Banach J. Math. Anal. 9 (2015), no. 3, 243–247
http://doi.org/10.15352/bjma/09-3-17
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

DETERMINANTAL INEQUALITIES FOR HADAMARD PRODUCT OF M-MATRICES AND INVERSE M-MATRICES

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Communicated by F. Zhang

Abstract. In this paper, we generalize some determinantal inequalities for Hadamard product of M-matrices and inverse M-matrices obtained by Chen [Linear Algebra Appl. **426** (2007), no. 2-3, 610–618.].

1. Introduction

Let $C^{m\times n}$ ($R^{m\times n}$) be the set of all complex (real) matrices. A real matrix is nonnegative if every entry is nonnegative. For two $m\times n$ matrices $A=(a_{ij}), B=(b_{ij})\in R^{m\times n}$, we write $A\geq B$ if $A-B$ is nonnegative. The Hadamard product of $A=(a_{ij}), B=(b_{ij})\in C^{m\times n}$ is $A\circ B=(a_{ij}b_{ij})\in C^{m\times n}$.

Suppose that $N=\{1,2,\ldots,n\}$ and the index set $\alpha=\{i_1,i_2,\ldots,i_s\}\subseteq N$. The principal submatrix of an $n\times n$ matrix A including rows and columns $i_1,i_2,\ldots,i_s$ is denoted by $A[i_1,i_2,\ldots,i_s]$ or $A[\alpha]$. In particular, the $k\times k$ leading principal submatrix of A is denoted by $A_k=A[1,2,\ldots,k]$ ($k\in N$) and we set $A(k)=A[N\backslash\{k\}]$.

Let $Z^{n\times n}=\{A=(a_{ij})\in R^{n\times n}: a_{ij}\leq 0, i\neq j, i,j\in N\}$. $A\in Z^{n\times n}$ is called an M-matrix if it is nonsingular and its inverse is a nonnegative matrix. We denote by M_n the class of all $n\times n$ M-matrices. If $A^{-1}\in M_n$, then we say A is an inverse M-matrix. The class of $n\times n$ inverse M-matrices is denoted by M_n^{-1}. Clearly, if $A\in M_n^{-1}$, then A is necessarily nonnegative.

Date: Received: Oct. 9, 2014; Revised: Nov. 3, 2014; Accepted: Nov. 10, 2014.
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2010 *Mathematics Subject Classification.* Primary 15A15; Secondary 47A63.
Key words and phrases. Hadamard product, determinantal inequality, M-matrix, inverse M-matrix.

Chen [1, Theorem 3.3] proved the following determinantal inequalities for the Hadamard product of an M-matrix and an inverse M-matrix: if $A = (a_{ij}) \in M_n$, $B = (b_{ij}) \in M_n^{-1}$, then $A \circ B \in M_n$, and for any permutation $i_1, i_2, \ldots, i_n$ of N, the following inequalities hold:

$$\det(A \circ B) \geq \det(AB) \times \prod_{s=2}^{n} \left(\frac{a_{i_s i_s} \det A[i_1, \ldots, i_{s-1}]}{\det A[i_1, \ldots, i_{s-1}, i_s]} + \frac{b_{i_s i_s} \det B[i_1, \ldots, i_{s-1}]}{\det B[i_1, \ldots, i_{s-1}, i_s]} - 1 \right), \tag{1.1}$$

and

$$\det(A \circ B) \leq \left(\prod_{i=1}^{n} a_{ii} b_{ii} \right) \times \prod_{s=2}^{n} \left(1 - \frac{|a_{i_1 i_2} \cdots a_{i_{s-1} i_s} a_{i_s i_1}|}{a_{i_1 i_1} a_{i_2 i_2} \cdots a_{i_s i_s}} \cdot \frac{b_{i_1 i_2} \cdots b_{i_{s-1} i_s} b_{i_s i_1}}{b_{i_1 i_1} b_{i_2 i_2} \cdots b_{i_s i_s}} \right). \tag{1.2}$$

Very recently, Lin [3] proved that for the block positive definite matrices, a similar result to (1.1) holds for a block Hadamard product.

