Banach J. Math. Anal. 9 (2015), no. 3, 24–42
http://doi.org/10.15352/bjma/09-3-3
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

FREDHOLMNESS AND INDEX OF SIMPLEST WEIGHTED SINGULAR INTEGRAL OPERATORS WITH TWO SLOWLY OSCILLATING SHIFTS

ALEXEI YU. KARLOVICH

Dedicated to Professor Yuri I. Karlovich on his 65th anniversary

Communicated by L. P. Castro

Abstract. Let α and β be orientation-preserving diffeomorphisms (shifts) of $\mathbb{R}_+ = (0, \infty)$ onto itself with the only fixed points 0 and ∞, where the derivatives α' and β' may have discontinuities of slowly oscillating type at 0 and ∞. For $p \in (1, \infty)$, we consider the weighted shift operators U_α and U_β given on the Lebesgue space $L^p(\mathbb{R}_+)$ by $U_\alpha f = (\alpha')^{1/p}(f \circ \alpha)$ and $U_\beta f = (\beta')^{1/p}(f \circ \beta)$. For $i, j \in \mathbb{Z}$ we study the simplest weighted singular integral operators with two shifts $A_{ij} = U_\alpha^i P_\gamma^+ + U_\beta^j P_\gamma^-$ on $L^p(\mathbb{R}_+)$, where $P_\gamma^\pm = (I \pm S_\gamma)/2$ are operators associated to the weighted Cauchy singular integral operator

$$(S_\gamma f)(t) = \frac{1}{\pi i} \int_{\mathbb{R}_+} \left(\frac{t}{\tau}\right)^\gamma \frac{f(\tau)}{\tau - t} d\tau$$

with $\gamma \in \mathbb{C}$ satisfying $0 < 1/p + \Re\gamma < 1$. We prove that the operator A_{ij} is a Fredholm operator on $L^p(\mathbb{R}_+)$ and has zero index if

$$0 < \frac{1}{p} + \Re\gamma + \frac{1}{2\pi} \inf_{t \in \mathbb{R}_+} (\omega_{ij}(t)\Im\gamma), \quad \frac{1}{p} + \Re\gamma + \frac{1}{2\pi} \sup_{t \in \mathbb{R}_+} (\omega_{ij}(t)\Im\gamma) < 1,$$

where $\omega_{ij}(t) = \log[\alpha_i(\beta_{-j}(t))/t]$ and α_i, β_{-j} are iterations of α, β. This statement extends an earlier result obtained by the author, Yuri Karlovich, and Amarino Lebre for $\gamma = 0$.

Date: Received: May 2, 2014; Accepted: Sep. 16, 2014.
2010 *Mathematics Subject Classification.* Primary 47B35; Secondary 45E05, 47A53, 47G10, 47G30.
Key words and phrases. Fredholmness, index, slowly oscillating shift, weighted singular integral operator, Mellin pseudodifferential operator.

1. Introduction

Let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators acting on a Banach space X and let $\mathcal{K}(X)$ be the ideal of all compact operators in $\mathcal{B}(X)$. An operator $A \in \mathcal{B}(X)$ is called *Fredholm* if its image is closed and the spaces $\ker A$ and $\ker A^*$ are finite-dimensional. In that case the number

$$\operatorname{Ind} A := \dim\ker A - \dim\ker A^*$$

is referred to as the *index* of A (see, e.g., [7, Chap. 4]). For $A, B \in \mathcal{B}(X)$, we will write $A \simeq B$ if $A - B \in \mathcal{K}(X)$. Recall that an operator $B_r \in \mathcal{B}(X)$ (resp. $B_l \in \mathcal{B}(X)$) is said to be a right (resp. left) regularizer for A if

$$AB_r \simeq I \quad (\text{resp.} \quad B_l A \simeq I).$$

It is well known that an operator A is Fredholm on X if and only if it admits simultaneously a right and a left regularizers. Moreover, each right regularizer differs from each left regularizer by a compact operator (see, e.g., [7, Chap. 4, Section 7]).

Following Sarason [32, p. 820], a bounded continuous function f on $\mathbb{R}_+ = (0, \infty)$ is called slowly oscillating (at 0 and ∞) if for each (equivalently, for some) $\lambda \in (0, 1)$,

$$\lim_{r\to s} \sup_{t,\tau\in[\lambda r,r]} |f(t) - f(\tau)| = 0 \quad \text{for} \quad s \in \{0, \infty\}.$$

The set $SO(\mathbb{R}_+)$ of all slowly oscillating functions forms a C^*-algebra. This algebra properly contains $C(\overline{\mathbb{R}}_+)$, the C^*-algebra of all continuous functions on $\overline{\mathbb{R}}_+ := [0, +\infty]$. Note that this notion of slow oscillation does not involve any differentiability. Various modifications of it were studied in many works, just to mention a few, see [15, 21, 22, 26, 27, 33]. On the other hand, there is also another (stronger) notion of slow oscillation (or slow variation) in the literature. That notion is defined by using at least the first derivative of a function and goes back to Grushin [9], it was extensively used since then in works on pseudodifferential operators and their applications, see e.g., [24, 28, 29, 30].

Suppose α is an orientation-preserving diffeomorphism of $\mathbb{R}_+$ onto itself, which has only two fixed points 0 and ∞. We say that α is a slowly oscillating shift if $\log \alpha'$ is bounded and $\alpha' \in SO(\mathbb{R}_+)$. The set of all slowly oscillating shifts is denoted by $SOS(\mathbb{R}_+)$.

We suppose that $1 < p < \infty$. It is easily seen that if $\alpha \in SOS(\mathbb{R}_+)$, then the shift operator W_α defined by $W_\alpha f = f \circ \alpha$ is bounded and invertible on all spaces $L^p(\mathbb{R}_+)$ and its inverse is given by $W_\alpha^{-1} = W_{\alpha_{-1}}$, where α_{-1} is the inverse function to α. Along with W_α we consider the weighted shift operator

$$U_\alpha := (\alpha')^{1/p} W_\alpha$$

being an isometric isomorphism of the Lebesgue space $L^p(\mathbb{R}_+)$ onto itself. It is clear that $U_\alpha^{-1} = U_{\alpha_{-1}}$.

Let $\Re\gamma$ and $\Im\gamma$ denote the real and imaginary part of $\gamma \in \mathbb{C}$, respectively. As usual, $\overline{\gamma} = \Re\gamma - i\Im\gamma$ denotes the complex conjugate of γ. If $\gamma \in \mathbb{C}$ satisfies $0 < 1/p + \Re\gamma < 1$, then the weighted Cauchy singular integral operator S_γ, given by

$$(S_\gamma f)(t) := \frac{1}{\pi i} \int_{\mathbb{R}_+} \left(\frac{t}{\tau}\right)^\gamma \frac{f(\tau)}{\tau - t} d\tau,$$

where the integral is understood in the principal value sense, is bounded on the Lebesgue space $L^p(\mathbb{R}_+)$ (see, e.g., [2, Section 1.10.2], [3], [10, Section 2.1.2], [31, Proposition 4.2.11]). Put

$$P_\gamma^\pm := (I \pm S_\gamma)/2.$$

Consider the weighted singular integral operators with two shifts $\alpha, \beta \in SOS(\mathbb{R}_+)$ given by

$$A_{ij} := U_\alpha^i P_\gamma^+ + U_\beta^j P_\gamma^-, \quad i, j \in \mathbb{Z}. \tag{1.1}$$

Here, by definition, $U_\alpha^i = (U_\alpha^{-1})^{|i|}$ and $U_\beta^j = (U_\beta^{-1})^{|j|}$ for negative i and j, respectively.

In [14] we called such operators (with $\gamma = 0$) simplest singular integral operators with two shifts because they do not involve functional coefficients, in contrast to as it is traditionally considered (see, e.g., [12, 13, 23, 25]). In that paper we proved that if $\gamma = 0$, then all operators (1.1) are Fredholm and their indices are equal to zero. This paper is a sequel of [14], here our aim is to extend that result to the case of $\gamma \in \mathbb{C}$ satisfying $0 < 1/p + \Re\gamma < 1$.

For a shift $\alpha \in SOS(\mathbb{R}_+)$, put $\alpha_0(t) := t$ and $\alpha_i(t) := \alpha[\alpha_{i-1}(t)]$ for every $i \in \mathbb{Z}$ and $t \in \mathbb{R}_+$.

Theorem 1.1 (Main result)**.** *Suppose $1 < p < \infty$ and $\gamma \in \mathbb{C}$ is such that $0 < 1/p + \Re\gamma < 1$. If $\alpha, \beta \in SOS(\mathbb{R}_+)$, $i, j \in \mathbb{Z}$, and*

$$0 < \frac{1}{p} + \Re\gamma + \frac{1}{2\pi} \inf_{t \in \mathbb{R}_+} (\omega_{ij}(t) \Im\gamma), \quad \frac{1}{p} + \Re\gamma + \frac{1}{2\pi} \sup_{t \in \mathbb{R}_+} (\omega_{ij}(t) \Im\gamma) < 1,$$

where $\omega_{ij}(t) = \log[\alpha_i(\beta_{-j}(t))/t]$, then the operator A_{ij} given by (1.1) is Fredholm on the space $L^p(\mathbb{R}_+)$ and its index is equal to zero.

By [14, Corollary 2.5], if $\alpha, \beta \in SOS(\mathbb{R}_+)$, then $\delta_{ij} := \alpha_i \circ \beta_{-j}$ belongs to $SOS(\mathbb{R}_+)$ for all $i, j \in \mathbb{Z}$. Since $A_{ij} = U_\beta^j (U_{\delta_{ij}} P_\gamma^+ + P_\gamma^-)$, where $U_{\delta_{ij}} = U_{\beta_{-j}} U_{\alpha_i} = U_\beta^{-j} U_\alpha^i$, the proof of Theorem 1.1 is immediately reduced to the proof of the following partial case (treated for $\gamma = 0$ in [14, Theorem 5.2]).

