represent an important tool in the study of the existence results and developing numerical methods for solving variational inequalities and related optimization problems, the origin of which can be traced back to Lions and Stampacchia [10]. The main idea in this technique is to establish the equivalence between the variational inequalities and the fixed point problems. Essentially using the projection technique, Noor [13] has proved that the quasi variational inequalities are equivalent to the fixed-point problems. This alternative equivalence formulation has been used to develop some numerical methods, sensitivity analysis framework and dynamical system for quasi variational inequalities, see [15]–[19] and the references therein. To the best of our knowledge, this approach has not been used to discuss the existence of a solution of quasi variational inequalities. Inspired and motivated by the research going on this filed, we again use the fixed point approach to discuss the existence of a solution of quasi variational inequalities under some mild conditions. As special cases, we also obtain the existence results for the classical variational inequalities under some mild conditions. In facts, results proved in this paper may be viewed as an improvement and refinement of the previously known results.

2. Formulations and Basic Facts

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $K(u)$ be a closed and convex-valued set in H and $T : H \longrightarrow H$ be a nonlinear operator.

A *quasi variational inequality* consists in finding $u \in K(u)$, such that

$$\langle Tu, v - u \rangle \geq 0 \quad (v \in K(u)). \tag{2.1}$$

It is well known [1]–[10], [15]–[28] that a large class of obstacle, unilateral, contact, free, moving, and equilibrium problems arising in economics, finance, physical, mathematical, engineering and applied sciences can be studied in the unifying and general framework of (2.1).

To convey an idea of the applications of the quasi variational inequalities, we consider the second-order implicit obstacle boundary value problem of finding u such that

$$\left. \begin{array}{ll} -u'' \geq f(x) & \text{on } \Omega = [a, b] \\ u \geq M(u) & \text{on } \Omega = [a, b] \\ [-u'' - f(x)][u - M(u)] = 0 & \text{on } \Omega = [a, b] \\ u(a) = 0, \quad u(b) = 0. \end{array} \right\} \tag{2.2}$$

where $f(x)$ is a continuous function and $M(u)$ is the cost (obstacle) function. The prototype encountered [2] is

$$M(u) = k + \inf_i \{u^i\}. \tag{2.3}$$

In (2.3), k represents the switching cost. It is positive when the unit is turned on and equal to zero when the unit is turned off. Note that the operator M provides the coupling between the unknowns $u = (u^1, u^2, \ldots, u^i)$. We study the problem (2.2) in the framework of the quasi variational inequality approach. To do so, we first define the set $K(u)$ as

$$K(u) = \{v : v \in H_0^1(\Omega) : v \geq M(u), \quad \text{on } \Omega\}, \tag{2.4}$$

which is a closed convex-valued set in $H_0^1(\Omega)$, where $H_0^1(\Omega)$ is a Sobolev (Hilbert) space. One can easily show that the energy functional associated with the problem (2.2) is

$$
\begin{aligned}
I[v] &= -\int_a^b \left(\frac{d^2 v}{dx^2}\right) v dx - 2 \int_a^b f(x)(v) dx \quad (v \in K(u)) \\
&= \int_a^b \left(\frac{dv}{dx}\right)^2 dx - 2 \int_a^b f(x)(v) dx \\
&= \langle Tv, v \rangle - 2 \langle f, v \rangle
\end{aligned}
\tag{2.5}
$$

where

$$
\begin{aligned}
\langle Tu, v \rangle &= \int_a^b \left(\frac{d^2 u}{dx^2}\right)(v) dx = \int_a^b \frac{du}{dx}\frac{dv}{dx} dx \\
\langle f, v \rangle &= \int_a^b f(x)(v) dx.
\end{aligned}
\tag{2.6}
$$

It is clear that the operator T defined by (2.6) is linear, symmetric and positive. Using the technique of Noor [19, 20], one can show that the minimum of the functional $I[v]$ defined by (2.5) associated with the problem (2.2) on the closed convex-valued set $K(u)$ can be characterized by the inequality of type (2.1). See also [1]–[29] for the formulation, applications, numerical methods and sensitivity analysis of the quasi variational inequalities.

