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TRUNCATION METHOD FOR RANDOM BOUNDED SELF-ADJOINT OPERATORS

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Abstract. This article addresses the following question; 'how to approximate the spectrum of random bounded self-adjoint operators on separable Hilbert spaces'. This is an attempt to establish a link between the spectral theory of random operators and the rich theory of random matrices; including various notions of convergence. This study tries to develop a random version of the truncation method, which is useful in approximating spectrum of bounded self-adjoint operators. It is proved that the eigenvalue sequences of the truncations converge in distribution to the eigenvalues of the random bounded self-adjoint operator. The convergence of moments are also proved with some examples. In addition, the article discusses some new methods to predict the existence of spectral gaps between the bounds of essential spectrum. Some important open problems are also stated at the end.

1. Introduction and preliminary results

Let A be a bounded self-adjoint operator defined on a complex separable Hilbert space $\mathbb{H}$. Consider the orthogonal projection P_n of $\mathbb{H}$ onto the finite dimensional subspace spanned by the first n elements of the orthonormal basis $\{e_1, e_2, \ldots\}$. The truncation $A_n = P_n A P_n$ of A can be treated as a finite matrix by restricting the domain to the image of P_n. It is known that the eigenvalue sequence of A_n are useful in approximating the spectrum $\sigma(A)$ and the essential spectrum $\sigma_e(A)$ (see [1, 3] and references therein and [11, 14] for new perspectives).

Here, in the place of the single operator A, a one parameter family of operators $A(\omega)$ is considered, where the parameter ω varies in some suitable probability

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space. Such operators arise naturally in many practical problems and the knowledge of their spectrum is very important. Observe that the spectrum and the essential spectrum of $A(\omega)$ depend on the parameter ω. Here we approximate the set valued functions $\omega \mapsto \sigma(A(\omega))$ and $\omega \mapsto \sigma_e(A(\omega))$, using the sequence of functions $\omega \mapsto \sigma(A(\omega)_n)$.

The truncations $A(\omega)_n$ are finite matrices and depending on the parameter ω. The eigenvalue distribution of such random matrices, with the matrix entries following certain distributions has a rich theory (see [19, 20] and references therein). Physicists have been interested in certain types of $n \times n$ random matrices as n increases to infinity (see [5, 12] and references therein). Also, there are various notions of convergence in this case such as convergence in distribution, convergence in probability, etc. This article is an attempt to use this random matrix theory in the spectral approximation problem of random operators. Also, the various notions of convergence have to be analyzed to obtain information about the spectrum of infinite dimensional operators.

The notion of a random operator is defined below.

Definition 1.1. Let $(\Omega, \mathbb{F}, \alpha)$ be a probability measure space and $\mathbb{H}$ be a complex separable Hilbert space. A random operator is a mapping

$$A(.,.) : \Omega \times \mathbb{H} \to \mathbb{H}$$

such that $A(\omega, .)$ is a linear operator on $\mathbb{H}$ for almost all ω, and the functions $f_{x,y} : \Omega \to \mathbb{C}$, defined by $f_{x,y}(\omega) = \langle A(\omega)x, y\rangle$ are measurable for every $x, y \in \mathbb{H}$.

In this article, a random operator is denoted by $A(\omega)$ and all the operators considered are bounded.

1.1. **The Main Results.** As it is mentioned above, the aim of this article is to use the eigenvalue distribution of the random matrices $A(\omega)_n$ to obtain information about the spectrum and essential spectrum of the random operator $A(\omega)$. It is well known that the approximation numbers of an infinite dimensional bounded operator can be approximated by the approximation number sequence of its truncations (Theorem 1.1 in [3], Lemma 2.4 in [14]). The following result is proved in this article; if the approximation number sequences of truncations converge with a uniform rate of convergence, then the expectations of eigenvalues of truncations converge to the expectations of the discrete spectral values of the operator at a uniform rate of convergence. It is also proved that the sequence of random eigenvalues of truncations $A(\omega)_n$, converge in distribution to the discrete eigenvalues of the bounded random self-adjoint operator $A(\omega)$. The convergence of moments leads to an interesting question related to the classical moment problem and Carleman's condition. This open problem is discussed in detail at the end of the article.

This article also addresses the problem of predicting spectral gaps that may exist between the bounds of the essential spectrum of a bounded self-adjoint operator. The random version of these results can be one possible area of research in future. Examples which indicate the instability of spectral gaps under a random perturbation are given here.

The following is a brief account of some results in the linear algebraic techniques used in the spectral approximation problem.

1.2. **Truncation method.** Let A be a bounded self-adjoint operator defined on a complex separable Hilbert space $\mathbb{H}$. The spectrum of A is denoted by $\sigma(A)$ with m, M as its lower and upper bounds. Firstly, let's recall the following notations and definitions:

- $|A| = (A^*A)^{\frac{1}{2}}$
- The essential norm of A, $\|A\|_{ess} = \inf\{\|A-K\|\,; K \text{ compact }\}$.
- The k^{th} approximation number of A,

$$s_k(A) = \inf\{\|A-F\|\,;\ \text{rank } F \leq k-1\}.$$

The following result, taken from [9], is useful in understanding the location of discrete eigenvalues of a bounded self-adjoint operator (for the proof, see pp. 204, 212-214 of [9]).

Theorem 1.2. [9] *The set* $\sigma(|A|) - [0, \|A\|_{ess}]$ *is at most countable,* $\|A\|_{ess}$ *is the only possible accumulation point, and all the points in the set are eigenvalues with finite multiplicity of* $|A|$. *Furthermore if*

$$\lambda_1(|A|) \geq \lambda_2(|A|) \geq \dots \tag{1.2}$$

are those eigenvalues in non increasing order and let $N \in \{1, 2, \dots\} \cup \{\infty\}$ *be the number of terms in* (1.2), *then*

$$s_k\left(A\right) = \begin{cases} \lambda_k\left(|A|\right),\ if\, N = \infty\ \ or\, 1 \leq k \leq N \\ \|A\|_{ess},\ if\, N < \infty\ and\, k \geq N+1 \end{cases} \tag{1.3}$$

By considering the positive operators $A - mI$ and $MI - A$, the above result implies that there exist at most countably many discrete eigenvalues of A, outside the bounds of essential spectrum of A. Also, the possible accumulation points are the upper and lower bounds of the essential spectrum, $\sigma_e(A)$. Let ν, μ be the lower and upper bounds of $\sigma_e(A)$ respectively. Also denote the discrete eigenvalues of A lying above μ by

$$\lambda_1^+(A) \geq \lambda_2^+(A) \dots (\text{ total } R \text{ terms })$$

and the discrete eigenvalues of A lying below ν by

$$\lambda_1^-(A) \leq \lambda_2^-(A) \leq \dots (\text{ total } S \text{ terms }),$$

where $R, S \in \{1, 2, \dots\} \cup \{\infty\}$.

