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POSITIONS IN ℓ_1

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Abstract. We treat several questions related to the positions of subspaces of ℓ_1. Among them, we show that all quotients ℓ_1/ℓ_1 have the Schur property and that a nontrivial twisted sum of ℓ_1 and c_0 cannot be isomorphic to the direct product $\ell_1 \oplus c_0$.

1. Introduction

This paper can be considered a spin-off of [11] motivated by a question of Pełczyński: Inside ℓ_1 there are complemented copies of ℓ_1 that yield quotients isomorphic to ℓ_1. On the other hand, after Bourgain's paper [3], we know that ℓ_1 contains uncomplemented copies of itself and that the quotient space $\mathcal{B}^* = \ell_1/\ell_1$ thus obtained is not even an $\mathcal{L}_1$-space. How many different quotients ℓ_1/ℓ_1 are there?

Let us put the question in proper context. Let Y and X be Banach spaces. An embedding $i : Y \to X$ is an into isomorphism with infinite codimensional range; i.e., $X/i(Y)$ is infinite dimensional. A *position* of Y in X comes defined by a given embedding $i : Y \to X$.

Definition 1. Two positions $a : Y \to X$ and $b : Y \to X$ are said to be *equivalent* if there exists an automorphism $\sigma : X \to X$ such that $\sigma a = b$.

This definition has an homological root. Recall that an *exact sequence*

$$0 \longrightarrow Y \xrightarrow{\ j\ } X \xrightarrow{\ q\ } Z \longrightarrow 0 \tag{1.1}$$

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is a diagram formed by Banach spaces and linear continuous operators in which the kernel of each arrow coincides with the image of the preceding. Which, thanks to the open mapping theorem, is a rather visual form of saying that Y is isomorphic to a subspace of X so that the corresponding quotient $X/j(Y)$ is isomorphic to Z. The exact sequence above is said to split, or trivial, if $j(Y)$ is complemented in X; and it is said to be non-trivial if it does not split. The middle space X is usually called a twisted sum of Y and Z. When the sequence splits then X is isomorphic to the product space $Y \oplus Z$ something we write as $X \simeq Y \oplus Z$. When the sequence is nontrivial the twisted sum space is usually called $Y \oplus_\Omega Z$, to indicate that the direct sum space is "twisted" in some way Ω. We use the notation $\mathrm{Ext}(Z, Y) = 0$ to mean that every exact sequence $0 \to Y \to X \to Z \to 0$ splits.

Equivalent embeddings corresponds to the existence of isomorphisms σ, γ making commutative the diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "a"] \arrow[d, equal] & X \arrow[r] \arrow[d, "\sigma"] & X/a(Y) \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & Y \arrow[r, "b"'] & X \arrow[r] & X/b(Y) \arrow[r] & 0. \end{tikzcd} \tag{1.2}$$

The definition of equivalent positions corresponds to Kalton's notion of "strongly equivalent" embeddings [15], and is consistent with Moreno's notion of automorphy index introduced in [20] as an attempt to quantify the problem of how many equivalent positions Y admits in X.

Definition 2. Two positions a, b of Y into X are said to be *isomorphic* if there is an automorphism σ in X such that $\sigma(aY) = bY$.

This definition corresponds to Kalton's notion [15] of "equivalent embeddings". Or else, in the language of [10], that the two exact sequences in (1.3) are isomorphically equivalent; i.e., : there exist isomorphisms α, σ, γ making commutative the diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "i"] \arrow[d, "\alpha"'] & X \arrow[r] \arrow[d, "\sigma"'] & X/iY \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & Y \arrow[r, "j"'] & X \arrow[r] & X/jY \arrow[r] & 0. \end{tikzcd} \tag{1.3}$$

It is thus clear that equivalent positions are isomorphic, although isomorphic positions can be non-equivalent:

