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INEQUALITIES FOR INTERPOLATION FUNCTIONS

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Abstract. In this paper, in relation with interpolation functions we study some generalized Powers-Størmer's type inequalities and monotonicity inequality of indefinite type which generalizes a result of Ando.

1. Introduction and preliminaries

Throughout this paper, M_n stands for the algebra of all $n \times n$ matrices. Denote by M_n^+ the set of all positive semi-definite matrices. A continuous function f on I ($\subset \mathbb{R}$) is called *matrix convex of order* n (or n*-convex*) if the inequality

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B)$$

holds for all self-adjoint matrices $A, B \in M_n$ with $\sigma(A), \sigma(B) \subset I$ and for all $\lambda \in [0,1]$, where $\sigma(A)$ stands for the spectrum of A. Also, f is called a n-concave on I if $-f$ is n-convex on I.

A continuous function f on I is called *matrix monotone of order* n or n*-monotone*, if

$$A \leq B \quad \Longrightarrow \quad f(A) \leq f(B)$$

for any pair of self-adjoint matrices $A, B \in M_n$ with $\sigma(A), \sigma(B) \subset I$. We call a function f *operator convex* (resp. *operator concave*) if f is k-convex (resp. k-concave) for any $k \in \mathbb{N}$, and *operator monotone* if f is k-monotone for any $k \in \mathbb{N}$.

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A function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ (where $\mathbb{R}_+ = (0,\infty)$) is called an *interpolation function of order n* if for any $T, A \in M_n$ with $A > 0$ and $T^*T \le 1$,

$$T^*AT \le A \quad \Longrightarrow \quad T^*f(A)T \le f(A).$$

We denote by $\mathcal{C}_n$ the class of all interpolation functions of order n.

Let $\mathcal{P}(\mathbb{R}_+)$ be the set of all Pick functions on $\mathbb{R}_+$, and $\mathcal{P}'$ the set of all positive Pick functions on $\mathbb{R}_+$, i.e., functions of the form

$$h(s) = \int_{[0,\infty]} \frac{(1+t)s}{s+t} d\rho(t), \quad s > 0,$$

where ρ is some positive Radon measure on $[0,\infty]$.

Denote by $\mathcal{P}'_n$ the set of all strictly positive n-monotone functions on $(0,\infty)$. Let us recall a well-known characterization of functions in $\mathcal{C}_n$ that actually is due to Ameur [1] and Ameur, Kaijser, and Sergei [2] (see also [8]).

Theorem 1.1. ([2, Corollary 2.4]) *A function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ belongs to $\mathcal{C}_n$ if and only if for every n-set $\{\lambda_i\}_{i=1}^n \subset \mathbb{R}_+$ there exists a function h from $\mathcal{P}'$ such that $f(\lambda_i) = h(\lambda_i)$ for $i = 1, \ldots, n$.*

Corollary 1.2. *Let A be a positive definite matrix in M_n and $f \in \mathcal{C}_n$. Then there exists a positive Radon measure ρ on $[0,\infty]$ such that*

$$f(A) = \int_{[0,\infty]} A(1+s)(A+s)^{-1} d\rho(s).$$

Remark 1.3.

(i) $\mathcal{P}' = \cap_{n=1}^\infty \mathcal{P}'_n$ [13], $\mathcal{P}' = \cap_{n=1}^\infty \mathcal{C}_n$ [7];
(ii) $\mathcal{C}_{n+1} \subseteq \mathcal{C}_n$;
(iii) $\mathcal{P}'_{n+1} \subseteq \mathcal{C}_{2n+1} \subseteq \mathcal{C}_{2n} \subseteq \mathcal{P}'_n$, $\mathcal{P}'_n \subsetneq \mathcal{C}_n$ [2];
(iv) $\mathcal{C}_{2n} \subsetneq \mathcal{P}'_n$ [14];
(v) $\mathcal{C}_n \circ \mathcal{C}_n \subset \mathcal{C}_n$;
(vi) A function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ belongs to $\mathcal{C}_n$ if and only if $\dfrac{t}{f(t)}$ belongs to $\mathcal{C}_n$.

It is not known whether $\mathcal{P}'_{n+1} \subsetneq \mathcal{C}_{2n+1}$ or not.