Consider the finite dimensional truncations of A, that is $A_n = P_n A P_n$, where P_n is the projection of $\mathbb{H}$ onto the span of first n elements $\{e_1, e_2, \dots, e_n\}$ of the basis. Denote the eigenvalues of A_n by $\lambda_1(A_n) \geq \lambda_2(A_n) \geq \dots \geq \lambda_n(A_n)$. The following results are proved in [3], with the help of Theorem 1.2 (Corollary 2.2 and Theorem 3.1 of [3]).

Theorem 1.3. [3]

$$\lim_{n\to\infty} \lambda_k\left(|A_n|\right) = \lim_{n\to\infty} s_k\left(A_n\right) = s_k\left(A\right)\ = \begin{cases} \lambda_k\left(|A|\right)\, if\, N = \infty\, or\, 1 \leq k \leq N \\ \|A\|_{ess}\, if\ N < \infty\, and\, k \geq N+1 \end{cases}$$

Theorem 1.4. [3] *For every fixed integer k,*

$$\lim_{n\to\infty}\lambda_k(A_n)=\begin{cases}\lambda_k^+(A), & if\, R=\infty \;\; or\, 1\le k\le R,\\ \mu, & if\, R<\infty\; and\, k\ge R+1,\end{cases}$$

$$\lim_{n\to\infty}\lambda_{n+1-k}(A_n)=\begin{cases}\lambda_k^-(A), & if\, S=\infty \;\; or\, 1\le k\le S,\\ \nu, & if\, S<\infty \;\; and\, k\ge S+1.\end{cases}$$

In particular, $\lim_{k\to\infty}\lim_{n\to\infty}\lambda_k(A_n)=\mu$ *and* $\lim_{k\to\infty}\lim_{n\to\infty}\lambda_{n+1-k}(A_n)=\nu$.

Denote $\Lambda=\{\lambda\in R;\lambda=\lim\lambda_n,\lambda_n\in\sigma(A_n)\}$. The subsequent theorem taken from [3] denies the existence of spurious eigenvalues (points in Λ which are not spectral values) if the essential spectrum is connected.

Theorem 1.5. *If A is a bounded self-adjoint operator and if $\sigma_e(A)$ is connected, then $\sigma(A)=\Lambda$.*

Hence the remaining task is to predict the existence of spectral gaps that may occur in between the bounds of essential spectrum. Attempts have been done in this direction using the truncation method (see [14]). This problem is addressed in section 3.

The structure of this article is as follows. In the next section, results on the random version of the spectral approximation are discussed. There it is proved that the sequence of random eigenvalues of truncations converge in distribution to the discrete eigenvalues of the bounded random self-adjoint operator. Some examples of random operators and their spectral approximation are also given there. In the third section, a new method is proposed to predict the spectral gaps using eigenvalues of truncations. The article ends with a discussion on the main results and the further possibilities.

2. Spectral Approximation - Random Case

Here onwards, we consider the case where $A(\omega)$ is a bounded self-adjoint operator for almost all ω. Results on random version of spectral approximation, in particular Theorem 1.4, are proved with respect to different modes of convergence. That is the convergence of eigenvalue sequence of truncations $A(\omega)_n$ to the discrete eigenvalues and to the upper and lower bounds of essential spectrum of $A(\omega)$ are proved with respect to various notions of convergence such as convergence in distribution, convergence in probability, etc.

Firstly, assume that the moments of all orders exist and are finite for the random variable that maps $\omega\mapsto\|A(\omega)\|$. This assumption is too strong and one can prove the results of this section with some weaker assumptions. However, this assumption will help us to reduce many technical steps involved in the proofs of some results. Recall the notations:

- $\|A(\omega,.)\|_{ess}=inf\{\|A(\omega)-K\|\,;K \text{ compact}\}$.
- $s_k(\omega)=inf\{\|A(\omega)-F\|\,;\text{ rank } F<k\}$.

Since each $s_k(\omega)$ are bounded by $\|A(\omega)\|$, for which the moments exist by our assumption, the moments of all orders exist for $s_k(\omega)$. Now by Theorem 1.3, $s_k(\omega)$ converges to $\|A(\omega)\|_{ess}$ almost everywhere as k tends to infinity. Also $s_k(\omega)$ is a monotone decreasing sequence of nonnegative functions. Hence by Monotone Convergence Theorem, the expectations $E(s_k) - E(\|A(.,.)\|_{ess}) = o(1)$. Similarly, moments of all orders converge.

Next lemma is the random version of the approximation of approximation numbers, proved in [3].

Lemma 2.1. *If $s_{k,n}(\omega) = inf\{\|A(\omega)_n - F\|;\ rank\ F < k\}$, then the expectation of $s_{k,n}$ converges to the expectation of s_k for each positive integer k. That is $E(s_{k,n}) \to E(s_k)$ as $n \to \infty$, for each $k = 1, 2, 3 \ldots$.*

Proof. Observe that, for each k, $s_{k,n}(.)$ converges to $s_k(.)$ almost everywhere as n goes to ∞ by Theorem 1.3. From the interlacing theorem for singular values (refer e.g [2]), for each k, $\{s_{k,n}(.); n \in \mathbb{N}\}$ is a monotone decreasing sequence of nonnegative functions. Hence by Monotone Convergence Theorem,

$$E(s_{k,n}) \to E(s_k) \text{ as } n \text{ goes to } \infty, \text{ for each } k.$$

The existence of the integral is a consequence of our assumption on the moments of $\|A(\omega)\|$. Hence the proof is completed. □

Now let $\nu(\omega), \mu(\omega)$ denote the lower and upper bounds of the essential spectrum $\sigma_e(A(\omega))$ respectively, and also let the numbers

$$\lambda_1^+(A(\omega)) \geq \lambda_2^+(A(\omega)) \geq \ldots (\text{ total } R \text{ terms })$$

be the discrete eigenvalues of $A(\omega)$ lying above $\mu(\omega)$, and

$$\lambda_1^-(A(\omega)) \leq \lambda_2^-(A(\omega)) \leq \ldots (\text{ total } S \text{ terms })$$

be the eigenvalues lying below $\nu(\omega)$. Here R and S can be infinite and they depend on ω. The quantities $\lambda_{1,n}(\omega) \geq \lambda_{2,n}(\omega) \geq \ldots \geq \lambda_{n,n}(\omega)$ denote the eigenvalues of $A(\omega)_n$ in non increasing order. Also, it is possible that the above inequalities may not hold in a set of measure zero.