An example of two isomorphically equivalent but not equivalent sequences. Consider $\imath : \ell_1 \to \ell_1$ the identity and $u : \ell_1 \to \ell_1$ an embedding providing an uncomplemented copy of ℓ_1 in ℓ_1. The exact sequences

$$0 \longrightarrow \ell_1 \oplus \ell_1 \xrightarrow{\imath,u} \ell_1 \oplus \ell_1 \longrightarrow Z \longrightarrow 0$$

and

$$0 \longrightarrow \ell_1 \oplus \ell_1 \xrightarrow{u,\imath} \ell_1 \oplus \ell_1 \longrightarrow Z \longrightarrow 0$$

are isomorphically equivalent: the isomorphism $\sigma(x,y)=(y,x)$ makes the diagram

$$\begin{tikzcd} 0 \arrow[r] & \ell_1\oplus\ell_1 \arrow[r, "\imath,u"] \arrow[d, "\sigma"'] & \ell_1\oplus\ell_1 \arrow[r] \arrow[d, "\sigma"] & Z \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \ell_1\oplus\ell_1 \arrow[r, "u,\imath"] & \ell_1\oplus\ell_1 \arrow[r] & Z \arrow[r] & 0 \end{tikzcd}$$

commutative. The sequences are not equivalent since no automorphism $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ of $\ell_1\oplus\ell_1$ can verify $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}(\imath,u)=(u,\imath)$, since this means

$$\begin{aligned}\alpha\imath x+\beta uy&=ux\\ \gamma\imath x+\delta uy&=\imath y.\end{aligned}$$

Taking $x=0$ one gets $\delta u=\imath$, so there is a projection through u, against the hypothesis.

2. Preliminaries

We will need a few facts concerning the behavior of exact sequences and commutative diagrams. The first of them is that given an exact sequence $0\to Y\to X\to Z\to 0$ and an operator $t:Y\to B$ there is a commutative diagram (usually called a push-out diagram)

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, "t"'] & X \arrow[r] \arrow[d, "t'"] & Z \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & PO \arrow[r] & Z \arrow[r] & 0. \end{tikzcd}$$

In a push-out diagram, the lower exact sequence splits if and only if t can be extended to an operator $X\to B$; that is, there is an operator $T:X\to B$ so that $T_{|Y}=t$ [9]. Since every separable Banach space Z admits a quotient map $q:\ell_1\to Z$, the standard lifting properties of ℓ_1 yield that every exact sequence $0\to Y\to X\to Z\to 0$ is part of a push-out diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker q \arrow[r] \arrow[d, "t"'] & \ell_1 \arrow[r] \arrow[d, "t'"] & Z \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r] & X \arrow[r] & Z \arrow[r] & 0. \end{tikzcd}$$

and thus the exact sequence $0\to Y\to X\to Z\to 0$ splits if and only if t can be extended to an operator $\ell_1\to Y$. The second fact we will need is the diagonal principle obtained in [10]

Proposition 2.1. *Let $\imath:Y\to X$ and $j:Y\to X'$ be into isomorphisms between Banach spaces. If there exist operators $I:X'\to X$ and $J:X\to X'$ such that $Ij=\imath$ and $J\imath=j$ then the exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & X\oplus X' \arrow[r] & (X/Y)\oplus X' \arrow[r] & 0 \\ 0 \arrow[r] & Y \arrow[r] & X'\oplus X \arrow[r] & (X'/Y)\oplus X \arrow[r] & 0 \end{tikzcd}$$

are isomorphically equivalent.

See [10] for a homological proof and [2] for a proof in the classical language.

3. Pełczyński's problem about positions in ℓ_1

Given two spaces Y, X it is apparently unknown whether the fact that all positions of Y into X are isomorphic implies that all of them are equivalent. It is also clear that a complemented and an uncomplemented copy of Y inside X are in non-isomorphic positions. This –in combination with Bourgain's construction of an uncomplemented copy of ℓ_1 inside ℓ_1– give form to what is probably the most interesting open problem, formulated by Pełczyński [21]: Are there other "different" copies of ℓ_1 inside ℓ_1? Which can thus be transformed in: How many non-isomorphic positions does ℓ_1 admit in ℓ_1? Recall that in [11, Cor. 3.16] it was proved that there is a continuum of non-equivalent positions of ℓ_1 inside ℓ_1. One has:

Lemma 3.1. *Two isomorphic subspaces A, B of ℓ_1 are in isomorphic positions if and only if ℓ_1/A and ℓ_1/B are isomorphic*

Proof. Let $i : Y \to \ell_1$ be a position of Y into ℓ_1 and let $j : Y \to \ell_1$ be another. It is clear that if they are isomorphic then ℓ_1/iY is isomorphic to ℓ_1/jY. Conversely: As it is well known, every separable Banach space X admits a quotient map $q : \ell_1 \to X$, and its kernel $\ker q$ is uniquely defined up to isomorphisms [19, Thm. 2.f.8]. Moreover, it can be proved that all the kernels of all quotients $\ell_1 \to X$ are in isomorphic positions [8]. This concludes the proof. □

Thus, Pełczyński's question can be reformulated as:

Problem 1. How many non-isomorphic quotients ℓ_1/ℓ_1 do there exist?