Theorem 1.2. *Suppose $1 < p < \infty$ and $\gamma \in \mathbb{C}$ is such that $0 < 1/p + \Re\gamma < 1$. If $\alpha \in SOS(\mathbb{R}_+)$ and*

$$0 < \frac{1}{p} + \Re\gamma + \frac{1}{2\pi} \inf_{t \in \mathbb{R}_+} (\omega(t) \Im\gamma), \quad \frac{1}{p} + \Re\gamma + \frac{1}{2\pi} \sup_{t \in \mathbb{R}_+} (\omega(t) \Im\gamma) < 1, \tag{1.2}$$

where $\omega(t) = \log[\alpha(t)/t]$, then the operators $U_\alpha P_\gamma^+ + P_\gamma^-$ and $U_\alpha^{-1} P_{\overline{\gamma}}^+ + P_{\overline{\gamma}}^-$ are Fredholm on the space $L^p(\mathbb{R}_+)$ and

$$\operatorname{Ind}(U_\alpha P_\gamma^+ + P_\gamma^-) = \operatorname{Ind}(U_\alpha^{-1} P_{\overline{\gamma}}^+ + P_{\overline{\gamma}}^-) = 0.$$

The rest of the paper is devoted to the proof of Theorem 1.2. In Section 2 we reduce the proof of Theorem 1.2 to the study of the Fredholmness and the calculation of the index of an operator $A_{\alpha,\gamma}$ involving the operators U_α, U_α^{-1} and R_γ, $R_{\overline{\gamma}}$, where

$$(R_\gamma f)(t) := \frac{1}{\pi i}\int_{\mathbb{R}_+}\left(\frac{t}{\tau}\right)^\gamma \frac{f(\tau)}{\tau+t}d\tau, \quad \gamma\in\mathbb{C}, \quad 0<1/p+\Re\gamma<1.$$

This reduction is based on the well known fact that S_γ and R_γ with $\gamma\in\mathbb{C}$ satisfying $0<1/p+\Re\gamma<1$ are similar to certain Mellin convolution operators. To study the operator $A_{\alpha,\gamma}$, we need a more sophisticated machinery of Mellin pseudodifferential operators. In Section 3 we formulate several results on Mellin pseudodifferential operators concerning their boundedeness, compactness, and Fredholmness. In Section 4 we prove that the operator $A_{\alpha,\gamma}$ is similar to a Mellin pseudodifferential operator $\mathrm{Op}(\mathfrak{g}_{\omega,\gamma})$ with certain explicitly given symbol $\mathfrak{g}_{\omega,\gamma}$. At this step the proof of Theorem 1.2 is reduced to the study of the operator $\mathrm{Op}(\mathfrak{g}_{\omega,\gamma})$. The important point is that the Mellin pseudodifferential operator $\mathrm{Op}(\mathfrak{g}_{\omega,\gamma})$ belongs to the class $\{\mathrm{Op}(\mathfrak{a}) : \mathfrak{a}\in\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))\}$, for which a Fredholm criterion and an index formula are available. Here $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ is the Banach algebra of slowly oscillating symbols of limited smoothness introduced by Yuri Karlovich in [20]. In Section 5 we show that $\mathrm{Op}(\mathfrak{g}_{\omega,\gamma})$ is Fredholm and $\operatorname{Ind}\mathrm{Op}(\mathfrak{g}_{\omega,\gamma})=0$. This completes the proof of Theorem 1.2 and, thus, of Theorem 1.1.

2. Reduction

In this section we introduce an operator $A_{\alpha,\gamma}$ involving the operators U_α, U_α^{-1}, R_γ, and $R_{\overline{\gamma}}$. We show that if the operator $A_{\alpha,\gamma}$ is Fredholm and its index is equal to zero, then the operators $U_\alpha P_\gamma^+ + P_\gamma^-$ and $U_\alpha^{-1}P_{\overline{\gamma}}^+ + P_{\overline{\gamma}}^-$ are so.

2.1. **Fourier and Mellin convolution operators.** Let $\mathcal{F}: L^2(\mathbb{R})\to L^2(\mathbb{R})$ denote the Fourier transform,

$$(\mathcal{F}f)(x) := \int_{\mathbb{R}} f(y)e^{-ixy}dy, \quad x\in\mathbb{R},$$

and let $\mathcal{F}^{-1}: L^2(\mathbb{R})\to L^2(\mathbb{R})$ be the inverse of $\mathcal{F}$. A function $a\in L^\infty(\mathbb{R})$ is called a Fourier multiplier on $L^p(\mathbb{R})$ if the mapping $f\mapsto \mathcal{F}^{-1}a\mathcal{F}f$ maps $L^2(\mathbb{R})\cap L^p(\mathbb{R})$ onto itself and extends to a bounded operator on $L^p(\mathbb{R})$. The latter operator is then denoted by $W^0(a)$. We let $\mathcal{M}_p(\mathbb{R})$ stand for the set of all Fourier multipliers on $L^p(\mathbb{R})$. One can show that $\mathcal{M}_p(\mathbb{R})$ is a Banach algebra under the norm

$$\|a\|_{\mathcal{M}_p(\mathbb{R})} := \|W^0(a)\|_{\mathcal{B}(L^p(\mathbb{R}))}.$$

Let $d\mu(t)=dt/t$ be the (normalized) invariant measure on $\mathbb{R}_+$. Consider the Fourier transform on $L^2(\mathbb{R}_+, d\mu)$, which is usually referred to as the Mellin transform and is defined by

$$\mathcal{M}: L^2(\mathbb{R}_+, d\mu)\to L^2(\mathbb{R}), \quad (\mathcal{M}f)(x) := \int_{\mathbb{R}_+} f(t)t^{-ix}\frac{dt}{t}.$$

It is an invertible operator, with inverse given by

$$\mathcal{M}^{-1}: L^2(\mathbb{R})\to L^2(\mathbb{R}_+, d\mu), \quad (\mathcal{M}^{-1}g)(t) = \frac{1}{2\pi}\int_{\mathbb{R}} g(x)t^{ix}\,dx.$$

Let E be the isometric isomorphism

$$E : L^p(\mathbb{R}_+, d\mu) \to L^p(\mathbb{R}), \quad (Ef)(x) := f(e^x), \quad x \in \mathbb{R}. \tag{2.1}$$

Then the map

$$A \mapsto E^{-1}AE$$

transforms the Fourier convolution operator $W^0(a) = \mathcal{F}^{-1}a\mathcal{F}$ to the Mellin convolution operator

$$\mathrm{Co}(a) := \mathcal{M}^{-1}a\mathcal{M}$$

with the same symbol a. Hence the class of Fourier multipliers on $L^p(\mathbb{R})$ coincides with the class of Mellin multipliers on $L^p(\mathbb{R}_+, d\mu)$.

2.2. **Some operator identities.** Let $\mathcal{A}$ be the smallest closed subalgebra of $\mathcal{B}(L^p(\mathbb{R}_+))$ that contains the operators I and S_0.

Lemma 2.1 ([14, Lemma 2.7])**.** *If $\alpha \in SOS(\mathbb{R}_+)$, $A \in \mathcal{A}$, then $U_\alpha^{\pm 1}A \simeq AU_\alpha^{\pm 1}$.*

Consider the isometric isomorphism

$$\Phi : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+, d\mu), \quad (\Phi f)(t) := t^{1/p}f(t), \quad t \in \mathbb{R}_+.$$

The following two statements are well known and go back to Duduchava [3, 4] (see also [2, Section 1.10.2], [10, Section 2.1.2], and [31, Sections 4.2.2–4.2.3]).

Lemma 2.2. *The algebra $\mathcal{A}$ is commutative.*

Lemma 2.3. *Let $1 < p < \infty$ and $\gamma \in \mathbb{C}$ be such that $0 < 1/p + \Re\gamma < 1$. The functions s_γ and r_γ, given for $x \in \mathbb{R}$ by*

$$s_\gamma(x) := \coth[\pi(x + i/p + i\gamma)], \quad r_\gamma(x) := 1/\sinh[\pi(x + i/p + i\gamma)],$$

belong to $\mathcal{M}_p(\mathbb{R})$; the operators S_γ and R_γ belong to $\mathcal{A}$; and

$$S_\gamma = \Phi^{-1}\,\mathrm{Co}(s_\gamma)\Phi, \quad R_\gamma = \Phi^{-1}\,\mathrm{Co}(r_\gamma)\Phi.$$

The above lemma is a source for various relations involving operators S_γ and R_γ. We will need the following two identities, which we were unable to find in the literature (although similar results and/or techniques were used, for instance, in [4], [10, p. 50]).

Lemma 2.4. *Let $1 < p < \infty$ and $\gamma, \delta \in \mathbb{C}$ be such that $0 < 1/p + \Re\gamma < 1$ and $0 < 1/p + \Re\delta < 1$. Then*

$$P_\gamma^\pm P_\delta^\pm = \frac{1}{2}P_\gamma^\pm + \frac{1}{2}P_\delta^\pm + \frac{\cos[\pi(\gamma-\delta)]}{4}R_\gamma R_\delta, \quad P_\gamma^- P_\delta^+ = -\frac{e^{i\pi(\delta-\gamma)}}{4}R_\gamma R_\delta. \tag{2.2}$$

Proof. Let $x \in \mathbb{R}$. Put

$$x_\gamma := \pi(x + i/p + i\gamma), \quad x_\delta := \pi(x + i/p + i\delta).$$

It is easy to see that

$$\begin{aligned} s_\gamma(x)s_\delta(x) - 1 &= \frac{\cosh(x_\gamma)\cosh(x_\delta) - \sinh(x_\gamma)\sinh(x_\delta)}{\sinh(x_\gamma)\sinh(x_\delta)} = \frac{\cosh(x_\gamma - x_\delta)}{\sinh(x_\gamma)\sinh(x_\delta)} \\ &= \cosh[i\pi(\gamma-\delta)]r_\gamma(x)r_\delta(x) = \cos[\pi(\gamma-\delta)]r_\gamma(x)r_\delta(x) \end{aligned} \tag{2.3}$$

and

$$s_\gamma(x) - s_\delta(x) = \frac{\cosh(x_\gamma)\sinh(x_\delta) - \cosh(x_\delta)\sinh(x_\gamma)}{\sinh(x_\gamma)\sinh(x_\delta)} = \frac{\sinh(x_\delta - x_\gamma)}{\sinh(x_\gamma)\sinh(x_\delta)}$$
$$= \sinh[i\pi(\delta-\gamma)]r_\gamma(x)r_\delta(x) = -i\sin[\pi(\gamma-\delta)]r_\gamma(x)r_\delta(x). \quad (2.4)$$