The following couple of theorems are the main results in this section. In the first one, uniform rate of convergence is assumed for $s_k(A(\omega)_n) - s_k(A(\omega))$, which leads to an estimate of the first moment sequence of eigenvalues of $A(\omega)_n$. Such an estimate may be useful for some special class of operators. The second theorem is of theoretical interest, where the convergence in distribution is obtained for eigenvalues of truncations.

Theorem 2.2. *If $s_k(A(\omega)_n) - s_k(A(\omega)) = O(\theta_n)$, where θ_n goes to zero as n goes to infinity, then*

$$E(\lambda_{k,n}(.)) - E((\lambda_k^+)) = O(\theta_n), \quad \text{if } R = \infty \text{ and}$$
$$E(\lambda_{n+1-k,n}(.)) - E((\lambda_k^-)) = O(\theta_n), \quad \text{if } S = \infty.$$

Proof. First we prove that if $s_k(A_n) - s_k(A) = O(\theta_n)$, where θ_n goes to 0 as n tends to ∞, then

$$\lambda_k(A_n) = \begin{cases} \lambda_k^+(A) + O(\theta_n), & \text{if } R = \infty \text{ or } 1 \leq k \leq R \\ \mu + O(\theta_n), & \text{if } R < \infty \text{ and } k \geq R+1 \end{cases}$$

$$\lambda_{n+1-k}(A_n) = \begin{cases} \lambda_k^-(A) + O(\theta_n), & \text{if } S = \infty \text{ or } 1 \leq k \leq S \\ \nu + O(\theta_n), & \text{if } S < \infty \text{ and } k \geq S+1 \end{cases}$$

where R and S are defined as in Theorem 1.4.

For if N is the number of eigenvalues lying in $\sigma(|A|) - [0, \|A\|_{ess}]$, then from identity (1.3), and the the fact that $s_k(A_n) = \lambda_k(|A_n|)$, we get the following.

$$s_k(A_n) - s_k(A) = \begin{cases} \lambda_k(|A_n|) - \lambda_k(|A|), \text{ if } N = \infty \text{ or } 1 \leq k \leq N \\ \lambda_k(|A_n|) - \|A\|_{ess}, \text{ if } N < \infty \text{ and } k \geq N+1 \end{cases}$$

Since $s_k(A_n) - s_k(A) = O(\theta_n)$,

$$\lambda_k(|A_n|) - \lambda_k(|A|) = O(\theta_n), \quad \text{if } N = \infty \text{ or } 1 \leq k \leq N,$$

$$\lambda_k(|A_n|) - \|A\|_{ess} = O(\theta_n), \quad \text{if } N < \infty \text{ and } k \geq N+1.$$

Applying this to the positive operators $A - mI$, and $MI - A$, with the notations used in Theorem 1.4, we get the following conclusions.

$$\lambda_k(A_n - mI_n) = \begin{cases} \lambda_k(A - mI) + O(\theta_n), \text{ if } R = \infty \text{ or } 1 \leq k \leq R \\ \|A - mI\|_{ess} + O(\theta_n), \text{ if } R < \infty \text{ and } k \geq R+1 \end{cases}$$

and

$$\lambda_k(MI_n - A_n) = \begin{cases} \lambda_k(MI - A) + O(\theta_n), \text{ if } S = \infty \text{ or } 1 \leq k \leq S \\ \|MI - A\|_{ess} + O(\theta_n), \text{ if } S < \infty \text{ and } k \geq S+1 \end{cases}$$

Also by spectral mapping theorem, the upper bound of the essential spectrum of the positive operators $A - mI$ and $MI - A$ are $\mu - m$ and $M - \nu$ respectively. Now the following identities are easy consequences of Theorem 1.2.

$$\|A - mI\|_{ess} = \mu - m, \quad \|MI - A\|_{ess} = M - \nu.$$

$$\lambda_k(A_n - mI_n) = \lambda_k(A_n) - m, \quad \lambda_k(MI_n - A_n) = M - \lambda_{n+1-k}(A_n).$$

$$\lambda_k(A - mI) = \lambda_k^+(A) - m, \quad \lambda_k(MI - A) = M - \lambda_k^-(A).$$

Hence we get the desired conclusions

$$\lambda_k(A_n) = \begin{cases} \lambda_k^+(A) + O(\theta_n), & \text{if } R = \infty \text{ or } 1 \leq k \leq R \\ \mu + O(\theta_n), & \text{if } R < \infty \text{ and } k \geq R+1 \end{cases}$$

$$\lambda_{n+1-k}(A_n) = \begin{cases} \lambda_k^-(A) + O(\theta_n), & \text{if } S = \infty \text{ or } 1 \leq k \leq S \\ \nu + O(\theta_n), & \text{if } S < \infty \text{ and } k \geq S+1 \end{cases}$$

Now by hypothesis, $s_k(A(\omega)_n) - s_k(A(\omega)) = O(\theta_n)$. Therefore

$$\lambda_{n,k}(\omega) = \lambda_k^+(\omega) + O(\theta_n), \text{ if } R = \infty,$$

$$\lambda_{n,n+1-k}(\omega) = \lambda_k^-(\omega) + O(\theta_n), \text{ if } S = \infty.$$

Now observe that the order θ_n is independent of ω. Consequently, we obtain

$$E(\lambda_{k,n}) - E(\lambda_k^+) = E(\lambda_{k,n} - \lambda_k^+) = O(\theta_n), \text{ if } R = \infty,$$

$$E(\lambda_{n,n+1-k}) - E(\lambda_k^-) = E(\lambda_{n,n+1-k} - \lambda_k^-) = O(\theta_n), \text{ if } S = \infty.$$

$\square$

Before proving the next theorem, recall the notion of convergence in distribution. Let Ω be the set of all real numbers.