Since the kernel $\ker q$ of a quotient map $q : \ell_1 \to X$ is uniquely defined, let us therefore call it $K(X)$. This space $K(X)$ can however admit different positions in ℓ_1. Actually, it is tempting to conjecture that no infinite dimensional closed subspace of ℓ_1 admits a unique isomorphic position. If one, additionally fixes some condition on the quotient, there is a nice result of Lindenstrauss [18], see also [10], asserting:

Proposition 3.2. *Let X, Y be two separable $\mathcal{L}_1$-spaces. Then $K(X) \simeq K(Y)$ if and only if $X \simeq Y$.*

Since there are [18, 12] actually $2^{\aleph_0}$ non-isomorphic separable $\mathcal{L}_1$ spaces, there are $2^{\aleph_0}$ non-isomorphic subspaces of ℓ_1 in a "restrictedly unique" (whatever this could mean) isomorphic position. But, as Bourgain's example shows, ℓ_1 admits two non-isomorphic positions; in one of them the quotient space is ℓ_1 and in the other it cannot be even an $\mathcal{L}_1$-space.

Thus, an approach to the problem would be to study which properties of the quotient space could distinguish two copies of ℓ_1 inside ℓ_1. The two currently known examples of quotients ℓ_1/ℓ_1 are both duals of subspaces of c_0, hence they have the Schur property. Actually one has:

Proposition 3.3. *Every quotient ℓ_1/ℓ_1 has the Schur property.*

Proof. This is a combination of two beautiful results of Kalton: The first one is [13, Prop.5.1], asserting that if $\mathrm{Ext}(X, C[0,1]) = 0$ then X has the Schur property. This can be reformulated as: if every operator t in a push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker q & \longrightarrow & \ell_1 & \xrightarrow{q} & X & \longrightarrow & 0 \\ & & {\scriptstyle t}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & C(K) & \longrightarrow & E & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

can be extended to a linear continuous operator $\ell_1 \to C(K)$ then X has the Schur property. The second one is [14, Cor. 10.2] asserting that given an embedding $\ell_1 \to E$ of ℓ_1 into a separable space E, any operator $\ell_1 \to C[0,1]$ can be extended to an operator $E \to C[0,1]$. Consider now an embedding $j : \ell_1 \to \ell_1$ and an exact sequence

$$0 \longrightarrow C[0,1] \longrightarrow X \longrightarrow \ell_1/j\ell_1 \longrightarrow 0.$$

Form the push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_1 & \xrightarrow{j} & \ell_1 & \longrightarrow & \ell_1/j\ell_1 & \longrightarrow & 0 \\ & & {\scriptstyle t}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & C[0,1] & \longrightarrow & X & \longrightarrow & \ell_1/j\ell_1 & \longrightarrow & 0 \end{array}$$

and observe that t admits an extension $\ell_1 \to C[0,1]$; which immediately implies that the lower sequence splits and thus $\ell_1/j\ell_1$ has the Schur property. $\square$

A similar argument yields a partial solution to [10, Problem B and Conjecture C]: Does $C[0,1]/\ell_1$ have the Dunford-Pettis property? Recall that a Banach space X is said to have the Dunford-Pettis property (in short, DPP) if weakly compact operators $X \to c_0$ are completely continuous. Two simple facts about the DPP that we need to know are that complemented subspaces and products of spaces with DPP have DPP and that spaces with the Schur property have DPP. The simplest examples of spaces having DPP are the $\mathcal{L}_1$ and $\mathcal{L}_\infty$ spaces. Observe that $C[0,1]/\ell_1$ is not a space uniquely defined since it depends on the position of ℓ_1 inside $C[0,1]$. One however has:

Proposition 3.4. *All quotients $C[0,1]/\ell_1$ have or fail the DPP simultaneously.*

Proof. Indeed, consider two positions i, j of ℓ_1 inside $C[0,1]$ and look at the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_1 & \xrightarrow{i} & C[0,1] & \longrightarrow & C[0,1]/i\ell_1 & \longrightarrow & 0 \\ & & \| & & & & & & \\ 0 & \longrightarrow & \ell_1 & \xrightarrow[j]{} & C[0,1] & \longrightarrow & C[0,1]/j\ell_1 & \longrightarrow & 0, \end{array}$$