In this paper, we will present two determinantal inequalities for the Hadamard product of an M-matrix and some inverse M-matrices which are generalizations of (1.1) and (1.2).

2. Main results

We give some lemmas before we give the main theorem of this paper.

Lemma 2.1. [2, p. 117 and p. 127 Problem 9] *If $A = (a_{ij}) \in M_n \cup M_n^{-1}$, then for any i*

$$\frac{a_{ii} \det A(i)}{\det A} \geq 1.$$

Remark 2.2. Chen [1, Theorem 3.1] proved a stronger result as follows:
If $A = (a_{ij}) \in M_n \cup M_n^{-1}$, then for any $\alpha = \{i_1, i_2, \ldots, i_s\} \subseteq N$, where $i_1, i_2, \ldots, i_s$ are mutually distinct,

$$0 < \frac{\det A}{a_{i_s i_s} \det A(i_s)} \leq 1 - \frac{|a_{i_1 i_2} \cdots a_{i_{s-1} i_s} a_{i_s i_1}|}{a_{i_1 i_1} a_{i_2 i_2} \cdots a_{i_s i_s}}.$$

Lemma 2.3. [1, Theorem 3.3] *If $A = (a_{ij}) \in M_n$, $B = (b_{ij}) \in M_n^{-1}$, then $A \circ B \in M_n$.*

Now we present our main result:

Theorem 2.4. *If $A_1 = (a_{ij}^{(1)}) \in M_n$ and $A_i = (a_{ij}^{(i)})$ $(i = 2, \ldots, m) \in M_n^{-1}$, then the Hadamard product $A_1 \circ A_2 \circ \cdots \circ A_m \in M_n$, and for any permutation $i_1, i_2, \cdots, i_n$ of N, the following determinantal inequalities hold:*

$$\det(A_1 \circ A_2 \circ \cdots \circ A_m) \geq \det(A_1 \cdot A_2 \cdot \cdots \cdot A_m) \times \prod_{s=2}^{n} \left(\sum_{j=1}^{m} \frac{a_{i_s i_s}^{(j)} \det A_j[i_1, \ldots, i_{s-1}]}{\det A_j[i_1, \ldots, i_{s-1}, i_s]} - (m-1) \right), \tag{2.1}$$

and

$$\det(A_1 \circ A_2 \circ \cdots \circ A_m) \le \left(\prod_{i=1}^{n} a_{ii}^{(1)} a_{ii}^{(2)} \cdots a_{ii}^{(m)}\right) \times \prod_{s=2}^{n}\left(1 - \frac{|a_{i_1 i_2}^{(1)} \cdots a_{i_{s-1} i_s}^{(1)} a_{i_s i_1}^{(1)}|}{a_{i_1 i_1}^{(1)} a_{i_2 i_2}^{(1)} \cdots a_{i_s i_s}^{(1)}} \cdot \prod_{j=2}^{m} \frac{a_{i_1 i_2}^{(j)} \cdots a_{i_{s-1} i_s}^{(j)} a_{i_s i_1}^{(j)}}{a_{i_1 i_1}^{(j)} a_{i_2 i_2}^{(j)} \cdots a_{i_s i_s}^{(j)}}\right). \tag{2.2}$$

Proof. By Lemma 2.3, it is straightforward to observe that the Hadamard product $A_1 \circ A_2 \circ \cdots \circ A_m$ is an M-matrix. Use induction on m. When $m = 2$, the result is (1.1). Suppose that (2.1) holds when $m = k - 1$:

$$\det(A_1 \circ \cdots \circ A_{k-1}) \ge \det(A_1 \cdots A_{k-1}) \times \prod_{s=2}^{n}\left(\frac{a_{i_s i_s}^{(1)} \det A_1[i_1, \ldots, i_{s-1}]}{\det A_1[i_1, \ldots, i_{s-1}, i_s]} + \cdots + \frac{a_{i_s i_s}^{(k-1)} \det A_{k-1}[i_1, \ldots, i_{s-1}]}{\det A_{k-1}[i_1, \ldots, i_{s-1}, i_s]} - (k-2)\right).$$