Definition 2.3. A sequence of random variables X_n converges in distribution to a random variable X if, for every bounded continuous function $F : \mathbb{R} \to \mathbb{R}$ with compact support, one has

$$\lim_{n\to\infty} E(F(X_n)) = E(F(X))$$

Theorem 2.4. *Let $A(\omega)$ be a random bounded self-adjoint operator with infinitely many eigenvalues above and below the bounds of essential spectrum. Then for each k, the sequence of k^{th} eigenvalues $\lambda_{k,n}$ of the truncations $A(\omega)_n$ converges in distribution to the k^{th} eigenvalue ${\lambda_k}^+$ above the upper bound of the essential spectrum of $A(\omega)$. Also for each k, the sequence of $(n+1-k)^{th}$ eigenvalues $\lambda_{n+1-k,n}$ of the truncations $A(\omega)_n$ converges in distribution to the k^{th} eigenvalue ${\lambda_k}^-$ below the lower bound of the essential spectrum of $A(\omega)$.*

Proof. First we claim that the Fourier transform $F_{\lambda_{k,n}}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{it.\lambda_{k,n}(\omega)}d\omega$ converges to $F_{\lambda_k^+}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{it.\lambda_k^+(\omega)}d\omega$ pointwise. For this, note that $\lambda_{k,n}(.)$ converges to $\lambda_k^+(.)$ almost surely and the exponential function is continuous. Therefore,

$$e^{it.\lambda_{k,n}} \text{ converges to } e^{it.\lambda_k^+} \text{ almost surely.}$$

Also, $\left|e^{it.\lambda_{k,n}}\right|$ is dominated by the constant function 1. Hence by Lebesgue dominated convergence theorem,

$$F_{\lambda_{k,n}}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{it.\lambda_{k,n}(\omega)}d\omega \text{ converges to } F_{\lambda_k^+}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{it.\lambda_k^+(\omega)}d\omega \text{ pointwise.}$$

From the Fourier inversion formula:

$$\Phi(\lambda_{k,n})(\omega) = \int_{\mathbb{R}} \hat{\Phi}(t)e^{it.\lambda_{k,n}(\omega)}dt,$$

for every Schwartz function Φ on $\mathbb{R}$. Therefore,

$$\int_{\mathbb{R}} \Phi(\lambda_{k,n}(\omega))d\omega = \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{\Phi}(t)e^{it.\lambda_{k,n}(\omega)}dtd\omega$$

By Fubini-Tonelli theorem, this gives

$$\int_{\mathbb{R}} \Phi(\lambda_{k,n}(\omega))d\omega = \int_{\mathbb{R}} \hat{\Phi}(t)F_{\lambda_{k,n}}(t)dt$$

Now the integrand $\hat{\Phi}(t)F_{\lambda_{k,n}}(t)$ converges to $\hat{\Phi}(t)F_{\lambda_k^+}(t)$ pointwise.

Also, $\left|\hat{\Phi}(t)F_{\lambda_{k,n}}(t)\right|$ is dominated by the integrable function $\hat{\Phi}(t)$. Again using the Lebesgue dominated convergence theorem, we get

$$\int_{\mathbb{R}} \hat{\Phi}(t)F_{\lambda_{k,n}}(t)dt \text{ converges to } \int_{\mathbb{R}} \hat{\Phi}(t)F_{\lambda_k^+}(t)dt.$$

That is the integral $\int_{\mathbb{R}} \Phi(\lambda_{k,n}(\omega))d\omega$ converges to $\int_{\mathbb{R}} \Phi(\lambda_k^+(\omega))d\omega$ for every Schwartz function Φ on $\mathbb{R}$.

Now by Stone-Weierstrass theorem, every continuous function with compact support can be uniformly approximated by sequence of Schwartz functions. Since uniform convergence will allow us to interchange the limit and the integral, we get the integral $\int_{\mathbb{R}} f(\lambda_{k,n}(\omega))d\omega$ converges to $\int_{\mathbb{R}} f(\lambda_k^+(\omega))d\omega$ for every continuous function f on $\mathbb{R}$, with compact support. Hence the proof is completed. $\square$

Remark 2.5. The motivation for the above proof technique is the proof of Levy continuity theorem (see [18]).

Remark 2.6. It can be observed that the convergence of moments of eigenvalues of truncations to the eigenvalue of the operator can be proved without the assumption of uniform rate of convergence of approximation numbers. The arguments are as follows:

Theorem 2.7. *Let $A(\omega)$ be a bounded random self-adjoint operator with infinitely many discrete eigenvalues lying outside the bounds of essential spectrum for almost all ω. Then*

$$E(\lambda_{k,n}(.)) - E(\lambda_k^+(.)) = o(1), \text{ for each } k,$$

$$E(\lambda_{n+1-k,n}(.)) - E(\lambda_k^-(.)) = o(1), \text{ for each } k,$$

Also,

$$\lim_{k\to\infty}\lim_{n\to\infty} E(\lambda_{k,n}(.)) = E(\mu(.))$$

and

$$\lim_{k\to\infty}\lim_{n\to\infty} E(\lambda_{n+1-k,n}(.) = E(\nu(.))$$

Proof. For almost every ω, by Theorem 1.4,

$$\lim_{n\to\infty} \lambda_{k,n}(\omega) = \begin{cases} \lambda_k^+(\omega), & \text{if } R = \infty \text{ or } 1 \leq k \leq R, \\ \mu(\omega), & \text{if } R < \infty \text{ and } k \geq R+1, \end{cases}$$

$$\lim_{n\to\infty} \lambda_{n+1-k,n}(\omega) = \begin{cases} \lambda_k^-(x), & \text{if } S = \infty \text{ or } 1 \leq k \leq S, \\ \nu(\omega), & \text{if } S < \infty \text{ and } k \geq S+1. \end{cases}$$

In particular,

$$\lim_{k\to\infty}\lim_{n\to\infty} \lambda_{k,n}(\omega) = \mu(\omega) \text{ and } \lim_{k\to\infty}\lim_{n\to\infty} \lambda_{n+1-k,n}(\omega) = \nu(\omega) \ a.e.$$

Now observe that for each k and n,

$$|\lambda_{k,n}(\omega)| \leq \|A(\omega)\| \text{ for almost every } \omega.$$

Since the moments of $\|A(\omega)\|$ exist and are finite, by Lebesgue dominated convergence theorem, the proof is completed. $\square$

Remark 2.8. Complex analytic techniques to compute integrals involving eigenvalues using matrix entries (see for eg. [13]), are useful in the computation of moment sequences appearing in the above theorem.