By the above mentioned result in [14], j admits an extension to an operator $\phi_j : C[0,1] \to C[0,1]$ through i, and i has an extension $\phi_i : C[0,1] \to C[0,1]$

through j. In that situation, the diagonal principle [10, Thm.2] applies to get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ell_1 & \xrightarrow{i} & C[0,1]\oplus C[0,1] & \longrightarrow & C[0,1]/i\ell_1\oplus C[0,1] & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & \ell_1 & \xrightarrow[j]{} & C[0,1]\oplus C[0,1] & \longrightarrow & C[0,1]/j\ell_1\oplus C[0,1] & \longrightarrow & 0,
\end{array}$$

in which both β, γ are isomorphisms. So $C[0,1]/i\ell_1 \oplus C[0,1] \simeq C[0,1]/j\ell_1 \oplus C[0,1]$ and since the DPP is stable by products and complemented subspaces the conclusion is clear. $\square$

This completes the results in [10]:

Corollary 3.5. *$C[0,1]/\ell_1$ has DPP if and only if ℓ_∞/ℓ_1 has DPP.*

Proof. Indeed, from [10, Prop. 4.3 (2)] it follows that if ℓ_∞/ℓ_1 has DPP then also $C[0,1]/\ell_1$ has DPP (regardless of the embedding $\ell_1 \to C[0,1]$). For the converse implication apply (3) $\Rightarrow$ (1) in [10, Prop. 4.4] with the choice $X = c_0$. $\square$

Which is interesting since the space ℓ_∞/ℓ_1 is uniquely defined [19, Thm.2.f.12]. Let us moreover observe that the role of $C[0,1]$ in the results above cannot be played by an arbitrary $\mathcal{L}_\infty$-space since not every operator $\ell_1 \to \mathcal{L}_\infty$ extends to separable superspaces: Following [4], there is an embedding $j : \ell_1 \to \mathcal{L}_\infty(\ell_1)$ of ℓ_1 into an $\mathcal{L}_\infty$-space having the Schur property. Consider any embedding $i : \ell_1 \to C[0,1]$ and form the diagram

$$\begin{array}{ccc}
\ell_1 & \xrightarrow{i} & C[0,1]\\
\| & & \\
\ell_1 & \xrightarrow[j]{} & \mathcal{L}_\infty(\ell_1).
\end{array}$$

The operator j cannot be extended through i since operators $C[0,1] \to \mathcal{L}_\infty(\ell_1)$ must be compact.

4. Subspaces of ℓ_1 in different non-isomorphic positions

In general, it is clear that for any Banach space X not containing ℓ_1 complemented, $K(X)$ admits at least two non isomorphic positions: one gives X as quotient and the other $X \oplus \ell_1$. A specially interesting case is that of $X = c_0$. In this case a position of $K(c_0)$ yields as quotient c_0, and other yields $\ell_1 \oplus c_0$. We will show below that $K(c_0) \simeq K(\ell_1 \oplus_\Omega c_0)$ for all twisted sum spaces $\ell_1 \oplus_\Omega c_0$. Moreover, while no current method is nowadays able to decide when two twisted sum spaces $Y \oplus_\Omega Z$ and $Y \oplus_\omega Z$ are isomorphic, we are able to show:

Proposition 4.1. *A nontrivial twisted sum $\ell_1 \oplus_\Omega c_0$ cannot be isomorphic to $A_1 \oplus A_0$ with $A_1 \simeq \ell_1$ and $A_0 \simeq c_0$.*

Proof. Otherwise, the exact sequence $0 \to \ell_1 \to \ell_1 \oplus_\Omega c_0 \to c_0 \to 0$ adopts the form $0 \to \ell_1 \to A_1 \oplus A_0 \to c_0 \to 0$. The following claim implies that the sequence splits:

Claim *A subspace Y of $\ell_1 \oplus c_0$ isomorphic to ℓ_1 such that $(\ell_1 \oplus c_0)/Y$ is isomorphic to c_0 is complemented.*