When $m = k$, we need to prove the following inequality:

$$\det(A_1 \circ A_2 \circ \cdots \circ A_k) \ge \det(A_1 \cdot A_2 \cdots A_k) \times \prod_{s=2}^{n}\left(\frac{a_{i_s i_s}^{(1)} \det A_1[i_1, \ldots, i_{s-1}]}{\det A_1[i_1, \ldots, i_{s-1}, i_s]} + \cdots + \frac{a_{i_s i_s}^{(k)} \det A_k[i_1, \ldots, i_{s-1}]}{\det A_k[i_1, \ldots, i_{s-1}, i_s]} - (k-1)\right).$$

By (1.1), we have

$$\det((A_1 \circ \cdots \circ A_{k-1}) \circ A_k) \ge \det((A_1 \circ \cdots \circ A_{k-1}) A_k) \times \prod_{s=2}^{n}\left(\frac{a_{i_s i_s}^{(1)} \cdots a_{i_s i_s}^{(k-1)} \det(A_1 \circ \cdots \circ A_{k-1})[i_1, \ldots, i_{s-1}]}{\det(A_1 \circ \cdots \circ A_{k-1})[i_1, \ldots, i_{s-1}, i_s]} + \frac{a_{i_s i_s}^{(k)} \det A_k[i_1, \ldots, i_{s-1}]}{\det A_k[i_1, \ldots, i_{s-1}, i_s]} - 1\right).$$

By the induction hypothesis, it follows

$$\begin{aligned} &\det((A_1 \circ \cdots \circ A_{k-1}) A_k) \times \prod_{s=2}^{n}\left(\frac{a_{i_s i_s}^{(1)} \cdots a_{i_s i_s}^{(k-1)} \det(A_1 \circ \cdots A_{k-1})[i_1, \ldots, i_{s-1}]}{\det(A_1 \circ \cdots \circ A_{k-1})[i_1, \ldots, i_{s-1}, i_s]} + \frac{a_{i_s i_s}^{(k)} \det A_k[i_1, \ldots, i_{s-1}]}{\det A_k[i_1, \ldots, i_{s-1}, i_s]} - 1\right) \\ &\ge \det(A_1 \cdots A_{k-1}) \det A_k \times \\ &\prod_{s=2}^{n}\left(\frac{a_{i_s i_s}^{(1)} \det A_1[i_1, \ldots, i_{s-1}]}{\det A_1[i_1, \ldots, i_{s-1}, i_s]} + \cdots + \frac{a_{i_s i_s}^{(k-1)} \det A_{k-1}[i_1, \ldots, i_{s-1}]}{\det A_{k-1}[i_1, \ldots, i_{s-1}, i_s]} - (k-2)\right) \times \\ &\prod_{s=2}^{n}\left(\frac{a_{i_s i_s}^{(1)} \cdots a_{i_s i_s}^{(k-1)} \det(A_1 \circ \cdots \circ A_{k-1})[i_1, \ldots, i_{s-1}]}{\det(A_1 \circ \cdots \circ A_{k-1})[i_1, \ldots, i_{s-1}, i_s]} + \frac{a_{i_s i_s}^{(k)} \det A_k[i_1, \ldots, i_{s-1}]}{\det A_k[i_1, \ldots, i_{s-1}, i_s]} - 1\right). \end{aligned} \tag{2.3}$$

Let

$$a_s = \frac{a_{i_s i_s}^{(1)} \det A_1[i_1, \ldots, i_{s-1}]}{\det A_1[i_1, \ldots, i_{s-1}, i_s]} + \cdots + \frac{a_{i_s i_s}^{(k-1)} \det A_{k-1}[i_1, \ldots, i_{s-1}]}{\det A_{k-1}[i_1, \ldots, i_{s-1}, i_s]} - (k-2),$$

$$b_s = \frac{a^{(1)}_{i_s i_s} \cdots a^{(k-1)}_{i_s i_s} \det(A_1 \circ \cdots \circ A_{k-1})[i_1, \ldots, i_{s-1}]}{\det(A_1 \circ \cdots \circ A_{k-1})[i_1, \ldots, i_{s-1}, i_s]} + \frac{a^{(k)}_{i_s i_s} \det A_k[i_1, \ldots, i_{s-1}]}{\det A_k[i_1, \ldots, i_{s-1}, i_s]} - 1.$$