2.1. **Examples.** Now let's discuss some examples of random bounded self-adjoint operators and their spectral approximation.

Random Toeplitz operators:

Let f_ω be a family of continuous functions with ω varying in some probability space. Consider the family of Toeplitz operators with the one parameter family of continuous symbols f_ω. That is the operators T_{f_ω} acting on $l^2(\mathbb{N})$ defined by the infinite matrices obtained from the Fourier coefficients of the continuous functions f_ω. The $(i,j)^{th}$ entry in the infinite matrix is $a_{i-j}(\omega)$, where $a_k(\omega)$ is the k^{th} Fourier coefficient of f_ω.

The spectrum of T_{f_ω} is the essential range of f_ω. It is known that the spectrum of such operators is completely determined by the limits of the eigenvalues of their truncations (Theorem 4.1 and Example 4.3 of [3]). We shall consider some concrete examples which are compact perturbations of random Toeplitz operators.

Example 2.9. Let $f_\omega(x) = sin(x+\omega); x,\omega \in [-\pi,\pi]$. Here the spectrum is independent of ω, and $\sigma(T_{f_\omega}) = \sigma_e(T_{f_\omega}) = [-1,1]$. Now if $A(\omega)$ is a compact perturbation of the operator T_{f_ω}; that is $A(\omega) = T_{f_\omega} + A$ for some compact operator A, then considering the truncations $A(\omega)_n$, we get uniform rate of convergence for approximation numbers.

That is $s_k(A(\omega)_n) - s_k(A(\omega)) = O(\theta_n)$, where θ_n goes to zero as n goes to infinity and therefore the hypothesis of Theorem 2.2 is satisfied. Hence we have

$$E(\lambda_{k,n}(.)) - E((\lambda_k^+)) = O(\theta_n), \quad \text{if } R = \infty \text{ and}$$
$$E(\lambda_{n+1-k,n}(.)) - E((\lambda_k^-)) = O(\theta_n), \quad \text{if } S = \infty.$$

Example 2.10. Let $f_\omega(x) = sin(\omega)sin(x); x,\omega \in [-\pi,\pi]$. Here the spectrum depends on ω for almost all ω and $\sigma(T_{f_\omega}) = \sigma_e(T_{f_\omega}) = [-sin(\omega), sin(\omega)]$. As in the previous example, we shall consider the compact perturbations of this operator; that is $A(\omega) = T_{f_\omega} + A$ for some compact operator A. This will provide some nontrivial examples of the truncation method for the spectral approximation discussed in this section.

3. Gap prediction methods

In this section, a new method is proposed to predict the spectral gaps that may occur between the bounds of the essential spectrum. We begin with two lemmas which are the modified versions of the results in [17]. These results can be used to compute one end point of a spectral gap if the other end point is known.

Lemma 3.1. *Let A be a bounded self-adjoint operator with the essential spectrum, $\sigma_e(A) = [a,b] \cup \{c\}$ where $a < b < c$. Assume that b is not an accumulation point of the discrete spectra of A. Then a, b, c can be computed by truncation method.*

Proof. The numbers a and c can be computed by truncation method, with the help of Theorem 1.4, as they are the lower and upper bounds of essential spectrum. Assume that A is positive so that $a \geq 0$. Now consider the continuous function ϕ defined as follows.

$$\phi(t) = \begin{cases} t, & t \in [a,b] \\ \frac{b(t-c)}{b-c}, & t \in [b,c] \end{cases}$$

Then by spectral mapping theorem, we have

$$\sigma_e(\phi(A)) = \{0\} \cup [a, b].$$

Hence b can also be computed as the upper bound of $\sigma_e(\phi(A))$. If A is not a positive operator, then consider the positive operator $A - mI$ and apply the same technique to compute b. This completes the proof. □

Lemma 3.2. *Let A be a bounded self-adjoint operator and $\sigma_e(A) = [a,b] \cup [c,d]$, where $a < b < c < d$. Assume that b is known and not an accumulation point of the discrete spectra of A. Then c can be computed by truncation method.*

Proof. Here it requires to consider the function ϕ defined as

$$\phi(t) = \begin{cases} 0, & t \in [a, b] \\ t - b & t \in [c, d] \end{cases}$$

Then $\sigma_e(\phi(A)) = \{0\} \cup [c - b, d - b]$. Hence the numbers $c - b$ and $d - b$ can be obtained by truncation method. Therefore since b is assumed to be known, c and d are computable. Hence the proof. □

The concepts of second order relative spectra and quadratic projection method, which are almost synonyms, were used in the spectral pollution problems and to determine eigenvalues in the gaps (see [6, 7, 15, 16]). Analogous to this, here a new method is proposed which can be used in the spectral gap prediction problems. In short, the spectral gap prediction problem is reduced to determining nonzero values of a particular function. This particular function can be uniformly approximated by a sequence of functions, which are calculated using the eigenvalues of truncations of the operator under concern.

The idea is to open the gap by translating and squaring the operator and identifying each number in the interval (ν, μ) as the lower bound of essential spectrum of a positive definite operator, which is computed using the truncation method, in particular by Theorem 1.4.

Define the nonnegative valued function f on the real line $\mathbb{R}$ as follows.

$$f(\lambda) = \nu_\lambda = \inf \sigma_e((A - \lambda I)^2).$$

The first observation is that one can predict the existence of a gap inside the essential spectrum by evaluating the function and checking whether it attains a nonzero value. Each nonzero value of this function gives indication of a spectral gap.