In this paper, we consider some inequalities with interpolation functions. More precisely, in Section 2, we extend Petz's trace inequality [15, Theorem 11.18] (Theorem 2.1) to the class of interpolation functions and give a new trace inequality (Theorem 2.5) which might play an important role in the quantum information theory. Moreover, in Section 3 we extend an Ando's monotonicity inequality of indefinite type. We show that for $f \in \mathcal{C}_{2n}$ and any pair of J-selfadjoint matrices $A, B \in M_n$ such that $\sigma(A), \sigma(B) \subset (0,\infty)$,

$$A \le^J B \quad \Longrightarrow \quad f(A) \le^J f(B),$$

where J is a selfadjoint involution and $A \le^J B$ means that $JA^*J = A$, $JB^*J = B$, and $JA \le JB$.

Theorem 1.4. *Let $f \in \mathcal{C}_{2n}$. For positive definite matrices K and L in M_n, let Q the projection onto the range of $(K-L)_+$. We have, then,*

$$\operatorname{Tr}(QL(f(K)-f(L))) \geq 0. \tag{1.1}$$

Proof. Let $\{\lambda_i\}_{i=1}^n$ and $\{\mu_i\}_{i=1}^n$ be sets of eigenvalues of K and L, respectively. Then by Theorem 1.1 there exists an interpolation function $h \in \mathcal{P}'$ such that $f(\lambda) = h(\lambda)$ for $\lambda \in \{\lambda_i\}_{i=1}^n \cup \{\mu_i\}_{i=1}^n$. By Corollary 1.2 there is some positive Radon measure ρ on $[0,\infty]$ such that

$$\begin{aligned} f(K)-f(L) &= \int_{[0,\infty]} K(1+s)(K+s)^{-1} d\rho(s) - \int_{[0,\infty]} L(1+s)(L+s)^{-1} d\rho(s) \\ &= \int_{[0,\infty]} [(1+s)(K+s)^{-1}K - L(1+s)(L+s)^{-1}] d\rho(s) \\ &= \int_{[0,\infty]} (1+s)s(K+s)^{-1}(K-L)(L+s)^{-1} d\rho(s). \end{aligned}$$

Hence

$$\operatorname{Tr}(QL(f(K)-f(L)) = \int_{[0,\infty]} (1+s)s\operatorname{Tr}(QL(K+s)^{-1}(K-L)(L+s)^{-1}) d\rho(s)$$

Repeat the same steps in [15, Theorem 11.18], we get the conclusion. □

Corollary 1.5. *Let $f \in \mathcal{P}'_{n+1}$. For positive definite matrices K and L in M_n, let Q be the projection onto the range of $(K-L)_+$. We have, then,*

$$\operatorname{Tr}(QL(f(K)-f(L))) \geq 0.$$

Proof. It is suffices to mention that $\mathcal{P}'_{n+1} \subset \mathcal{C}_{2n}$ by Remark 1.3. The conclusion follows from Theorem 1.4. □

Using Theorem 1.4 we get a generalized Powers-Størmer's type inequality. Another generalization of Powers-Størmer inequality can be found in [12]. We need the following lemmas.

Lemma 1.6. *Let $h\colon (0,\infty) \to (0,\infty)$ be a function such that the function $th(t)$ is operator monotone. Then the inverse of $\dfrac{t}{h(t)}$ is operator monotone.*

Proof. Since $th(t)$ is operator monotone, the function $\dfrac{1}{h(t)} = \dfrac{t}{th(t)}$ is operator monotone by [11, Corollary 2.6]. Hence the inverse of $t\dfrac{1}{h(t)}$ is operator monotone from by [3, Lemma 5]. □

Lemma 1.7. *Let f be a function from $(0,\infty)$ into itself such that $tf(t) \in \mathcal{C}_{2n}$. Then the inverse of $g(t) = \dfrac{t}{f(t)}$ $(t>0)$ belongs to $\mathcal{C}_{2n}|_{g((0,\infty))}$.*

Proof. Indeed, for any set $T \subset g((0,\infty))$ with $|T| = 2n$ we can write

$$T = \{g(t_1), g(t_2), \ldots, g(t_{2n})\},$$

where $t_i \in (0,\infty)$ for $1 \le i \le 2n$. Since $tf(t) \in \mathcal{C}_{2n}$, there is an interpolation map $k_T \in \mathcal{P}'$ such that $t_i f(t_i) = k_T(t_i)$ for $1 \le i \le 2n$. Then we have

$$g(t_i) = \frac{t_i}{f(t_i)} = t_i \frac{t_i}{k_T(t_i)} \quad (1 \le i \le 2n).$$