Put

$$p_\gamma^\pm(x) := (1 \pm s_\gamma(x))/2, \quad p_\delta^\pm(x) := (1 \pm s_\delta(x))/2. \quad (2.5)$$

Taking into account (2.3), we obtain

$$\begin{aligned} p_\gamma^\pm(x)p_\delta^\pm(x) &= \frac{1}{4}(1 \pm s_\gamma(x) \pm s_\delta(x) + s_\gamma(x)s_\delta(x)) \\ &= \frac{1}{2}p_\gamma^\pm(x) + \frac{1}{2}p_\delta^\pm(x) + \frac{1}{4}(s_\gamma(x)s_\delta(x) - 1) \\ &= \frac{1}{2}p_\gamma^\pm(x) + \frac{1}{2}p_\delta^\pm(x) + \frac{\cos[\pi(\gamma-\delta)]}{4}r_\gamma(x)r_\delta(x). \end{aligned} \quad (2.6)$$

From (2.3)–(2.4) we get

$$\begin{aligned} p_\gamma^-(x)p_\delta^+(x) &= \frac{1}{4}(1 + s_\delta(x) - s_\gamma(x) - s_\gamma(x)s_\delta(x)) \\ &= -\frac{1}{4}(s_\gamma(x) - s_\delta(x)) - \frac{1}{4}(s_\gamma(x)s_\delta(x) - 1) \\ &= -\frac{1}{4}(-i\sin[\pi(\gamma-\delta)]r_\gamma(x)r_\delta(x)) - \frac{\cos[\pi(\gamma-\delta)]}{4}r_\gamma(x)r_\delta(x) \\ &= -\frac{1}{4}(\cos[\pi(\delta-\gamma)] + i\sin[\pi(\delta-\gamma)])r_\gamma(x)r_\delta(x) \\ &= -\frac{e^{i\pi(\delta-\gamma)}}{4}r_\gamma(x)r_\delta(x). \end{aligned} \quad (2.7)$$

Combining identities (2.5)–(2.7) with Lemma 2.3, we arrive at (2.2). ☐

As it is pointed out by the anonymous referee, the results of the above lemma can also be derived by the completely different techniques of Duduchava, Kvergelidze, Tsaava [5, Theorem 2.2] based on the Poincare-Beltrami and Muskhelishvili formulas.

Lemma 2.5. *Let $1 < p < \infty$ and $\mathcal{C}$ be the operator of complex conjugation given on $L^p(\mathbb{R}_+)$ by*

$$(\mathcal{C}f)(t) := \overline{f(t)}, \quad t \in \mathbb{R}_+.$$

If $\alpha \in SOS(\mathbb{R}_+)$ and $\gamma \in \mathbb{C}$ is such that $0 < 1/p + \Re\gamma < 1$, then

$$\mathcal{C}U_\alpha\mathcal{C} = U_\alpha, \quad \mathcal{C}S_\gamma\mathcal{C} = -S_{\overline{\gamma}}, \quad \mathcal{C}P_\gamma^+\mathcal{C} = P_{\overline{\gamma}}^-, \quad \mathcal{C}P_\gamma^-\mathcal{C} = P_{\overline{\gamma}}^+.$$

Proof. The first identity is trivial. The second equality follows from $\overline{1/(\pi i)} = -1/(\pi i)$ and $\overline{(t/\tau)^\gamma} = (t/\tau)^{\overline{\gamma}}$, where $t, \tau \in \mathbb{R}_+$. The remaining identities are immediate corollaries of the second equality. ☐

2.3. **Starting the proof of Theorem 1.2.** Now we are in a position to prove the main result of this section.

Theorem 2.6. *Suppose* $1 < p < \infty$ *and* $\gamma \in \mathbb{C}$ *is such that* $0 < 1/p + \Re\gamma < 1$ *and* $\alpha \in SOS(\mathbb{R}_+)$. *If the operator*

$$A_{\alpha,\gamma} := I + \frac{1}{4}\big[e^{2\pi\Im\gamma}(I - U_\alpha^{-1}) + e^{-2\pi\Im\gamma}(I - U_\alpha)\big]R_\gamma R_{\overline{\gamma}}. \tag{2.8}$$

is Fredholm on the space $L^p(\mathbb{R}_+)$ *and* $\operatorname{Ind} A_{\alpha,\gamma} = 0$, *then the operators* $U_\alpha P_\gamma^+ + P_\gamma^-$ *and* $U_\alpha^{-1} P_{\overline{\gamma}}^+ + P_{\overline{\gamma}}^-$ *are Fredholm on the space* $L^p(\mathbb{R}_+)$ *and*

$$\operatorname{Ind}(U_\alpha P_\gamma^+ + P_\gamma^-) = \operatorname{Ind}(U_\alpha^{-1} P_{\overline{\gamma}}^+ + P_{\overline{\gamma}}^-) = 0. \tag{2.9}$$

Proof. The idea of the proof is borrowed from [18, Theorem 9.4] (see also [19, Theorem 6.1]). From Lemmas 2.2 and 2.3 it follows that the operators P_γ^+, P_γ^-, $P_{\overline{\gamma}}^+$, $P_{\overline{\gamma}}^-$, $R_\gamma, R_{\overline{\gamma}}$ pairwise commute. From this observation and Lemma 2.1 we obtain

$$(U_\alpha P_\gamma^+ + P_\gamma^-)(U_\alpha^{-1} P_{\overline{\gamma}}^+ + P_{\overline{\gamma}}^-) \simeq \widetilde{A}_{\alpha,\gamma}, \tag{2.10}$$

$$(U_\alpha^{-1} P_{\overline{\gamma}}^+ + P_{\overline{\gamma}}^-)(U_\alpha P_\gamma^+ + P_\gamma^-) \simeq \widetilde{A}_{\alpha,\gamma}. \tag{2.11}$$

where

$$\widetilde{A}_{\alpha,\gamma} := P_\gamma^+ P_{\overline{\gamma}}^+ + U_\alpha^{-1} P_\gamma^- P_{\overline{\gamma}}^+ + U_\alpha P_{\overline{\gamma}}^- P_\gamma^+ + P_\gamma^- P_{\overline{\gamma}}^-.$$

Since $\gamma - \overline{\gamma} = 2i\Im\gamma$, we have

$$\cos[\pi(\gamma - \overline{\gamma})] = \frac{e^{2\pi\Im\gamma} + e^{-2\pi\Im\gamma}}{2}, \quad e^{i\pi(\gamma-\overline{\gamma})} = e^{-2\pi\Im\gamma}, \quad e^{i\pi(\overline{\gamma}-\gamma)} = e^{2\pi\Im\gamma}.$$

From Lemma 2.4 and the above identities we get

$$\begin{aligned}
\widetilde{A}_{\alpha,\gamma} &= \left(\frac{1}{2}P_\gamma^+ + \frac{1}{2}P_{\overline{\gamma}}^+ + \frac{\cos[\pi(\gamma-\overline{\gamma})]}{4}R_\gamma R_{\overline{\gamma}}\right) - \frac{e^{i\pi(\overline{\gamma}-\gamma)}}{4}U_\alpha^{-1}R_\gamma R_{\overline{\gamma}} \\
&\quad + \left(\frac{1}{2}P_\gamma^- + \frac{1}{2}P_{\overline{\gamma}}^- + \frac{\cos[\pi(\gamma-\overline{\gamma})]}{4}R_\gamma R_{\overline{\gamma}}\right) - \frac{e^{i\pi(\gamma-\overline{\gamma})}}{4}U_\alpha R_\gamma R_{\overline{\gamma}} \\
&= I + \frac{e^{2\pi\Im\gamma} + e^{-2\pi\Im\gamma}}{4}R_\gamma R_{\overline{\gamma}} - \frac{e^{2\pi\Im\gamma}}{4}U_\alpha^{-1}R_\gamma R_{\overline{\gamma}} - \frac{e^{-2\pi\Im\gamma}}{4}U_\alpha R_\gamma R_{\overline{\gamma}} \\
&= A_{\alpha,\gamma}.
\end{aligned} \tag{2.12}$$

By the hypothesis, $A_{\alpha,\gamma}$ is Fredholm. Therefore there exist $B_r, B_l \in \mathcal{B}(L^p(\mathbb{R}_+))$ such that

$$A_{\alpha,\gamma}B_r \simeq B_l A_{\alpha,\gamma} \simeq I. \tag{2.13}$$

Multiplying (2.10) from the right by B_r and from the left by B_l and taking into account (2.12)–(2.13), we see that $(U_\alpha^{-1} P_{\overline{\gamma}}^+ + P_{\overline{\gamma}}^-)B_r$ is a right regularizer for the operator $U_\alpha P_\gamma^+ + P_\gamma^-$ and $B_l(U_\alpha P_\gamma^+ + P_\gamma^-)$ is a left regularizer for the operator $U_\alpha^{-1} P_{\overline{\gamma}}^+ + P_{\overline{\gamma}}^-$. Analogously, from (2.11)–(2.13) we deduce that $B_l(U_\alpha^{-1} P_{\overline{\gamma}}^+ + P_{\overline{\gamma}}^-)$ is a left regularizer for $U_\alpha P_\gamma^+ + P_\gamma^-$ and $(U_\alpha P_\gamma^+ + P_\gamma^-)B_r$ is a right regularizer for $U_\alpha^{-1} P_{\overline{\gamma}}^+ + P_{\overline{\gamma}}^-$. Thus, both $U_\alpha P_\gamma^+ + P_\gamma^-$ and $U_\alpha^{-1} P_{\overline{\gamma}}^+ + P_{\overline{\gamma}}^-$ are Fredholm.