By Lemma 2.1, we have

$$\frac{a^{(l)}_{i_s i_s} \det A_l[i_1, \ldots, i_{s-1}]}{\det A_l[i_1, \ldots, i_{s-1}, i_s]} \geq 1, \quad l = 1, \ldots, k,$$

$$\frac{a^{(1)}_{i_s i_s} \cdots a^{(k-1)}_{i_s i_s} \det(A_1 \circ \cdots \circ A_{k-1})[i_1, \ldots, i_{s-1}]}{\det(A_1 \circ \cdots \circ A_{k-1})[i_1, \ldots, i_{s-1}, i_s]} \geq 1, \tag{2.4}$$

and so

$$a_s \geq (k-1) - (k-2) = 1, \quad b_s \geq 1.$$

Using a simple scalar inequality $ab \geq a+b-1$ for $a, b \geq 1$, we obtain the following inequality:

$$\begin{aligned}
&\det(A_1 \circ \cdots \circ A_k) \\
&\geq \det(A_1 \cdots A_{k-1}) \det A_k \times \prod_{s=2}^{n} a_s b_s \ (\text{by } (2.3)) \\
&\geq \det(A_1 \cdots A_k) \times \prod_{s=2}^{n} (a_s + b_s - 1) \\
&\geq \det(A_1 \cdots A_k) \times \\
&\prod_{s=2}^{n} \left(\frac{a^{(1)}_{i_s i_s} \det A_1[i_1, \ldots, i_{s-1}]}{\det A_1[i_1, \ldots, i_{s-1}, i_s]} + \cdots + \frac{a^{(k)}_{i_s i_s} \det A_k[i_1, \ldots, i_{s-1}]}{\det A_k[i_1, \ldots, i_{s-1}, i_s]} - (k-1) \right) \\
&\qquad .(\text{by } (2.4))
\end{aligned}$$

This completes the proof of (2.1).

Next we verify that the inequality (2.2) holds by induction on m.

When $m = 2$, the inequality (2.2) is (1.2). We assume that it is true when $m = k-1$; namely,

$$\det(A_1 \circ A_2 \circ \cdots \circ A_{k-1}) \leq \left(\prod_{i=1}^{n} a^{(1)}_{ii} a^{(2)}_{ii} \cdots a^{(k-1)}_{ii} \right) \times$$
$$\prod_{s=2}^{n} \left(1 - \frac{|a^{(1)}_{i_1 i_2} \cdots a^{(1)}_{i_{s-1} i_s} a^{(1)}_{i_s i_1}|}{a^{(1)}_{i_1 i_1} a^{(1)}_{i_2 i_2} \cdots a^{(1)}_{i_s i_s}} \cdot \frac{a^{(2)}_{i_1 i_2} \cdots a^{(2)}_{i_{s-1} i_s} a^{(2)}_{i_s i_1}}{a^{(2)}_{i_1 i_1} a^{(2)}_{i_2 i_2} \cdots a^{(2)}_{i_s i_s}} \cdots \frac{a^{(k-1)}_{i_1 i_2} \cdots a^{(k-1)}_{i_{s-1} i_s} a^{(k-1)}_{i_s i_1}}{a^{(k-1)}_{i_1 i_1} a^{(k-1)}_{i_2 i_2} \cdots a^{(k-1)}_{i_s i_s}} \right).$$