Theorem 3.3. *The number λ in the interval (ν, μ) is in the gap if and only if $f(\lambda) > 0$. Also one end point of the gap will be $\lambda \pm \sqrt{f(\lambda)}$.*

Proof. Using the spectral mapping theorem, observe that $f(\lambda)$ is the square of the distance of λ to the essential spectrum of A. The details are given below.

$$\inf \sigma_e((A - \lambda I)^2) = d(0, \sigma_e((A - \lambda I)^2)) = d(0, \sigma_e(A - \lambda I))^2 = d(\lambda, \sigma_e(A))^2$$

Hence λ is in the essential spectrum of A if and only if $f(\lambda) = 0$, since essential spectrum is a closed set. Therefore the number λ in the interval (ν, μ) is in the gap if and only if $f(\lambda) > 0$. Now if λ is in the gap, then one of the end points will be at a distance $\sqrt{f(\lambda)}$ from λ. Hence that end point will be $\lambda \pm \sqrt{f(\lambda)}$. □

The advantage of considering $f(\lambda)$ is that, it is the lower bound of the essential spectrum of the operator $(A-\lambda I)^2$, which can be computed using the finite dimensional truncations with the help of Theorem 1.4. So the computation of $f(\lambda)$ is possible for each λ. This enables us to predict the gap using truncations. Also in Theorem 3.3, it is possible to compute one end point of a gap. The other end point can be computed as discussed in Lemma 3.2. Below, the function $f(.)$ is approximated by a double sequence of functions, which arise from the eigenvalues of truncations of operators.

Theorem 3.4. *Let $f_{n,k}$ be the sequence of functions defined by*

$$f_{n,k}(\lambda)=\lambda_{n+1-k}\left(P_n(A-\lambda I)^2 P_n\right).$$

Then $f(.)$ is the uniform limit of a subsequence of $\{f_{n,k}(.)\}$ on all compact subsets of the real line.

Proof. By Theorem 1.4, for each λ,

$$f(\lambda)=\lim_{k\to\infty}\lim_{n\to\infty} f_{n,k}(\lambda), \text{ where } f_{n,k}(\lambda) = \lambda_{n+1-k}\left(P_n(A-\lambda I)^2 P_n\right).$$

Now the quantity $\Delta=|f_{n,k}(\lambda)-f_{n,k}(\lambda_0)|$ can be estimated as follows.

$$\begin{aligned}\Delta &= \left|\lambda_{n+1-k}\left(P_n(A-\lambda I)^2 P_n\right)-\lambda_{n+1-k}\left(P_n(A-\lambda_0 I)^2 P_n\right)\right| \\ &\le \left\|P_n(A-\lambda I)^2 P_n - P_n(A-\lambda_0 I)^2 P_n\right\| \\ &\le \left\|(A-\lambda I)^2-(A-\lambda_0 I)^2\right\| = \left\|(\lambda^2-\lambda_0{}^2)I-2(\lambda-\lambda_0)A\right\| \le M|\lambda-\lambda_0|,\end{aligned}$$

where $M=2(|\mu|+\|A\|)$. For the first inequality, recall an important inequality concerning the eigenvalues of self-adjoint matrices A,B (refer e.g. to [2])

$$|\lambda_k(A)-\lambda_k(B)|\le\|A-B\|. \tag{3.3}$$

and the second one from the fact that $\|P_n\|=1$. Hence,

$$|f_{n,k}(\lambda)-f_{n,k}(\lambda_0)|\le M|\lambda-\lambda_0|.$$

Since the constant M above is independent of n,k or λ, $\{f_{n,k}(.)\}$ forms an equicontinuous family of functions, also it is pointwise bounded. Hence $\{f_{n,k}(.)\}$ has a subsequence that converges uniformly on all compact subsets by Arzela-Ascoli theorem. Hence the proof is completed. ☐

The following result makes the computation of $f(\lambda)$ much easier for a particular class of operators. The difficulty of squaring a bounded operator is reduced by first truncating the operator and then squaring the truncation, rather truncating the square of the operator. The computation needs only squaring the finite matrices.

Theorem 3.5. *If $\|P_nA-AP_n\|\to 0\, as\, n\to\infty$, then*

$$\lim_{k\to\infty}\lim_{n\to\infty}\lambda_{n+1-k}\left(P_n(A-\lambda I)^2 P_n\right)=\lim_{k\to\infty}\lim_{n\to\infty}\lambda_{n+1-k}\left(P_n(A-\lambda I)P_n\right)^2.$$

Proof. The notation A_λ is used for $A-\lambda I$. Observe the following chain of equalities;

$$\left\|P_n(A_\lambda)^2P_n-(P_n(A_\lambda)P_n)^2\right\| = \|P_n(A_\lambda)(A_\lambda)P_n-(P_n(A_\lambda)P_n)(P_n(A_\lambda)P_n)\|$$
$$=\|P_n(A_\lambda)(A_\lambda)P_n-(A_\lambda)P_n(A_\lambda)P_n+(A_\lambda)P_nP_n(A_\lambda)P_n-P_n(A_\lambda)P_n(A_\lambda)P_n\ \|$$

using ${P_n}^2 = P_n$ and adding and subtracting $(A_\lambda) P_n (A_\lambda) P_n$. And notice that the latter is equal to

$$\|[P_n (A_\lambda) - (A_\lambda) P_n] (A_\lambda) P_n - [P_n (A_\lambda) - (A_\lambda) P_n] P_n (A_\lambda) P_n\| =$$

$$\|[P_n (A_\lambda) - (A_\lambda) P_n] [(A_\lambda) P_n - P_n (A_\lambda) P_n]\| \leq 2 \|A_\lambda\| \|P_n (A_\lambda) - (A_\lambda) P_n\| =$$

$$2 \|A_\lambda\| \|P_n A - A P_n\| \to 0,$$

as the dimension n tends to infinity.

Applying (3.3) to the matrices $\left(P_n (A - \lambda I)^2 P_n\right)$ and $(P_n (A - \lambda I) P_n)^2$, we get

$$\left|\lambda_{n+1-k} \left(P_n (A_\lambda)^2 P_n\right) - \lambda_{n+1-k} (P_n (A_\lambda) P_n)^2\right| \leq \left\|\left(P_n (A_\lambda)^2 P_n\right) - (P_n (A_\lambda) P_n)^2\right\|.$$

Since the right hand side goes to zero as n tends to infinity, we get the desired conclusion. □

Remark 3.6. By observing the proof of the last theorem, it can be noticed that the last assertion is independent of k. That means a similar conclusion holds for each k^{th} eigenvalue of A. Hence for a large class of operators, instead of squaring the operator and truncating, it is enough to square the truncation. The following is an example.