From this observation, relations (2.10), (2.12), and well known properties of indices of Fredholm operators we derive that

$$\mathrm{Ind}(U_\alpha P_\gamma^+ + P_\gamma^-) + \mathrm{Ind}(U_\alpha^{-1} P_{\overline{\gamma}}^+ + P_{\overline{\gamma}}^-) = \mathrm{Ind}\, A_{\alpha,\gamma} = 0. \tag{2.14}$$

Since the operator of complex conjugation $\mathcal{C}$ is isometric and anti-linear on $L^p(\mathbb{R}_+)$, we have

$$\mathrm{Ind}(U_\alpha P_\gamma^+ + P_\gamma^-) = \mathrm{Ind}[\mathcal{C}(U_\alpha P_\gamma^+ + P_\gamma^-)\mathcal{C}]. \tag{2.15}$$

From Lemma 2.5 we obtain

$$\mathcal{C}(U_\alpha P_\gamma^+ + P_\gamma^-)\mathcal{C} = U_\alpha P_{\overline{\gamma}}^- + P_{\overline{\gamma}}^+ = U_\alpha(U_\alpha^{-1} P_{\overline{\gamma}}^+ + P_{\overline{\gamma}}^-),$$

whence

$$\mathrm{Ind}[\mathcal{C}(U_\alpha P_\gamma^+ + P_\gamma^-)\mathcal{C}] = \mathrm{Ind}\, U_\alpha + \mathrm{Ind}(U_\alpha^{-1} P_{\overline{\gamma}}^+ + P_{\overline{\gamma}}^-) = \mathrm{Ind}(U_\alpha^{-1} P_{\overline{\gamma}}^+ + P_{\overline{\gamma}}^-). \tag{2.16}$$

Combining (2.14)–(2.16), we arrive at (2.9). □

Due to Theorem 2.6, in order to complete the proof of Theorem 1.2, it remains to show that the operator $A_{\alpha,\gamma}$ given by (2.8) is Fredholm and $\mathrm{Ind}\, A_{\alpha,\gamma} = 0$.

3. Mellin pseudodifferential operators and their symbols

3.1. **Melin PDO's: overview.** Mellin pseudodifferential operators are generalizations of Mellin convolution operators. Let $\mathfrak{a}$ be a sufficiently smooth function defined on $\mathbb{R}_+ \times \mathbb{R}$. The Mellin pseudodifferential operator (shortly, Mellin PDO) with symbol $\mathfrak{a}$ is initially defined for smooth functions f of compact support by the iterated integral

$$\begin{aligned}[]
[\mathrm{Op}(\mathfrak{a})f](t) &= [\mathcal{M}^{-1}\mathfrak{a}(t,\cdot)\mathcal{M}f](t) \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} dx \int_{\mathbb{R}_+} \mathfrak{a}(t,x)\left(\frac{t}{\tau}\right)^{ix} f(\tau)\frac{d\tau}{\tau} \quad \text{for} \quad t \in \mathbb{R}_+.
\end{aligned} \tag{3.1}$$

Obviously, if $\mathfrak{a}(t,x) = a(x)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, then the Mellin pseudodifferential operator $\mathrm{Op}(\mathfrak{a})$ becomes the Mellin convolution operator $\mathrm{Op}(\mathfrak{a}) = \mathrm{Co}(a)$.

In 1991 Vladimir Rabinovich [28] proposed to use Mellin pseudodifferential operators (shortly, Mellin PDO's) with C^∞ slowly oscillating symbols to study singular integral operators with slowly oscillating coefficients on L^p spaces. This idea was exploited in a series of papers by Rabinovich and coauthors (see [30, Sections 4.6–4.7] for the complete history up to 2004). Rabinovich stated in [29, Theorem 2.6] a Fredholm criterion for Mellin PDO's with C^∞ slowly oscillating (or slowly varying) symbols on the spaces $L^p(\mathbb{R}_+, d\mu)$ for $1 < p < \infty$. Namely, he considered symbols $\mathfrak{a} \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ such that

$$\sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} \left|(t\partial_t)^j \partial_x^k \mathfrak{a}(t,x)\right|(1+x^2)^{k/2} < \infty \quad \text{for all} \quad j,k \in \mathbb{Z}_+ \tag{3.2}$$

and

$$\lim_{t\to s}\sup_{x\in\mathbb{R}} \left|(t\partial_t)^j \partial_x^k \mathfrak{a}(t,x)\right|(1+x^2)^{k/2} = 0 \text{ for all } \ j \in \mathbb{N},\ k \in \mathbb{Z}_+,\ s \in \{0,\infty\}. \tag{3.3}$$

Here and in what follows ∂_t and ∂_x denote the operators of partial differentiation with respect to t and to x. Notice that (3.2) defines nothing but the Mellin version of the Hörmander class $S^0_{1,0}(\mathbb{R})$ (see, e.g., [11], [24, Chap. 2, Section 1] for the definition of the Hörmander classes $S^m_{\varrho,\delta}(\mathbb{R}^n)$). If $\mathfrak{a}$ satisfies (3.2), then the Mellin PDO $\operatorname{Op}(\mathfrak{a})$ is bounded on the spaces $L^p(\mathbb{R}_+, d\mu)$ for $1 < p < \infty$ (see, e.g., [34, Chap. VI, Proposition 4] for the corresponding Fourier PDO's). Condition (3.3) is the Mellin version of Grushin's definition of slowly varying symbols in the first variable (see, e.g., [9], [24, Chap. 3, Defintion 5.11]).

The above mentioned results have a disadvantage that the smoothness conditions imposed on slowly oscillating symbols are very strong. In particular, they are not applicable directly to the problem we are considering in the paper.

In 2005 Yuri Karlovich [16] (see also [17, 20]) developed a Fredholm theory of Fourier pseudodifferential with slowly oscillating symbols of limited smoothness in the spirit of Sarason's definition [32, p. 820] of slow oscillation adopted in the present paper (much less restrictive than in [29] and in the works mentioned in [30]). Necessary for our purposes results from [16, 17, 20] were translated to the Mellin setting, for instance, in [14] with the aid of the transformation

$$A \mapsto E^{-1}AE,$$

where $A \in \mathcal{B}(L^p(\mathbb{R}))$ and the isometric isomorphism $E : L^p(\mathbb{R}_+, d\mu) \to L^p(\mathbb{R})$ is defined by (2.1). For the convenience of the reader, we reproduce these results here exactly in the same form as they were stated in [14], where more details on their proofs can be found.

3.2. **Boundedness of Mellin PDO's.** Let a be an absolutely continuous function of finite total variation

$$V(a) := \int_{\mathbb{R}} |a'(x)|dx$$

on $\mathbb{R}$. The set $V(\mathbb{R})$ of all absolutely continuous functions of finite total variation on $\mathbb{R}$ becomes a Banach algebra equipped with the norm

$$\|a\|_V := \|a\|_{L^\infty(\mathbb{R})} + V(a).$$

Let $C_b(\mathbb{R}_+, V(\mathbb{R}))$ denote the Banach algebra of all bounded continuous $V(\mathbb{R})$-valued functions on $\mathbb{R}_+$ with the norm

$$\|\mathfrak{a}(\cdot,\cdot)\|_{C_b(\mathbb{R}_+,V(\mathbb{R}))} = \sup_{t\in\mathbb{R}_+} \|\mathfrak{a}(t,\cdot)\|_V.$$

As usual, let $C_0^\infty(\mathbb{R}_+)$ be the set of all infinitely differentiable functions of compact support on $\mathbb{R}_+$.

Theorem 3.1 ([14, Theorem 3.1])**.** *If $\mathfrak{a} \in C_b(\mathbb{R}_+, V(\mathbb{R}))$, then the Mellin pseudodifferential operator $\operatorname{Op}(\mathfrak{a})$, defined for functions $f \in C_0^\infty(\mathbb{R}_+)$ by the iterated integral (3.1), extends to a bounded linear operator on the space $L^p(\mathbb{R}_+, d\mu)$ and there is a number $C_p \in (0,\infty)$ depending only on p such that*

$$\|\operatorname{Op}(\mathfrak{a})\|_{\mathcal{B}(L^p(\mathbb{R}_+,d\mu))} \le C_p \|\mathfrak{a}\|_{C_b(\mathbb{R}_+,V(\mathbb{R}))}.$$

3.3. **Compactness of Mellin PDO's and their semi-commutators.** Consider the Banach subalgebra $SO(\mathbb{R}_+, V(\mathbb{R}))$ of the algebra $C_b(\mathbb{R}_+, V(\mathbb{R}))$ consisting of all $V(\mathbb{R})$-valued functions $\mathfrak{a}$ on $\mathbb{R}_+$ that slowly oscillate at 0 and ∞, that is,

$$\lim_{r\to s}\max_{t,\tau\in[r,2r]}\|\mathfrak{a}(t,\cdot)-\mathfrak{a}(\tau,\cdot)\|_{L^\infty(\mathbb{R})}=0,\quad s\in\{0,\infty\}.$$

Let $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ be the Banach algebra of all $V(\mathbb{R})$-valued functions $\mathfrak{a}$ in the algebra $SO(\mathbb{R}_+, V(\mathbb{R}))$ such that

$$\lim_{|h|\to 0}\sup_{t\in\mathbb{R}_+}\left\|\mathfrak{a}(t,\cdot)-\mathfrak{a}^h(t,\cdot)\right\|_V=0$$

where $\mathfrak{a}^h(t,x):=\mathfrak{a}(t,x+h)$ for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$.

Theorem 3.2 ([14, Theorem 3.2])**.** *If* $\mathfrak{a}\in\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ *and*

$$\lim_{\ln^2 t+x^2\to\infty}\mathfrak{a}(t,x)=0,$$

then $\mathrm{Op}(\mathfrak{a})\in\mathcal{K}(L^p(\mathbb{R}_+,d\mu))$.

Theorem 3.3 ([14, Theorem 3.3])**.** *If* $\mathfrak{a},\mathfrak{b}\in\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$*, then*

$$\mathrm{Op}(\mathfrak{a})\,\mathrm{Op}(\mathfrak{b})\simeq\mathrm{Op}(\mathfrak{a}\mathfrak{b}).$$

3.4. **Fredholmness of Mellin PDO's.** For a unital commutative Banach algebra $\mathfrak{A}$, let $M(\mathfrak{A})$ denote its maximal ideal space. Identifying the points $t\in\overline{\mathbb{R}}_+$ with the evaluation functionals $t(f)=f(t)$ for $f\in C(\overline{\mathbb{R}}_+)$, we get

$$M(C(\overline{\mathbb{R}}_+))=\overline{\mathbb{R}}_+.$$

Consider the fibers

$$M_s(SO(\mathbb{R}_+)):=\left\{\xi\in M(SO(\mathbb{R}_+)):\xi|_{C(\overline{\mathbb{R}}_+)}=s\right\}$$

of the maximal ideal space $M(SO(\mathbb{R}_+))$ over the points $s\in\{0,\infty\}$. By [20, Proposition 2.1], the set

$$\Delta:=M_0(SO(\mathbb{R}_+))\cup M_\infty(SO(\mathbb{R}_+))$$

coincides with $(\mathrm{clos}_{SO^*}\mathbb{R}_+)\setminus\mathbb{R}_+$ where $\mathrm{clos}_{SO^*}\mathbb{R}_+$ is the weak-star closure of $\mathbb{R}_+$ in the dual space of $SO(\mathbb{R}_+)$. Then $M(SO(\mathbb{R}_+))=\Delta\cup\mathbb{R}_+$.