If $K^\star(u)$ is the dual (polar) cone of the convex-valued cone $K(u)$, then the quasi variational inequalities (2.1) are equivalent to finding u such that

$$u \in K(u), \quad Tu \in K^\star(u) \quad \text{and} \quad \langle u, Tu \rangle = 0, \tag{2.7}$$

which are called the quasi (implicit) complementarity problems. It is well known that a wide class of problems arising in various branches of pure and applied sciences can be studied via the implicit complementarity problems (2.7). For the applications, numerical methods and physical formulation, see the references.

If the convex-valued set $K(u)$ is independent of the solution u, that is, $K(u) = K$, a closed convex set, then problem (1) is equivalent to finding $u \in K$, such that

$$\langle Tu, v - u \rangle \geq 0 \quad (v \in K), \tag{2.8}$$

which is known as the classic variational inequality introduced and studied by Stampacchia [29] in 1964. For the state of the art in this theory; see [1]–[36].

We also need the following well-known concepts and results.

Lemma 2.1. *Let $K(u)$ be a closed convex-valued set in H. Then, for a given $z \in H$, $u \in K(u)$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0 \quad (v \in K(u)),$$

if and only if

$$u = P_{K(u)}z,$$

where $P_{K(u)}$ is the projection of H onto the closed convex-valued set $K(u)$.

It is worth mentioning that the implicit projection operator $P_{K(u)}$ is not an nonexpansive operator. This motivates us to consider the following assumption on the projection operator $P_{K(u)}$ as:

Assumption 2.2. *The projection operator $P_{K(u)}$ satisfies the following relation.*

$$\|P_{K(u)}w - P_{K(v)}w\| \leq \nu\|u - v\| \quad (v, u, w \in H),$$

where $\nu > 0$ is a constant.

We remark that Assumption 2.2 is true for the special case,

$$K(x) = m(x) + K, \tag{2.9}$$

which appears in many important applications [7], where m is a point-to-point mapping and K is a closed convex set in H. It is well known that

$$P_{K(u)}w = P_{m(u)+K}w = m(u) + P_K[w - m(u)] \quad (w, u \in H). \tag{2.10}$$

We remark that if the mapping $m(u)$ is a Lipschitz continuous with constant $\nu_1 > 0$, then, from (2.9) and (2.10), we have

$$\|P_{m(u)K}w - P_{m(v)+K}w\| = \|m(u) - m(v) + P_K[w - m(u)] - P_K[w - m(v)]\|$$
$$\leq 2\|m(u) - m(v)\| \leq 2\nu_1\|u - v\|.$$

This shows that the projection operator $P_{m(u)+K}$ is Lipschitz continuous with constant $2\nu_1 > 0$. and satisfies Assumption 2.2 with $\nu = 2\nu_1$.

Definition 2.3. An operator $T : H \to H$ is called μ-Lipschitzian if, there exists a constant $\mu > 0$, such that

$$\|Tx - Ty\| \leq \mu\|x - y\| \quad (x, y \in H).$$

Definition 2.4. An operator $T : H \to H$ is called α-inverse strongly monotone (or co-coercive)if, there exists a constant $\alpha > 0$, such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha\|Tx - Ty\|^2 \quad (x, y \in H).$$

Definition 2.5. An operator $T : H \to H$ is called r-strongly monotone if, there exists a constant $r > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq r\|x - y\|^2 \quad (x, y \in H).$$

Definition 2.6. An operator $T : H \to H$ is called relaxed (γ, r)-cocoercive if, there exists constants $\gamma > 0, r > 0$, such that

$$\langle Tx - Ty, x - y \rangle \geq -\gamma ||Tx - Ty||^2 + r||x - y||^2 \quad (x, y \in H).$$

Remark 2.7. Clearly a r-strongly monotone operator or a γ-inverse strongly monotone operator must be a relaxed (γ, r)-cocoercive operator, but the converse is not true. Therefore the class of the relaxed (γ, r)-cocoercive operators is the most general class, and hence Definition 2.6 includes both Definitions 2.4 and 2.5 as special cases.

Remark 2.8. From Definition 2.4, it follows that if T is α-inverse strongly monotone (or co-coercive), than T is also Lipschitz continuous with constant $\frac{1}{\alpha}$.

3. Main Results

In this Section, we show study the existence of a solution of the quasi variational inequalities (2.1) under some mild conditions. For this purpose, we first of all prove that the quasi variational inequalities are equivalent to the implicit fixed point problem using Lemma 2.1. This result is due to Noor [13].