When $m = k$, we need to show that

$$\det(A_1 \circ A_2 \circ \cdots \circ A_k) \leq \left(\prod_{i=1}^{n} a_{ii}^{(1)} a_{ii}^{(2)} \cdots a_{ii}^{(k)}\right) \times$$

$$\prod_{s=2}^{n}\left(1 - \frac{|a_{i_1i_2}^{(1)} \cdots a_{i_{s-1}i_s}^{(1)} a_{i_si_1}^{(1)}|}{a_{i_1i_1}^{(1)} a_{i_2i_2}^{(1)} \cdots a_{i_si_s}^{(1)}} \cdot \frac{a_{i_1i_2}^{(2)} \cdots a_{i_{s-1}i_s}^{(2)} a_{i_si_1}^{(2)}}{a_{i_1i_1}^{(2)} a_{i_2i_2}^{(2)} \cdots a_{i_si_s}^{(2)}} \cdots \frac{a_{i_1i_2}^{(k)} \cdots a_{i_{s-1}i_s}^{(k)} a_{i_si_1}^{(k)}}{a_{i_1i_1}^{(k)} a_{i_2i_2}^{(k)} \cdots a_{i_si_s}^{(k)}}\right).$$

By (1.2) and the induction hypothesis, we have

$$\begin{aligned}\det(A_1 \circ A_2 \circ \cdots \circ A_{k-1} \circ A_k) &= \det((A_1 \circ A_2 \circ \cdots \circ A_{k-1}) \circ A_k) \\ &\leq \left(\prod_{i=1}^{n} (A_1 \circ A_2 \circ \cdots \circ A_{k-1})_{ii} \cdot a_{ii}^{(k)}\right) \times\end{aligned}$$

$$\prod_{s=2}^{n}\left(1 - \frac{|(a_{i_1i_2}^{(1)} a_{i_1i_2}^{(2)} \cdots a_{i_1i_2}^{(k-1)})(a_{i_2i_3}^{(1)} a_{i_2i_3}^{(2)} \cdots a_{i_2i_3}^{(k-1)}) \cdots (a_{i_si_1}^{(1)} a_{i_si_1}^{(2)} \cdots a_{i_si_1}^{(k-1)})|}{(a_{i_1i_1}^{(1)} a_{i_1i_1}^{(2)} \cdots a_{i_1i_1}^{(k-1)})(a_{i_2i_2}^{(1)} a_{i_2i_2}^{(2)} \cdots a_{i_2i_2}^{(k-1)}) \cdots (a_{i_si_s}^{(1)} a_{i_si_s}^{(2)} \cdots a_{i_si_s}^{(k-1)})} \cdot \frac{a_{i_1i_2}^{(k)} \cdots a_{i_{s-1}i_s}^{(k)} a_{i_si_1}^{(k)}}{a_{i_1i_1}^{(k)} a_{i_2i_2}^{(k)} \cdots a_{i_si_s}^{(k)}}\right)$$

$$= \left(\prod_{i=1}^{n} a_{ii}^{(1)} a_{ii}^{(2)} \cdots a_{ii}^{(k)}\right) \times$$

$$\prod_{s=2}^{n}\left(1 - \frac{|a_{i_1i_2}^{(1)} \cdots a_{i_{s-1}i_s}^{(1)} a_{i_si_1}^{(1)}|}{a_{i_1i_1}^{(1)} a_{i_2i_2}^{(1)} \cdots a_{i_si_s}^{(1)}} \cdot \frac{a_{i_1i_2}^{(2)} \cdots a_{i_{s-1}i_s}^{(2)} a_{i_si_1}^{(2)}}{a_{i_1i_1}^{(2)} a_{i_2i_2}^{(2)} \cdots a_{i_si_s}^{(2)}} \cdots \frac{a_{i_1i_2}^{(k)} \cdots a_{i_{s-1}i_s}^{(k)} a_{i_si_1}^{(k)}}{a_{i_1i_1}^{(k)} a_{i_2i_2}^{(k)} \cdots a_{i_si_s}^{(k)}}\right).$$

This completes the proof of (2.2). □

Remark 2.5. The inequalities (2.1) and (2.2) in Theorem 2.4 are generalizations of the inequalities (1.1) and (1.2), respectively.

Acknowledgement. The authors thank the referee and the handling editor for their valuable comments and suggestions which improve the presentation of the paper. The first author was supported by the key project of the applied mathematics of Hainan Normal University and the natural science foundation of Hainan Province (No.114007).

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