Let $\mathfrak{a}\in\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$. For every $t\in\mathbb{R}_+$, the function $\mathfrak{a}(t,\cdot)$ belongs to $V(\mathbb{R})$ and, therefore, has finite limits at $\pm\infty$, which will be denoted by $\mathfrak{a}(t,\pm\infty)$. Now we explain how to extend the function $\mathfrak{a}$ to $\Delta\times\overline{\mathbb{R}}$.

Lemma 3.4 ([14, Lemma 3.5])**.** *Let* $s\in\{0,\infty\}$ *and* $\mathfrak{a}\in\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$*. For each* $\xi\in M_s(SO(\mathbb{R}_+))$ *there is a sequence* $\{t_n\}\subset\mathbb{R}_+$ *such that* $t_n\to s$ *as* $n\to\infty$ *and a function* $\mathfrak{a}(\xi,\cdot)\in V(\mathbb{R})$ *such that*

$$\mathfrak{a}(\xi,x)=\lim_{n\to\infty}\mathfrak{a}(t_n,x)\quad\textit{for every}\quad x\in\overline{\mathbb{R}}.$$

To study the Fredholmness of Mellin pseudodifferential operators, we need the Banach algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ consisting of all functions $\mathfrak{a}$ belonging to $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ and such that

$$\lim_{m\to\infty}\sup_{t\in\mathbb{R}_+}\int_{\mathbb{R}\setminus[-m,m]}|\partial_x\mathfrak{a}(t,x)|\,dx=0.$$

Theorem 3.5 ([14, Theorem 3.6]). *If $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$, then the Mellin pseudodifferential operator $\mathrm{Op}(\mathfrak{a})$ is Fredholm on the space $L^p(\mathbb{R}_+, d\mu)$ if and only if*

$$\mathfrak{a}(t, \pm\infty) \neq 0 \quad \text{for all} \quad t \in \mathbb{R}_+, \qquad \mathfrak{a}(\xi, x) \neq 0 \quad \text{for all} \quad (\xi, x) \in \Delta \times \overline{\mathbb{R}}. \tag{3.4}$$

In the case of Fredholmness

$$\mathrm{Ind}\,\mathrm{Op}(\mathfrak{a}) = \lim_{\tau\to+\infty} \frac{1}{2\pi} \{ \arg \mathfrak{a}(t, x) \}_{(t,x)\in\partial\Pi_\tau},$$

where $\Pi_\tau = [\tau^{-1}, \tau] \times \overline{\mathbb{R}}$ and $\{\arg \mathfrak{a}(t, x)\}_{(t,x)\in\partial\Pi_\tau}$ denotes the increment of the function $\arg \mathfrak{a}(t, x)$ when the point (t, x) traces the boundary $\partial\Pi_\tau$ of Π_τ counterclockwise.

This result follows from [20, Theorem 4.3]. Note that for infinitely differentiable slowly oscillating symbols such result was obtained earlier in [29, Theorem 2.6].

4. The operator $A_{\alpha,\gamma}$ is similar to a Mellin PDO $\mathrm{Op}(\mathfrak{g}_{\omega,\gamma})$ with symbol in the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$

In this section we show that the operator $A_{\alpha,\gamma}$ given by (2.8) satisfies the relation $A_{\alpha,\gamma} \simeq \Phi^{-1}\,\mathrm{Op}(\mathfrak{g}_{\omega,\gamma})\Phi$, where $\mathfrak{g}_{\omega,\gamma}$ is an explicitly given function in the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. Hence the operator $A_{\alpha,\gamma}$ is Fredholm if and only if $\mathrm{Op}(\mathfrak{g}_{\omega,\gamma})$ is Fredholm and $\mathrm{Ind}\, A_{\alpha,\gamma} = \mathrm{Ind}\,\mathrm{Op}(\mathfrak{g}_{\omega,\gamma})$.

4.1. Some important functions in the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

The following statement for $\gamma = 0$ was proved in [12, Lemma 7.1] and [14, Lemma 4.2]. For $\gamma \in \mathbb{C}$ satisfying $0 < 1/p + \Re\gamma < 1$ the proof is the same.

Lemma 4.1. *Suppose $f \in SO(\mathbb{R}_+)$, $1 < p < \infty$, and $\gamma \in \mathbb{C}$ is such that $0 < 1/p + \Re\gamma < 1$. Then the functions*

$$\mathfrak{f}(t, x) := f(t), \quad \mathfrak{s}_\gamma(t, x) := s_\gamma(x), \quad \mathfrak{r}_\gamma(t, x) := r_\gamma(x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R},$$

belong to the Banach algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

The next lemma for $\gamma = 0$ was proved in [14, Lemma 4.3] (see also [12, Lemmas 7.3–7.4]). For arbitrary $\gamma \in \mathbb{C}$ satisfying $0 < 1/p + \Re\gamma < 1$ the arguments remain unaltered.

Lemma 4.2. *Suppose $\omega \in SO(\mathbb{R}_+)$ is a real-valued function, $1 < p < \infty$, and $\gamma \in \mathbb{C}$ is such that $0 < 1/p + \Re\gamma < 1$. Then the function*

$$\mathfrak{b}_{\omega,\gamma}(t, x) := e^{i\omega(t)x} r_\gamma(x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R},$$

belongs to the Banach algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ and there is a positive constant $C(p, \gamma)$ depending only on p and γ such that

$$\|\mathfrak{b}_{\omega,\gamma}\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))} \le C(p, \gamma) \left(1 + \sup_{t\in\mathbb{R}_+} |\omega(t)|\right).$$

4.2. **Operators $U_\alpha R_\gamma$ and $U_\alpha^{-1} R_\gamma$.** Let $C^1(\mathbb{R}_+)$ denote the set of all continuously differentiable functions on $\mathbb{R}_+$. We start with the important characterization of orientation-preserving slowly oscillating shifts.

Lemma 4.3 ([12, Lemma 2.2])**.** *An orientation-preserving shift $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to $SOS(\mathbb{R}_+)$ if and only if*

$$\alpha(t) = te^{\omega(t)}, \quad t \in \mathbb{R}_+,$$

for some real-valued function $\omega \in SO(\mathbb{R}_+) \cap C^1(\mathbb{R}_+)$ such that $\psi(t) := t\omega'(t)$ also belongs to $SO(\mathbb{R}_+)$ and

$$\inf_{t\in\mathbb{R}_+} \big(1 + t\omega'(t)\big) > 0.$$

The following statement is crucial for our analysis.

Lemma 4.4. *Suppose $1 < p < \infty$ and $\gamma \in \mathbb{C}$ is such that $0 < 1/p + \Re\gamma < 1$. Let $\alpha \in SOS(\mathbb{R}_+)$ and U_α be the associated isometric shift operator on $L^p(\mathbb{R}_+)$. Then the operator $U_\alpha R_\gamma$ can be realized as the Mellin pseudodifferential operator:*

$$U_\alpha R_\gamma = \Phi^{-1} \operatorname{Op}(\mathfrak{c}_{\omega,\gamma})\Phi,$$

where the function $\mathfrak{c}_{\omega,\gamma}$, given for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ by

$$\mathfrak{c}_{\omega,\gamma}(t, x) := (1 + t\omega'(t))^{1/p} e^{i\omega(t)x} r_\gamma(x) \quad \text{with} \quad \omega(t) := \log[\alpha(t)/t], \tag{4.1}$$

belongs to the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

Proof. We follow the proof of [14, Lemma 4.4], where this statement was proved for $\gamma = 1/y - 1/p$ with $y \in (1, \infty)$ (see also [12, Lemma 8.3]). By Lemma 4.3, $\alpha(t) = te^{\omega(t)}$, where $\omega \in SO(\mathbb{R}_+) \cap C^1(\mathbb{R}_+)$ is a real-valued function. Hence

$$\alpha'(t) = \Omega(t)e^{\omega(t)}, \quad \text{where} \quad \Omega(t) := 1 + t\omega'(t), \quad t \in \mathbb{R}_+. \tag{4.2}$$

Assume that $f \in C_0^\infty(\mathbb{R}_+)$. Taking into account (4.2), we have for $t \in \mathbb{R}_+$,

$$\begin{aligned}
(\Phi U_\alpha R_\gamma \Phi^{-1} f)(t) &= \frac{(\alpha'(t))^{1/p}}{\pi i} \int_{\mathbb{R}_+} \left(\frac{\alpha(t)}{\tau}\right)^\gamma \frac{f(\tau)(t/\tau)^{1/p}}{\tau + \alpha(t)}\, d\tau \\
&= \frac{(\Omega(t))^{1/p} e^{\omega(t)/p}}{\pi i} \int_{\mathbb{R}_+} \left(\frac{te^{\omega(t)}}{\tau}\right)^\gamma \frac{f(\tau)(t/\tau)^{1/p}}{1 + e^{\omega(t)}(t/\tau)} \frac{d\tau}{\tau} \\
&= \frac{(\Omega(t))^{1/p} e^{\omega(t)(1/p+\gamma)}}{\pi i} \int_{\mathbb{R}_+} \frac{f(\tau)(t/\tau)^{1/p+\gamma}}{1 + e^{\omega(t)}(t/\tau)} \frac{d\tau}{\tau} \\
&= (\Omega(t))^{1/p} (I_\gamma f)(t),
\end{aligned} \tag{4.3}$$

where

$$(I_\gamma f)(t) := \frac{e^{\omega(t)(1/p+\gamma)}}{\pi i} \int_{\mathbb{R}_+} \frac{f(\tau)(t/\tau)^{1/p+\gamma}}{1 + e^{\omega(t)}(t/\tau)} \frac{d\tau}{\tau}. \tag{4.4}$$