Lemma 3.1. *The function $u \in K(u)$ is a solution of the quasi variational inequality (2.1) if and only if $u \in K(u)$ satisfies the relation*

$$u = P_{K(u)}[u - \rho Tu],$$

where $\rho > 0$ is a constant.

Lemma 3.1 implies that quasi variational inequalities and the fixed point problems are equivalent. This alternative equivalent formulation has played a significant role in the studies of the quasi variational inequalities and related optimization problems. We here use this equivalent formulation to prove the existence of a solution of the quasi variational inequalities (2.1), which is the main motivation of this paper.

Theorem 3.2. *Let $K(u)$ be a closed convex-valued subset of a Hilbert space H. Let T be a relaxed (γ, r)-cocoercive and μ-Lipschitzian mapping of $K(u)$ into H. If Assumption (2.2) holds and*

$$\left| \rho - \frac{r - \gamma\mu^2}{\mu^2} \right| < \frac{\sqrt{(r - \gamma\mu^2)^2 - \mu^2(2\nu - \nu^2)}}{\mu^2} \tag{3.1}$$

$$r > \gamma\mu^2 + \mu\sqrt{\nu(2 - \nu)}, \quad \nu \in (0, 1), \tag{3.2}$$

then there exists a solution $u \in K(u)$ satisfying the quasi variational inequality (2.1).

Proof. Let $u \in K(u)$ be a solution of the quasi variational inequality (2.1). Then, from Lemma 3.1, we have

$$F(u) = P_{K(u)}[u - \rho Tu]. \tag{3.3}$$

Thus it is enough to show that the mapping $F(u)$ defined by (3.3) has a fixed point. For all $u \neq v \in K(u)$, we consider

$$
\begin{aligned}
\|F(u) - F(v)\| &= \|P_{K(u)}[u - \rho Tu] - P_{K(v)}[v - \rho Tv]\| \\
&\leq \|P_{K(u)}[u - \rho Tu] - P_{K(v)}[u - \rho Tu]\| \\
&\quad + \|P_{K(v)}[u - \rho Tu] - P_{K(v)}[v - \rho Tv]\| \\
&\leq \nu\|u - v\| + \|u - v - \rho(Tu - Tv)\|,
\end{aligned}
\tag{3.4}
$$

where we have used Assumption 2.2.

Now using the relaxed (γ, r)-cocercivity and μ-Lipschitz continuity of the operator T, we have

$$
\begin{aligned}
\|u - v - \rho(Tu - Tv)\|^2 &= \|u - v\|^2 - 2\rho\langle Tu - Tv, u - v\rangle + \rho^2\|Tu - Tv\|^2 \\
&\leq \|u - v\|^2 - 2\rho[-\gamma\|Tu - Tv\|^2 + r\|u - v\|^2] \\
&\quad + \rho^2\|Tu - Tv\|^2 \\
&\leq \|u - v\|^2 + 2\rho\gamma\mu^2\|v - u\|^2 - 2\rho r\|u - v\|^2 \\
&\quad + \rho^2\mu^2\|u - v\|^2 \\
&= \left[1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2\right]\|u - v\|^2.
\end{aligned}
\tag{3.5}
$$

From (3.4) and (3.5), we have

$$
\begin{aligned}
\|F(u) - F(v)\| &\leq \left\{\nu + \sqrt{1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2}\right\}\|u - v\| \\
&= \theta\|u - v\|,
\end{aligned}
\tag{3.6}
$$

where

$$
\theta = \nu + \sqrt{1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2}.
\tag{3.7}
$$

From (3.1) and (3.2), it follows that $\theta < 1$. This shows that the mapping $F(u)$ defined by (3.3) is a contraction mapping and consequently has a fixed point $u \in K(u)$ satisfying the quasi variational inequality (2.1). $\qquad\square$

If the convex-value set $K(u)$ is independent of the solution u, that is, $K(u) = K$, a closed convex set in H, then Theorem 3.2 reduces to the following uniqueness and existence result for the classical variational inequalities (2.8) and appears to a be new one.