Proof of the Claim. Let p be the canonical projection of $\ell_1 \oplus c_0$ onto ℓ_1. Since Y and c_0 are incomparable, any common subspace is finite dimensional and thus we can assume it is 0. So the restriction of p to Y is an into isomorphism ϕ. We therefore get the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & \ell_1 \oplus c_0 & \longrightarrow & c_0 & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \\
0 & \longrightarrow & Y & \xrightarrow[\phi]{} & \ell_1 & \xrightarrow[Q]{} & H & \longrightarrow & 0
\end{array}$$

where q is surjective, so H is isomorphic to a subspace of c_0. Since Y is isomorphic to ℓ_1, we will conclude observing that a sequence $0 \to \ell_1 \to \ell_1 \to H \to 0$ cannot exist when H is an infinite dimensional subspace of c_0, which is clear after Proposition 3.3. Thus, it only remains the possibility that H is finite dimensional; in which case $\phi(Y)$ is complemented in ℓ_1 by a projection P, what makes Y complemented in $\ell_1 \oplus c_0$ by $\phi^{-1}Pp$. This concludes the proof of the claim, and that of the theorem. □

We now show that $K(c_0) \simeq K(\ell_1 \oplus c_0) \simeq K(\ell_1 \oplus_\Omega c_0)$ for every twisted sum of ℓ_1 and c_0: indeed, let $0 \to \ell_1 \to \ell_1 \oplus_\Omega c_0 \xrightarrow{\pi} c_0 \to 0$ be an exact sequence with quotient map π, and let $q : \ell_1 \to c_0$ be a quotient map. Observe the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ell_1 & \longrightarrow & \ell_1 \oplus \ell_1 & \longrightarrow & \ell_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id \oplus Q} & & \downarrow{\scriptstyle q} & & \\
0 & \longrightarrow & \ell_1 & \longrightarrow & \ell_1 \oplus_\Omega c_0 & \xrightarrow{\pi} & c_0 & \longrightarrow & 0
\end{array}$$

in which Q is a lifting of q through π (i.e., $\pi Q = q$) and $(id \oplus Q)(x, y) = x + Qy$. The standard use of the snake's lemma [8] allows one to complete the diagram with a new exact sequence formed by the kernels of the vertical maps. This yields the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & K(\ell_1 \oplus_\Omega c_0) \arrow[r] \arrow[d] & K(c_0) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ell_1 \arrow[r] \arrow[d, "id"] & \ell_1 \oplus \ell_1 \arrow[r] \arrow[d, "id \oplus Q"] & \ell_1 \arrow[r] \arrow[d, "q"] & 0 \\
0 \arrow[r] & \ell_1 \arrow[r] \arrow[d] & \ell_1 \oplus_\Omega c_0 \arrow[r, "p"] \arrow[d] & c_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

Therefore the arrow $K(\ell_1 \oplus_\Omega c_0) \to K(c_0)$ is an isomorphism. This shows that the subspace $K(c_0) \simeq K(\ell_1 \oplus c_0) \simeq K(\ell_1 \oplus_\Omega c_0)$ admits three non-isomorphic positions in ℓ_1.

A different type of non-isomorphic position can be obtained as follows: Bourgain's construction [3] of an uncomplemented copy of ℓ_1 inside ℓ_1 has a local nature and what in fact provides is an uncomplemented subspace $\mathcal{B}$ of c_0 so that $c_0/\mathcal{B} = c_0$, which by duality provides the nontrivial exact sequence $0 \to \ell_1 \to \ell_1 \to \mathcal{B}^* \to 0$. Observe that $\mathcal{B}^*$ cannot be an $\mathcal{L}_1$ space, so $\mathcal{B}$ cannot be an $\mathcal{L}_\infty$ space. The commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K(c_0) \arrow[r, equal] \arrow[d] & K(c_0) \arrow[d] & & \\
0 \arrow[r] & \ell_1 \arrow[r] \arrow[d] & \ell_1 \arrow[r] \arrow[d] & \mathcal{B}^* \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & c_0 \arrow[r] \arrow[d] & c_0 \oplus \mathcal{B}^* \arrow[r] \arrow[d] & \mathcal{B}^* \arrow[r] & 0 \\
 & 0 & 0, & &
\end{tikzcd}$$

in which the lower sequence splits by Sobczyk's theorem, shows that $K(c_0) \simeq K(c_0 \oplus \mathcal{B}^*)$. The space $\mathcal{B}^*$ is a dual of a subspace of c_0, and thus has the Schur property; hence $c_0 \oplus \mathcal{B}^*$ cannot be c_0 since it does not contain Schur subspaces. Now, $c_0 \oplus \mathcal{B}^*$ cannot be either $c_0 \oplus \ell_1$: Otherwise, $\mathcal{B}^*$ would be a complemented subspace of $c_0 \oplus \ell_1$; since these two spaces are incomparable, by [19], $\mathcal{B}^* = A \oplus B$, where A is a complemented subspace of c_0 and B a complemented subspace of