From [6, formula 6.2.6] or [8, formula 3.194.4] it follows that for $k > 0$, $\gamma \in \mathbb{C}$ such that $0 < 1/p + \Re\gamma < 1$, and $x \in \mathbb{R}$,

$$\begin{aligned}\frac{1}{\pi i}\int_{\mathbb{R}_+}\frac{t^{1/p+\gamma}}{1+kt}t^{-ix}\frac{dt}{t} &= \frac{1}{\pi i}\cdot\frac{\pi}{k^{1/p+\gamma-ix}}\cdot\frac{1}{\sin[\pi(1/p+\gamma-ix)]}\\ &= \frac{1}{k^{1/p+\gamma-ix}}\cdot\frac{1}{\sinh[\pi(x+i/p+i\gamma)]}\\ &= e^{i(x+i/p+i\gamma)\log k}r_\gamma(x).\end{aligned}$$

Taking the inverse Mellin transform, we get

$$\frac{1}{\pi i}\cdot\frac{t^{1/p+\gamma}}{1+kt} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x+i/p+i\gamma)\log k}r_\gamma(x)t^{ix}\,dx. \tag{4.5}$$

From (4.4)–(4.5) with $k = e^{\omega(t)}$ we obtain

$$\begin{aligned}(I_\gamma f)(t) &= \frac{e^{\omega(t)(1/p+\gamma)}}{2\pi}\int_{\mathbb{R}_+}\left(\int_{\mathbb{R}} e^{i\omega(t)(x+i/p+i\gamma)}r_\gamma(x)\left(\frac{t}{\tau}\right)^{ix}dx\right)f(\tau)\,\frac{d\tau}{\tau}\\ &= \frac{1}{2\pi}\int_{\mathbb{R}_+}\left(\int_{\mathbb{R}} e^{i\omega(t)x}r_\gamma(x)\left(\frac{t}{\tau}\right)^{ix}dx\right)f(\tau)\,\frac{d\tau}{\tau}\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}dx\int_{\mathbb{R}_+} e^{i\omega(t)x}r_\gamma(x)\left(\frac{t}{\tau}\right)^{ix}f(\tau)\frac{d\tau}{\tau}.\end{aligned} \tag{4.6}$$

From (4.3) and (4.6) we obtain for $f \in C_0^\infty(\mathbb{R}_+)$,

$$\Phi U_\alpha R_y \Phi^{-1} f = \operatorname{Op}(\mathfrak{c}_{\omega,\gamma})f. \tag{4.7}$$

By Lemma 4.3, the function Ω belongs to $SO(\mathbb{R}_+)$. Then $\Omega^{1/p}$ is also in $SO(\mathbb{R}_+)$. Therefore, from Lemmas 4.1 and 4.2 it follows that the function $\mathfrak{c}_{\omega,\gamma}$ belongs to the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R})) \subset C_b(\mathbb{R}_+, V(\mathbb{R}))$. Then Theorem 3.1 implies that $\operatorname{Op}(\mathfrak{c}_{\omega,\gamma})$ extends to a bounded operator on $L^p(\mathbb{R}_+, d\mu)$. Therefore, from (4.7) we obtain $\Phi U_\alpha R_\gamma \Phi^{-1} = \operatorname{Op}(\mathfrak{c}_{\omega,\gamma})$, which completes the proof. $\square$

From the above lemma and Theorem 3.2, making minor modifications in the proof of [14, Lemma 4.5], we can get the following.

Lemma 4.5. *Suppose $1 < p < \infty$ and $\gamma \in \mathbb{C}$ is such that $0 < 1/p + \Re\gamma < 1$. Let $\alpha \in SOS(\mathbb{R}_+)$ and U_α be the associated isometric shift operator on $L^p(\mathbb{R}_+)$. Then the operators $U_\alpha R_\gamma$ and $U_\alpha^{-1} R_\gamma$ can be realized as the Mellin pseudodifferential operators up to compact operators:*

$$U_\alpha^{\pm 1} R_\gamma \simeq \Phi^{-1}\operatorname{Op}(\mathfrak{c}^\pm_{\omega,\gamma})\Phi,$$

where the functions $\mathfrak{c}^\pm_{\omega,\gamma}$, given for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ by

$$\mathfrak{c}^\pm_{\omega,\gamma}(t,x) := e^{\pm i\omega(t)x}r_\gamma(x) \quad \textit{with} \quad \omega(t) := \log[\alpha(t)/t], \tag{4.8}$$

belong to the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

4.3. **Similarity relation.** We are ready to prove the main result of this section.

Theorem 4.6. *Suppose $1 < p < \infty$ and $\gamma \in \mathbb{C}$ is such that $0 < 1/p + \Re\gamma < 1$. Let $\alpha \in SOS(\mathbb{R}_+)$ and $A_{\alpha,\gamma}$ be the operator given by (2.8). Then*

$$A_{\alpha,\gamma} \simeq \Phi^{-1} \operatorname{Op}(\mathfrak{g}_{\omega,\gamma})\Phi,$$

where the function $\mathfrak{g}_{\omega,\gamma}$, given for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ by

$$\mathfrak{g}_{\omega,\gamma}(t,x) := 1 + \frac{1}{4}\big[e^{2\pi\Im\gamma}(1 - e^{-i\omega(t)x}) + e^{-2\pi\Im\gamma}(1 - e^{i\omega(t)x})\big] r_\gamma(x) r_{\overline{\gamma}}(x) \tag{4.9}$$

with $\omega(t) := \log[\alpha(t)/t]$, belongs to the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

Proof. From Lemmas 2.3 and 4.1 we know that the functions $\mathfrak{r}_\gamma(t,x) = r_\gamma(x)$ and $\mathfrak{r}_{\overline{\gamma}}(t,x) = r_{\overline{\gamma}}(x)$ with $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ belong to $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ and

$$R_\gamma = \Phi^{-1} \operatorname{Co}(r_\gamma)\Phi = \Phi^{-1} \operatorname{Op}(\mathfrak{r}_\gamma)\Phi, \;\; R_{\overline{\gamma}} = \Phi^{-1} \operatorname{Co}(r_{\overline{\gamma}})\Phi = \Phi^{-1} \operatorname{Op}(\mathfrak{r}_{\overline{\gamma}})\Phi. \tag{4.10}$$

From the first identity in (4.10) and Lemma 4.5 it follows that

$$(I - U_\alpha^{\pm 1})R_\gamma \simeq \Phi^{-1} \operatorname{Op}(1 - \mathfrak{c}^{\pm}_{\omega,\gamma})\Phi, \tag{4.11}$$

where the functions $\mathfrak{c}^{\pm}_{\omega,\gamma}$ given by (4.8) belong to the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. Then the function $\mathfrak{g}_{\omega,\gamma}$ given by (4.9) belongs to the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. Combining (2.8) and (4.9)–(4.11) with Theorem 3.3, we arrive at

$$\begin{aligned}
A_{\alpha,\gamma} &\simeq I + \frac{1}{4}\big[e^{2\pi\Im\gamma}\Phi^{-1} \operatorname{Op}(1 - \mathfrak{c}^{-}_{\omega,\gamma}) \operatorname{Op}(\mathfrak{r}_{\overline{\gamma}})\Phi + e^{-2\pi\Im\gamma}\Phi^{-1} \operatorname{Op}(1 - \mathfrak{c}^{+}_{\omega,\gamma}) \operatorname{Op}(\mathfrak{r}_{\overline{\gamma}})\Phi\big] \\
&\simeq I + \frac{1}{4}\big[e^{2\pi\Im\gamma}\Phi^{-1} \operatorname{Op}(1 - \mathfrak{c}^{-}_{\omega,\gamma}\mathfrak{r}_{\overline{\gamma}})\Phi + e^{-2\pi\Im\gamma}\Phi^{-1} \operatorname{Op}(1 - \mathfrak{c}^{+}_{\omega,\gamma}\mathfrak{r}_{\overline{\gamma}})\Phi\big] \\
&= \Phi^{-1} \operatorname{Op}(\mathfrak{g}_{\omega,\gamma})\Phi,
\end{aligned}$$

which completes the proof. ☐

Corollary 4.7. *Suppose $1 < p < \infty$ and $\gamma \in \mathbb{C}$ is such that $0 < 1/p + \Re\gamma < 1$ and $\alpha \in SOS(\mathbb{R}_+)$. Let $\omega(t) := \log[\alpha(t)/t]$ for $t \in \mathbb{R}_+$ and the function $\mathfrak{g}_{\omega,\gamma}$ be given for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ by (4.9). Then the operator $A_{\alpha,\gamma}$ given by (2.8) is Fredholm on the space $L^p(\mathbb{R}_+)$ if and only if the operator $\operatorname{Op}(\mathfrak{g}_{\omega,\gamma})$ is Fredholm on the space $L^p(\mathbb{R}_+, d\mu)$ and $\operatorname{Ind} A_{\alpha,\gamma} = \operatorname{Ind} \operatorname{Op}(\mathfrak{g}_{\omega,\gamma})$.*

Thus, in order to complete the proof of Theorem 1.2, it remains to show that the operator $\operatorname{Op}(\mathfrak{g}_{\omega,\gamma})$ is Fredholm and its index is equal to zero.

5. Fredholm theory for the operator $\operatorname{Op}(\mathfrak{g}_{\omega,\gamma})$

In this section we prove that the Mellin pseudodifferential operator with symbol given by (4.9) is Fredholm and its index is equal to zero. To do this, we employ Theorem 3.5.

5.1. **Technical lemma.** For the calculation of the index of operator $\operatorname{Op}(\mathfrak{g}_{\omega,\gamma})$, we are going to use a homotopic argument, linking the function $\mathfrak{g}_{\omega,\gamma}$ and the function that equals identically one via a family of functions $\widetilde{\mathfrak{g}}_{\omega,\gamma}(\cdot,\cdot,\theta)$ with $\theta\in[0,1]$ described below.