Theorem 3.3. *Let K be a closed convex subset of a Hilbert space H. Let T be a relaxed (γ, r)-cocoercive and μ-Lipschitzian mapping of $K(u)$ into H. If*

$$
\rho < 2\frac{r - \gamma\mu^2}{\mu^2}, \quad r > \gamma\mu^2,
$$

then there exists a unique solution $u \in K$ satisfying the variational inequality (2.8).

Proof. **Uniqueness.**   Let $u_1 \neq u_2 \in K$ be two solutions of the variational inequality (2.8). Then

$$
\langle Tu_1, v - u_1\rangle \geq 0 \quad (v \in K)
\tag{3.8}
$$

$$\langle Tu_2, v - u_2 \rangle \;\geq\; 0 \quad (v \in K). \tag{3.9}$$

Taking $v = u_2$ in (3.8) and $v = u_1$ in (3.9) and adding the resultant, we have

$$\langle Tu_1 - Tu_2, u_1 - u_2 \rangle \leq 0. \tag{3.10}$$

Using the relaxed $\gamma, r)$, of the operator T, and from (3.10), we have

$$(r - \gamma\mu^2)\|u_1 - u_2\|^2 \leq \langle Tu_1 - Tu_2, u_1, u_2 \rangle \leq 0,$$

from which it follows that, since $r > \gamma\mu^2$,

$$\|Tu_1 - Tu_2\|^2 \leq 0.$$

This is impossible. Thus, we have $u_1 = u_2$, the uniqueness of the solution.

Existence. Its proof follows from Theorem 3.2. $\square$

We now consider the existence of a solution of the quasi variational inequalities under the inverse strongly monotone operator T and this is the main motivation of our next result.

Theorem 3.4. *Let $K(u)$ be closed convex-valued set in H. Let the operator T be inverse strongly monotone with constant $\alpha > 0$. If Assumption 2.2 holds and*

$$|\rho - \alpha| < \alpha\sqrt{1 - \nu(2 - \nu)} \quad (\nu \in (0,1)), \tag{3.11}$$

then there exists a solution $u \in K(u)$ satisfying the quasi variational inequality (2.1).

Proof. Let $u \in K(u)$ be a solution of (2.1). Then, using Lemma 3.1, (2.1) is equivalent to solving the fixed point problem (3.3). Then, for $u \in K(u)$, and using the Assumption 2.2, we have (3.4).

Using the inverse strongly monotonicity of the operator T and from Remark 2.2, we have

$$
\begin{aligned}
\|u - v\rho(Tu - Tv)\|^2 &= \|u - v\|^2 + \rho^2\|Tu - Tv\|^2 - 2\rho\langle Tu - Tv, u - v\rangle \\
&\leq \|u - v\|^2 + \rho^2\|Tu - Tv\|^2 - 2\rho\alpha\|Tu - Tv\|^2 \\
&= \|u - v\|^2 + (\rho^2 - 2\rho\alpha)\|Tu - Tv\|^2 \\
&\leq \|u - v\|^2 + \left(\frac{\rho^2 - 2\rho\alpha}{\alpha^2}\right)\|u - v\|^2 \\
&= \left(1 + \frac{\rho^2 - 2\rho\alpha}{\alpha^2}\right)\|u - v\|^2.
\end{aligned}
\tag{3.12}
$$

From (3.4) and (3.12), we have

$$
\begin{aligned}
\|F(u) - F(v)\| &\leq \left\{\nu + \sqrt{1 + \frac{\rho^2 - 2\rho\alpha}{\alpha^2}}\right\}\|u - v\| \\
&= \theta_1\|u - v\|,
\end{aligned}
$$

where

$$\theta_1 = \nu + \sqrt{1 + \frac{\rho^2 - 2\rho\alpha}{\alpha^2}}.$$

From (3.11), it follows that $\theta_1 < 1$. Thus the mapping $F(u)$ defined by (3.3) is a contraction mapping and consequently has fixed-point $u \in K(u)$ satisfying the quasi variational inequality (2.1). $\square$

If the closed convex-valued set $K(u)$ is independent of the solution u, that is, $K(u) = K$, then Theorem 3.4 collapses to the following result for the classical variational inequality (2.8) and appears to be a new one.

Theorem 3.5. *Let K be a closed convex set in H and let the operator T be inverse strongly monotone with constant $\alpha > 0$. If $\rho \subset (0, 2\alpha)$, then there exists a unique solution $u \in K$ satisfying the variational inequality (2.8).*

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