ℓ_1; i.e., ℓ_1. Since $\mathcal{B}^*$ is Schur, A must be finite dimensional, so $\mathcal{B}^* = \ell_1$, which is impossible since $\mathcal{B}^*$ is not an $\mathcal{L}_1$-space.

Let us show that $c_0 \oplus \mathcal{B}^*$ cannot be any other twisted sum space $\ell_1 \oplus_\Omega c_0$ either. Assume the contrary: that $c_0 \oplus \mathcal{B} = \ell_1 \oplus_\Omega c_0$. Let p be the canonical projection of $c_0 \oplus \mathcal{B}^*$ onto $\mathcal{B}^*$. Since ℓ_1 and c_0 are incomparable, we can assume again that their intersection is 0. So the restriction of p to ℓ_1 is an isomorphism. We therefore get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_1 & \longrightarrow & \ell_1 \oplus_\Omega c_0 & \longrightarrow & c_0 & \longrightarrow & 0 \\ & & \| & & {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} & & \\ 0 & \longrightarrow & Y & \underset{\phi}{\longrightarrow} & \mathcal{B}^* & \underset{Q}{\longrightarrow} & H & \longrightarrow & 0 \end{array}$$

where q is surjective, so H is isomorphic to a subspace of c_0 by [1]. We will show below that a sequence $0 \to \ell_1 \to \mathcal{B}^* \to H \to 0$ cannot exist unless H is finite dimensional. In that case, $\mathcal{B}^* \simeq \ell_1$, which is impossible. To show that a sequence $0 \to \ell_1 \to \mathcal{B}^* \to H \to 0$ cannot exist, let us recall that Kalton and Pełczyński considered in [16] those Banach spaces X for which $\mathrm{Ext}(X, \ell_2) = 0$; and, accordingly, let us say that a Banach space X is a Kalton-Pełczyński space (in short, KP) if $\mathrm{Ext}(X, \ell_2) = 0$. The most prominent examples of KP-spaces are the $\mathcal{L}_1$-spaces. One can show:

Lemma 4.2. *The quotient of a KP space by an $\mathcal{L}_1$-space is a KP space.*

Proof. Let X be a KP-space and let $0 \to \mathcal{L}_1 \to X \xrightarrow{q} Q \to 0$ be an exact sequence. An application of [10] yields that given an exact sequence $0 \to \ell_2 \to E \to Q \to 0$ there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}_1 & \longrightarrow & X & \longrightarrow & Q & \longrightarrow & 0 \\ & & {\scriptstyle \phi}\big\downarrow & & \big\downarrow & & \| & & \\ 0 & \longrightarrow & \ell_2 & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 0. \end{array}$$

Since every operator $\mathcal{L}_1 \to \ell_2$ is 2-summing, it can be extended anywhere, which means that the lower sequence splits. □

We conclude with the proof that no exact sequence $0 \to \ell_1 \to \mathcal{B}^* \to H \to 0$ exists: In such a sequence the space H must be a KP-space. But, as we show now, $\mathrm{Ext}(H, \ell_2) \neq 0$ for every infinite dimensional subspace H of c_0: As it is well-known (see [6, 4.2]) there exist nontrivial exact sequences $0 \to \ell_2 \to E \to c_0 \to 0$. Since H must contain c_0, and complemented, we can assume $H = c_0 \oplus H$, thus $0 \to \ell_2 \to E \oplus H \to H \to 0$ is the desired nontrivial exact sequence. We conjecture that there is an uncountable quantity of non-isomorphic twisted sums $\ell_1 \oplus_\Omega c_0$, which would immediately imply that $K(c_0)$ can be placed in uncountably many non-isomorphic positions in ℓ_1. The problem here is that no current method is known to obtain a twisted sum $\ell_1 \oplus_\Omega c_0$, apart from its existence. Indeed, those obtained in [5, 6, 7] are actually non-constructive. Moreover, as we have already said, no current method is known to decide when two twisted sum spaces are isomorphic.

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