Lemma 5.1. *Let $1<p<\infty$, $\gamma\in\mathbb{C}$ be such that $0<1/p+\Re\gamma<1$, and ω be a real-valued function in $SO(\mathbb{R}_+)$. If*

$$\widetilde{\mathfrak{g}}_{\omega,\gamma}(t,x,\theta):=1+\frac{1}{4}\big[e^{2\pi\Im\gamma}(1-e^{-i\theta\omega(t)x})+e^{-2\pi\Im\gamma}(1-e^{i\theta\omega(t)x})\big]r_\gamma(x)r_{\overline{\gamma}}(x) \tag{5.1}$$

for $(t,x,\theta)\in\mathbb{R}_+\times\mathbb{R}\times[0,1]$,

$$\widetilde{\mathfrak{g}}_{\omega,\gamma}(t,\pm\infty,\theta):=\lim_{x\to\pm\infty}\widetilde{\mathfrak{g}}_{\omega,\gamma}(t,x,\theta)=1$$

for $(t,\theta)\in\mathbb{R}_+\times[0,1]$, and (1.2) is fulfilled, then there is a constant $c=c(p,\gamma,\omega)>0$, depending only on p, γ, and ω, and such that

$$|\widetilde{\mathfrak{g}}_{\omega,\gamma}(t,x,\theta)|\ge c \quad \text{for all} \quad (t,x,\theta)\in\mathbb{R}_+\times\overline{\mathbb{R}}\times[0,1]. \tag{5.2}$$

Proof. We follow the main lines of the proof of [14, Lemma 5.1]. It is easy to see that for every $(t,x)\in\mathbb{R}_+\times\mathbb{R}$,

$$\begin{aligned}\frac{4}{r_\gamma(x)r_{\overline{\gamma}}(x)}&=4\sinh[\pi(x+i/p+i\gamma)]\sinh[\pi(x+i/p+i\overline{\gamma})]\\&=e^{2\pi(x+i/p+i\Re\gamma)}+e^{-2\pi(x+i/p+i\Re\gamma)}-e^{2\pi\Im\gamma}-e^{-2\pi\Im\gamma}.\end{aligned}\tag{5.3}$$

From (5.1) and (5.3) we deduce that the function $\widetilde{\mathfrak{g}}_{\omega,\gamma}$ can be rewritten as follows:

$$\begin{aligned}\widetilde{\mathfrak{g}}_{\omega,\gamma}(t,x,\theta)&=\big[e^{2\pi(x+i/p+i\Re\gamma)}+e^{-2\pi(x+i/p+i\Re\gamma)}-e^{2\pi\Im\gamma}-e^{-2\pi\Im\gamma}\\&\quad+e^{2\pi\Im\gamma}(1-e^{-i\theta\omega(t)x})+e^{-2\pi\Im\gamma}(1-e^{i\theta\omega(t)x})\big]\frac{r_\gamma(x)r_{\overline{\gamma}}(x)}{4}\\&=\Big[\frac{e^{2\pi(x+i/p+i\Re\gamma)}+e^{-2\pi(x+i/p+i\Re\gamma)}}{2}-\frac{e^{i\theta\omega(t)x-2\pi\Im\gamma}+e^{2\pi\Im\gamma-i\theta\omega(t)x}}{2}\Big]\frac{r_\gamma(x)r_{\overline{\gamma}}(x)}{2}\\&=\frac{\cosh[2\pi(x+i/p+i\Re\gamma)]-\cosh[i\theta\omega(t)x-2\pi\Im\gamma]}{2}r_\gamma(x)r_{\overline{\gamma}}(x)\\&=\frac{\sinh[\pi(x-\Im\gamma+i/p+i\Re\gamma)+i\theta\omega(t)x/2]}{\sinh[\pi(x-\Im\gamma+i/p+i\Re\gamma)]}\\&\quad\times\frac{\sinh[\pi(x+\Im\gamma+i/p+i\Re\gamma)-i\theta\omega(t)x/2]}{\sinh[\pi(x+\Im\gamma+i/p+i\Re\gamma)]},\end{aligned}$$

whence, by [1, formula 4.5.54],

$$\begin{aligned}|\widetilde{\mathfrak{g}}_{\omega,\gamma}(t,x,\theta)|&=\sqrt{\frac{\sinh^2[\pi(x-\Im\gamma)]+\sin^2[\pi(1/p+\Re\gamma)+\theta\omega(t)x/2]}{\sinh^2[\pi(x-\Im\gamma)]+\sin^2[\pi(1/p+\Re\gamma)]}}\\&\quad\times\sqrt{\frac{\sinh^2[\pi(x+\Im\gamma)]+\sin^2[\pi(1/p+\Re\gamma)-\theta\omega(t)x/2]}{\sinh^2[\pi(x+\Im\gamma)]+\sin^2[\pi(1/p+\Re\gamma)]}}\\&=:\mathfrak{g}_+(t,x,\theta)\mathfrak{g}_-(t,x,\theta).\end{aligned}\tag{5.4}$$

Thus, we need to estimate from below the functions

$$\mathfrak{g}_\pm(t,x\pm\Im\gamma,\theta)=\sqrt{\frac{\sinh^2(\pi x)+\sin^2[\pi(1/p+\Re\gamma)+\theta\omega(t)\Im\gamma/2\pm\theta\omega(t)x/2]}{\sinh^2(\pi x)+\sin^2[\pi(1/p+\Re\gamma)]}}.$$

From $0<1/p+\Re\gamma<1$ and (1.2) it follows that

$$\mathcal{I}:=\frac{1}{p}+\Re\gamma+\min\left(0,\frac{1}{2\pi}\inf_{t\in\mathbb{R}_+}(\omega(t)\Im\gamma)\right)>0,$$
$$\mathcal{S}:=\frac{1}{p}+\Re\gamma+\max\left(0,\frac{1}{2\pi}\sup_{t\in\mathbb{R}_+}(\omega(t)\Im\gamma)\right)<1.$$

It is easy to see that

$$\mathcal{I}\le\frac{1}{p}+\Re\gamma+\frac{\theta\omega(t)\Im\gamma}{2\pi}\le\mathcal{S},\quad t\in\mathbb{R}_+,\quad\theta\in[0,1]. \tag{5.5}$$

Let

$$M(\omega):=\sup_{t\in\mathbb{R}_+}|\omega(t)|$$

and $q\in[2,+\infty)$ be defined by $1/q:=\min(\mathcal{I},1-\mathcal{S})$. Then

$$\left|\frac{\theta\omega(t)x}{2}\right|\le\frac{M(\omega)}{2}|x|\le\frac{\pi}{2q}\quad\text{if}\quad|x|\le\frac{\pi}{qM(\omega)},\quad t\in\mathbb{R}_+,\quad\theta\in[0,1]. \tag{5.6}$$

Further, from (5.5)–(5.6) we obtain

$$\begin{aligned}\frac{\pi}{2q}&=\pi\frac{1}{q}-\frac{\pi}{2q}\le\pi\mathcal{I}-\frac{\pi}{2q}\le\pi\left(\frac{1}{p}+\Re\gamma\right)+\frac{\theta\omega(t)\Im\gamma}{2}\pm\frac{\theta\omega(t)x}{2}\\&\le\pi\mathcal{S}+\frac{\pi}{2q}\le\pi\left(1-\frac{1}{q}\right)+\frac{\pi}{2q}=\pi-\frac{\pi}{2q}\end{aligned}$$

for all $|x|\le\frac{\pi}{qM(\omega)}$, $t\in\mathbb{R}_+$, and $\theta\in[0,1]$. Put

$$J_\pm:=\left[-\frac{\pi}{qM(\omega)}\pm\Im\gamma,\frac{\pi}{qM(\omega)}\pm\Im\gamma\right].$$

Hence

$$\frac{\pi}{2q}\le\pi\left(\frac{1}{\pi}+\Re\gamma\right)\pm\frac{\theta\omega(t)x}{2}\le\pi-\frac{\pi}{2q}\quad\text{if}\quad x\in J_\pm,\quad t\in\mathbb{R}_+,\quad\theta\in[0,1],$$

whence

$$\sin^2\left[\pi\left(\frac{1}{\pi}+\Re\gamma\right)\pm\frac{\theta\omega(t)x}{2}\right]\ge\sin^2\left(\frac{\pi}{2q}\right),\ x\in J_\pm,\ t\in\mathbb{R}_+,\ \theta\in[0,1]. \tag{5.7}$$

Since the functions $\varphi_\pm(x)=\sinh^2[\pi(x\pm\Im\gamma)]$ are increasing on $[\mp\Im\gamma,+\infty)$ and are decreasing on $(-\infty,\mp\Im\gamma]$, taking into account (5.7), we obtain

$$\mathfrak{g}_\pm(t,x,\theta)\ge\frac{\sin[\pi/(2q)]}{\sqrt{\sinh^2[\pi^2/(qM(\omega))]+\sin^2[\pi(1/p+\Re\gamma)]}}=:c_1(p,\gamma,\omega)>0 \tag{5.8}$$

for all $x \in J_\pm$, $t \in \mathbb{R}_+$, and $\theta \in [0,1]$. On the other hand,

$$\begin{aligned} \mathfrak{g}_\pm(t,x,\theta) &\geq \sqrt{\frac{\sinh^2[\pi(x \mp \Im\gamma)]}{\sinh^2[\pi(x \mp \Im\gamma)] + \sin^2[\pi(1/p+\Re\gamma)]}} \\ &= \left(1 + \sin^2[\pi(1/p+\Re\gamma)] \sinh^{-2}[\pi(x \mp \Im\gamma)]\right)^{-1/2} \\ &\geq \left(1 + \sin^2[\pi(1/p+\Re\gamma)] \sinh^{-2}[\pi^2/(qM(\omega))]\right)^{-1/2} \\ &=: c_2(p,\gamma,\omega) > 0 \end{aligned} \tag{5.9}$$

for all $x \in \mathbb{R} \setminus J_\pm$, $t \in \mathbb{R}_+$, and $\theta \in [0,1]$. Combining (5.4) and (5.8)–(5.9), we arrive at (5.2) with

$$c := c(p,\gamma,\omega) = \left(\min(c_1(p,\gamma,\omega), c_2(p,\gamma,\omega))\right)^2,$$

which completes the proof. □

5.2. **Fredholmness and index of the operator** $\mathrm{Op}(\mathfrak{g}_{\omega,\gamma})$**.** Now we are ready to finish the proof of Theorem 1.2.