Example 3.7. Let a_n be an enumeration of rational numbers in the set $[0, \frac{1}{2}] \cup [\frac{3}{4}, 1]$. Define the bounded self-adjoint operator A on $l^2(\mathbb{N})$ by

$$A(x_1, x_2, \ldots) = (a_1 x_1, a_2 x_2, \ldots)$$

It can be observed that the bounds of the essential spectrum are $\nu = 0$ and $\mu = 1$. Now

$$\sigma((A - \lambda I)_n^2) = \{(a_1 - \lambda)^2, (a_2 - \lambda)^2, \ldots (a_n - \lambda)^2\}.$$

Also, it can be inferred that the sequence of lowest eigenvalues of $((A - \lambda I)_n^2)$ goes to zero if and only if there exists a subsequence $a_{n,k}$ of a_n that approaches λ. This happens only when λ is in the set $[0, \frac{1}{2}] \cup [\frac{3}{4}, 1]$. Therefore for every λ in the interval $(\frac{1}{2}, \frac{3}{4})$, $f(\lambda)$ is positive and hence it is a spectral gap.

The spectral gap issues of random bounded self-adjoint operators is an important area for further research. Here we consider an example which indicates the instability of spectral gaps of random bounded self-adjoint operators. We consider the random block Toeplitz operator generated by a $p \times p$ random matrix valued symbol.

Example 3.8. Define a two parameter family of matrix valued symbols as follows.

$$f(\omega, \theta) = \begin{bmatrix} b_1(\omega) & 1 & & & & e^{-i\theta} \\ 1 & b_2(\omega) & 1 & & & \\ & 1 & b_3(\omega) & 1 & & \\ & & 1 & b_4(\omega) & 1 & \\ & & & \ddots & \ddots & \ddots \\ e^{i\theta} & & & & 1 & b_p(\omega) \end{bmatrix}$$

where $b_1(.), b_2(.) \dots b_p(.)$ are real valued measurable functions on a domain containing the interval $[0,1]$ and θ varying in the interval $[0, 2\pi]$. We consider the one parameter family of block Toeplitz operators arising from these symbols. Thus we get a random family of bounded self-adjoint operators $A(\omega)$. This corresponds to the discretized version of Schrödinger operator with periodic random potential.

The essential spectrum of $A(\omega_0)$ has no gaps if and only if $b_1(\omega_0) = b_2(\omega_0) \dots = b_p(\omega_0)$ (See [10], Section 4). Therefore it is clear that the existence of spectral gaps depend on the values of the random variables b_i's. That means the randomness in the prediction of spectral gaps is proportional to the randomness of b_i's. Hence the spectral gap prediction is highly instable in this example.

4. Discussion on the main results and further possibilities

In this section, we discuss some of the important features of the main results. Also, further possibilities of the considered theory are discussed here. We state some important open problems in this area.

4.1. **Wigner operators and error estimate.** The rate of convergence of the approximation in Theorem 1.4 and its random versions, is not known in the general case. Nevertheless one can expect a better rate of convergence in the case of some special class of operators. Here we consider Wigner operators and identify their truncations with the well known Wigner matrices. Unfortunately, these operators are unbounded. So the theory developed here is not applicable to this case. However this will be a source of many problems of practical interest. We give the details below.

Definition 4.1. A random operator $A(\omega)$ is called a Wigner operator if the following conditions are satisfied.

1. For $i \leq j$, $f_{i,j}(\omega) = \langle A(\omega)e_j, e_i\rangle$ are independent with mean zero and variance one, and $f_{i,j} = \bar{f}_{j,i}$.
2. For $i < j$, $f_{i,j}$ are identically distributed, with distribution η.
3. $f_{i,i}$ are identically distributed, with distribution $\tilde{\eta}$.
4. (Uniform exponential decay) There exists $C, C' > 0$ such that

$$P(|f_{i,j}| \geq t^C) \leq exp(-t), \text{ for every } t \geq C' \text{ and } i, j.$$

The classical example of Wigner matrices [20], is the motivation for the above definition. Wigner matrices are examples of random matrices for which the empirical distribution of eigenvalues converges to the semicircle distribution in the weak probability. If truncations $A(\omega)_n$ of a Wigner operator are considered, then resultant matrix is a Wigner matrix. The *Wigner semi circular law* asserts that the empirical spectral distribution

$$\mu_n = \frac{1}{n}\sum_{1}^{n} \delta_{\frac{1}{\sqrt{n}}}\lambda_{i,n}$$

converges to the semicircular distribution ρ_{sc} (in weak probability), where

$$\rho_{sc}(x) = \frac{1}{\pi}\sqrt{4-x^2}.\chi_{[-1,1]}$$

There were attempts to estimate the mean square difference of eigenvalues of Wigner matrices and the classical location of them. The article due to Terence Tao and Van Vu [19] is one among them. Under the assumption that the third moment vanishes, the following estimation was proved by Tao and Vu in [19].

Theorem 4.2. [19] *There is an absolute constant $c > 0$ such that the following holds for any constant $\epsilon > 0$. Let A_n be a Wigner matrix whose distribution η has vanishing third moment $E(\eta^3) = 0$. Then for all $1 \leq i \leq n$,*

$$E(\left|\lambda_{i,n} - \sqrt{n}\gamma_i\right|^2) = O(min(n^{-c}min(i, n+1-i)^{-2/3}n^{2/3}, n^{1/3+\epsilon})),$$

where γ_i is the classical location of the i^{th} eigenvalue of A_n, given by the formula

$$\int\limits_{-\infty}^{\gamma_i} \rho_{sc}(x)\, dx = \frac{i}{n}.$$

Remark 4.3. In the paper [8], a better bound

$$\sum_{i=1}^{n} E(\left|\lambda_{i,n} - \gamma_i\sqrt{n}\right|^2) = O(n^{1-c})$$

was established for some constant $c > 0$.

An interesting problem is to prove the operator theoretic version of these estimates. That means we need to estimate the mean square difference of eigenvalues of Wigner operator and the classical location of eigenvalues of Wigner matrices. It can be observed that the unboundedness of Wigner operator makes this task nontrivial. Otherwise these estimates are mere consequence of Theorem 1.4 and 2.2.

One can use the results in [19], to get good estimates of the spectrum and spectral gaps for Wigner operators. For a Wigner operator, the truncations are Wigner matrices. The following results for such matrices can be found in [19] and the references therein.

- $\lambda_i(A_n) - \gamma_i.\sqrt{n} = o(\sqrt{n});\ 1 \leq i \leq n$. Use this to get good estimates in the spectral approximation of the Wigner operator.
- Let $F_n(x) = \frac{1}{n}E(N_n[-2, x]), \Delta = Sup_x \left|F_n(x) - \int_{\infty}^{x} \rho_{sc}(t)dt\right|$, then $\Delta = O(n^{\frac{-1}{2-c}})$, if the third moment is 0. Use these to get information regarding essential points and integrated density of states.
- Four moment theorem in [19], can be used for the perturbation kind results.