Theorem 5.2. *Suppose $1 < p < \infty$ and $\gamma \in \mathbb{C}$ is such that $0 < 1/p + \Re\gamma < 1$. Let $\omega \in SO(\mathbb{R}_+)$ be a real-valued function and the function $\mathfrak{g}_{\omega,\gamma}$ be given for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ by (4.9). If (1.2) holds, then the operator $\mathrm{Op}(\mathfrak{g}_{\omega,\gamma})$ is Fredholm on the space $L^p(\mathbb{R}_+, d\mu)$ and $\mathrm{Ind}\,\mathrm{Op}(\mathfrak{g}_{\omega,\gamma}) = 0$.*

Proof. The proof is developed exactly as in [14, Theorem 5.2]. In view of Lemmas 4.1 and 4.2, the function $\mathfrak{g}_{\omega,\gamma}$ belongs to $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. From (4.9) and Lemmas 3.4 and 5.1 it follows that

$$\mathfrak{g}_{\omega,\gamma}(t,\pm\infty) = 1 \neq 0 \quad \text{for all} \quad t \in \mathbb{R}_+, \qquad \mathfrak{g}_{\omega,\gamma}(\xi,x) \neq 0 \quad \text{for all} \quad (\xi,x) \in \Delta \times \overline{\mathbb{R}}.$$

Thus, by Theorem 3.5, the operator $\mathrm{Op}(\mathfrak{g}_{\omega,\gamma})$ is Fredholm on $L^p(\mathbb{R}_+, d\mu)$.

For $\tau > 1$, consider $\Pi_\tau := [\tau^{-1}, \tau] \times \overline{\mathbb{R}}$. Since the function $\widetilde{\mathfrak{g}}_{\omega,\gamma}$ given by (5.1) is continuous and separated from 0 for all $(t,x,\theta) \in \mathbb{R}_+ \times \overline{\mathbb{R}} \times [0,1]$ in view of Lemma 5.1, we conclude that $\{\arg \widetilde{\mathfrak{g}}_{\omega,\gamma}(t,x,\theta)\}_{(t,x)\in\partial\Pi_\tau}$ does not depend on $\theta \in [0,1]$. Consequently,

$$\{\arg \mathfrak{g}_{\omega,\gamma}(t,x)\}_{(t,x)\in\partial\Pi_\tau} = \{\arg \widetilde{\mathfrak{g}}_{\omega,\gamma}(t,x,0)\}_{(t,x)\in\partial\Pi_\tau} = 0. \tag{5.10}$$

By Theorem 3.5 and (5.10),

$$\mathrm{Ind}\,\mathrm{Op}(\mathfrak{g}_{\omega,\gamma}) = \lim_{\tau\to\infty} \frac{1}{2\pi} \{\arg \mathfrak{g}_{\omega,\gamma}(t,x)\}_{(t,x)\in\partial\Pi_\tau} = 0,$$

which completes the proof of the theorem. □

Finally, Theorem 1.2 follows from Theorem 5.2, Corollary 4.7, and Theorem 2.6.

Acknowledgement. This work was partially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project PEst-OE/MAT/UI0297/2014 (Centro de Matemática e Aplicações). The authors would like to thank the anonymous referee for useful remarks and for informing about the work [5].

References

1. M. Abramowitz and I.A. Stegun (Eds) *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 10th printing, with corrections. National Bureau of Standards. A Wiley-Interscience Publication. John Wiley & Sons, New York etc., 1972.
2. V.D. Didenko and B. Silbermann, *Approximation of Additive Convolution-Like Operators. Real C*-Algebra Approach*, Frontiers in Mathematics, Birkhäuser, Basel, 2008.
3. R. Duduchava, *Integral Equations with Fixed Singularities*, Teubner Verlagsgesellschaft, Leipzig, 1979.
4. R. Duduchava, *On algebras generated by convolutions and discontinuous functions*, Integral Equ. Oper. Theory **10** (1987), 505–530.
5. R. Duduchava, N. Kverghelidze and M. Tsaava, *Singular integral operators on an open arc in spaces with weight*, Integral Equ. Oper. Theory **77** (2013), 39–56.
6. A. Erdélyi, W. Magnus, F. Oberhettinger and F.G Tricomi, *Tables of Integral Transforms.* Vol. I. Bateman Manuscript Project, California Institute of Technology, McGraw-Hill Book Co., New York, 1954.
7. I. Gohberg and N. Krupnik, *One-Dimensional Linear Singular Integral Equations. I. Introduction*, Operator Theory: Advances and Applications, vol. 53. Birkhäuser, Basel, 1992.
8. I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, 7th edn. Elsevier/Academic Press, Amsterdam, 2007.
9. V.V. Grushin, *Pseudodifferential operators on $\mathbb{R}^n$ with bounded symbols.* Funct. Anal. Appl. **4** (1970), 202–212.
10. R. Hagen, S. Roch and B. Silbermann, *Spectral Theory of Approximation Methods for Convolution Equations*, Operator Theory: Advances and Applications, vol. 74. Birkhäuser, Basel, 1994.
11. L. Hörmander, *Pseudo-differential operators and hypoelliptic equations*, In: "Singular integrals (Proc. Sympos. Pure Math., Vol. X, Chicago, Ill., 1966)", Amer. Math. Soc., Providence, R.I., 1967, pp. 138–183.
12. A.Yu. Karlovich, Yu.I. Karlovich and A.B. Lebre, *Sufficient conditions for Fredholmness of singular integral operators with shifts and slowly oscillating data*, Integr. Equ. Oper. Theory **70** (2011), 451–483.
13. A.Yu. Karlovich, Yu.I. Karlovich and A.B. Lebre, *Necessary conditions for Fredholmness of singular integral operators with shifts and slowly oscillating data*, Integr. Equ. Oper. Theory **71** (2011), 29–53.
14. A.Yu. Karlovich, Yu.I. Karlovich and A.B. Lebre, *Fredholmness and index of simplest singular integral operators with two slowly oscillating shifts*, Operators and Matrices, to appear. Available at http://oam.ele-math.com/forthcoming.
15. A.Yu. Karlovich and I.M. Spitkovsky, *On singular integral operators with semi-almost periodic coefficients on variable Lebesgue spaces*, J. Math. Anal. Appl. **384** (2011), 706–725.
16. Yu.I. Karlovich, *An algebra of pseudodifferential operators with slowly oscillating symbols*, Proc. London Math. Soc.(3) **92** (2006), 713–761.
17. Yu.I. Karlovich, *Pseudodifferential operators with compound slowly oscillating symbols*, In: "The Extended Field of Operator Theory". Operator Theory: Advances and Applications, vol. 171 (2006), pp. 189–224.
18. Yu.I. Karlovich, *Algebras of pseudo-differential operators with discontinuous symbols*, In: "Modern Trends in Pseudo-Differential Operators". Operator Theory: Advances and Applications, vol. 172 (2007), pp. 207–233.
19. Yu.I. Karlovich, *The Haseman boundary value problem with slowly oscillating data*, In: "Analytic Methods of Analysis and Differential Equations: Amade-2006", Cambridge Scientific Publishers, 2008, pp. 81–110.
20. Yu.I. Karlovich, *An algebra of shift-invariant singular integral operators with slowly oscillating data and its application to operators with a Carleman shift*, In: "Analysis, Partial

Differential Equations and Applications. The Vladimir Maz'ya Anniversary Volume". Operator Theory: Advances and Applications, vol. 193 (2009), pp. 81–95.
21. Yu.I. Karlovich and I. Loreto Hernández, *Algebras of convolution type operators with piecewise slowly oscillating data. I: Local and structural study*, Integral Equ. Oper. Theory **74** (2012), 377–415.
22. Yu.I. Karlovich and I. Loreto Hernández, *Algebras of convolution type operators with piecewise slowly oscillating data. II: Local spectra and Fredholmness*, Integral Equ. Oper. Theory **75** (2013), 49–86.
23. V.G. Kravchenko and G.S. Litvinchuk, *Introduction to the Theory of Singular Integral Operators with Shift*, Mathematics and its Applications, vol. 289. Kluwer Academic Publishers, Dordrecht, 1994.
24. H. Kumano-go, *Pseudo-Differential Operators*. The MIT Press. Cambridge, MA, 1982.
25. G.S. Litvinchuk, *Boundary Value Problems and Singular Integral Equations with Shift*, Nauka, Moscow, 1977 (in Russian).
26. H. Mascarenhas, P.A. Santos, and M. Seidel, *Quasi-banded operators, convolutions with almost periodic or quasi-continuous data, and their approximations*, J. Math. Anal. Appl. **418** (2014), 938–963.
27. S.C. Power, *Fredholm Toeplitz operators and slow oscillation*, Can. J. Math. **32** (1980), 1058–1071.
28. V.S. Rabinovich, *Singular integral operators on a composed contour with oscillating tangent and pseudodifferential Mellin operators*, Soviet Math. Dokl. **44** (1992), 791–796.
29. V.S. Rabinovich, *Mellin pseudodifferential operators techniques in the theory of singular integral operators on some Carleson curves*, In "Differential and Integral Operators (Regensburg, 1995)". Operator Theory: Advances and Applications, Vol. 102 (1998), 201–218.
30. V.S. Rabinovich, S. Roch and B. Silbermann, *Limit Operators and Their Applications in Operator Theory*. Operator Theory: Advances and Applications, vol. 150. Birkhäuser, Basel, 2004.
31. S. Roch, P.A. Santos and B. Silbermann, *Non-Commutative Gelfand Theories. A Toolkit for Operator Theorists and Numerical Analysts*, Universitext. Springer-Verlag London, London, 2011.
32. D. Sarason, *Toeplitz operators with piecewise quasicontinuous symbols*, Indiana Univ. Math. J. **26** (1977), 817–838.
33. D. Sarason, *The Banach algebra of slowly oscillating functions*, Houston J. Math. **33** (2007), 1161–1182.
34. E.M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Prinseton, NJ, 1993.

Centro de Matemática e Aplicações (CMA) and Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Quinta da Torre, 2829–516 Caparica, Portugal.

E-mail address: oyk@fct.unl.pt