4.2. **Spectral gap prediction.** The function $f(.)$ that is considered in section 3, is directly related to the distance from the essential spectrum, while the function used in [6, 7] is related to the distance from the spectrum. Here the approximation results in [3], especially Theorem 1.4 are used to approximate the function. But it is still not known to us whether these results are useful in a computational point of view. The methods in [6, 7] were applied in the case of some Schrödinger operators with a particular kind of potentials in [15, 16]. Hopefully, a combined use of both methods may give a better understanding of the spectrum.

To determine the gaps in the essential spectrum of a particular operator, using the method proposed in section 3, the following problems may arise. Checking the conditions for each λ in (ν, μ), is a difficult task in the computational point of view. The uniform convergence established in Theorem 3.4, may be useful to get around this difficulty.

Also taking truncations of the square of the operator may be difficult. Note that $(P_nAP_n)^2$ and $P_nA^2P_n$ are not equal, so that one has to do more computations to handle the problem. Theorem 3.5 solves this problem for a special class of operators, up to some extend.

4.3. **Moment Problem.** From the results proved in Section 2, the following observations can be made easily.

For each $h = 1, 2, 3, \ldots,$ and fixed k,

$$\lim_{n\to\infty} (\lambda_{k,n}(\omega))^h = (\lambda_k^+)^h (\omega), \quad \text{if } R = \infty,$$

for almost all ω. Thus, for each k, the h^{th} moment of eigenvalues converges. That is,

$$\lim_{n\to\infty} E(\lambda_{k,n}(.))^h = E((\lambda_k^+)^h), \quad \text{if } R = \infty.$$

Similarly,

$$\lim_{n\to\infty} E(\lambda_{n+1-k,n}(.))^h = E((\lambda_k^-)^h), \quad \text{if } S = \infty.$$

Recall that if Y_n is a sequence of real valued random variables and if there exists some (nonrandom) sequence β_h such that $E(Y_n^h) \to \beta_h$ for every positive integer h where β_h satisfies Carleman's condition,

$$\sum_{h=1}^{\infty} (\beta_{2h})^{-1/2h} = \infty,$$

then from [4], it is well-known that there exists a distribution function F, such that for all h,

$$\beta_h = \int x^h dF(x)$$

and Y_n converges to F in distribution. It is also known that this condition is not necessary in general. The following can be treated as a converse of the moment problem.

If β_h is used to denote $E((\lambda_k^{\pm})^h)$, does the sum $\sum_{h=1}^{\infty} (\beta_{2h})^{-1/2h}$ diverges to ∞? That is whether the Carleman's condition is satisfied?

We hope that the above mentioned open problems pave way for establishing a connection between the spectral theory of random operators and eigenvalue distribution of random matrices.

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References

1. W.B. Arveson, *C^*- algebras and numerical linear algebra*, J. Funct. Anal. **122** (1994), no.2 333–360.
2. R. Bhatia, *Matrix analysis*, Graduate text in Mathematics Springer Verlag, New York, 1997.
3. A. Böttcher, A.V. Chithra and M.N.N. Namboodiri, *Approximation of approximation numbers by truncation*, Integral Equations Operator Theory **39** (2001), 387–395.
4. T. Carleman, *Les fonctions quasi analytiques*, Collection Borel, GauthierVillars, Paris, 1926.
5. R. Carmona and J. Lacroix, *Spectral Theory of Random Schrödinger operators*, Birkhäuser, Boston, 1990.
6. E.B. Davies, *Spectral enclosures and complex resonances for general self-adjoint operators*, LMS J. Comput. Math. **1** (1998), 42–74.
7. E.B. Davies and M. Plum, *Spectral pollution*, IMA J. Numer. Anal. **24** (2004), no. 3, 417–438.
8. L.Erdös, H.T Yau and J. Yin, *Universality for generalized Wigner matrices with Bernoulli distribution*, J. Comb. **2** (2011), no. 1, 15–81.
9. I. Gohberg, S. Goldberg and M.A. Kaashoek, *Classes of linear operators*, Vol. I. Birkhguser Verlag, Basel, 1990.
10. L. Golinskii, K. Kumar, M.N.N. Namboodiri and S. Serra-Capizzano, *A note on a discrete version of Borg's Theorem via Toeplitz-Laurent operators with matrix-valued symbols*, Boll. Unione Mat. Ital. (9) **6** (2013), no. 1, 205–218.
11. A.C. Hansen, *On the approximation of spectra of linear operators on Hilbert spaces*, J. Funct. Anal. **254** (2008), 2092–2126.
12. P.D. Hislop, *Lectures on random Schrdinger operators*, Fourth Summer School in Analysis and Mathematical Physics, 41–131, Contemp. Math., 476, Amer. Math. Soc., Providence, RI, 2008.
13. T. Kato, *Perturbation theory of linear operators*, Springer-Verlag, Berlin, Heidelberg, New York, 1966.
14. K. Kumar, M.N.N. Namboodiri and S. Serra-Capizzano, *Perturbation of operators and approximation of spectrum*, Proc. Indian Acad. Sci. Math. Sci. **124** (2014), no. 2, 205–224.
15. L. Boulton and M. Levitin, *On approximation of the eigenvalues of perturbed periodic Schrdinger operators*, J. Phys. A **40** (2007), no. 31, 9319–9329.
16. M. Levitin and E. Shargorodsky *Spectral pollution and second-order relative spectra for self-adjoint operators*, IMA J. Numer. Anal. **24** (2004), 393–416.
17. M.N.N. Namboodiri, *Truncation method for operators with discounted essential spectrum*, Spectral and inverse spectral theory (Goa, 2000). Proc. Indian Acad. Sci. Math. Sci. 112 (2002), no. 1, 189–193.
18. Terence Tao, *Topics in random matrix theory*, Graduate Studies in Mathematics, 132. American Mathematical Society, Providence, RI, 2012.
19. Terence Tao and Van Vu, *Random Matrices: Localization of the eigenvalues and the necessity of four moments*, Acta Math. Vietnam. **36** (2011), no. 2, 431–449.
20. E.P. Wigner, *On the distribution of the roots of certain symmetric matrices*, Ann. of Math. (2) **67** (1958), 325–327.

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