Consequently,

$$g^{-1}(g(t_i)) = t_i = (\frac{t^2}{k_T(t)})^{-1}(g(t_i)) \quad (1 \le i \le 2n). \tag{1.2}$$

From the above argument, it is clear that $(\frac{t^2}{k_T(t)})^{-1}$ is operator monotone. From (1.2) we conclude that the inverse g^{-1} of g belongs to $C_{2n}|_{g((0,\infty))}$. $\square$

The main theorem of this section is as follows.

Theorem 1.8. *Let f be a function from $(0,\infty)$ into itself such that $tf(t) \in \mathcal{C}_{2n}$. Then for any pair of positive definite matrices $A, B \in M_n$,*

$$\mathrm{Tr}(A^2) + \mathrm{Tr}(B^2) - \mathrm{Tr}(|A^2 - B^2|) \le 2\,\mathrm{Tr}(Af(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}), \tag{1.3}$$

where $g(t) = \frac{t}{f(t)}$, $t \in (0,\infty)$.

Proof. Let A, B be positive definite matrices and $e(t) = tf(t)$ for $t \in (0,\infty)$. Let Q be the projection on the range of $(g(A) - g(B))_+$ and $L = g(B)$.

Let S be the set of eigenvalues of $g(A)$ and $g(B)$. Since $e \in \mathcal{C}_{2n}$, there is an interpolation map $h \in \mathcal{P}'$ such that $e(\lambda) = h(\lambda)$ for $\lambda \in S$. Since $t(h(t)/t) = h(t)$ is operator monotone, the inverse of $t^2/h(t)$ is operator monotone by Lemma 1.6. By Lemma 1.7 the inverse of g belongs to $\mathcal{C}_{2n}|_{g((0,\infty))}$. Consequently, $e \circ g^{-1} \in C_{2n}|_{g((0,\infty))}$ by Remark 1.3(v).

Apply Theorem 1.4 for the function $e \circ g^{-1}$, we get

$$\begin{aligned} 0 &\le \mathrm{Tr}(Qg(B)((e \circ g^{-1})(g(A)) - (e \circ g^{-1})(g(B))) \\ &= \mathrm{Tr}(Qg(B)(Af(A) - Bf(B))) \\ &= \mathrm{Tr}(Qg(B)Af(A)) - \mathrm{Tr}(QB^2). \end{aligned}$$

On the contrary,

$$\begin{aligned} &\mathrm{Tr}(Q(A^2 - B^2)) - \mathrm{Tr}(Af(A)Q(g(A) - g(B))) \\ &\quad = \mathrm{Tr}(QA^2) - \mathrm{Tr}(QB^2) - \mathrm{Tr}(Af(A)Qg(A)) + \mathrm{Tr}(Af(A)Qg(B)) \\ &\quad = \mathrm{Tr}(Qg(B)Af(A)) - \mathrm{Tr}(QB^2) \ge 0. \end{aligned} \tag{1.4}$$

Hence we have

$$\mathrm{Tr}(Af(A)Q(g(A) - g(B))) \le \mathrm{Tr}(Q(A^2 - B^2)) \le \mathrm{Tr}((A^2 - B^2)_+). \tag{1.5}$$

Therefore, from (1.4) and (1.5) we have

$$\begin{aligned}\operatorname{Tr}(Af(A)(g(A)-g(B))) &\le \operatorname{Tr}(Af(A)(g(A)-g(B))_+)\\ &= \operatorname{Tr}(Af(A)Q(g(A)-g(B)))\\ &\le \operatorname{Tr}((A^2-B^2)_+)\\ &= \frac{1}{2}\operatorname{Tr}((A^2-B^2)+|A^2-B^2|),\end{aligned}$$

and

$$\operatorname{Tr}(A^2+B^2-|A^2-B^2|) \le 2\operatorname{Tr}(Af(A)g(B)).$$

□

Corollary 1.9. *Let f be a function from $(0,\infty)$ into itself such that $tf(t) \in \mathcal{P}'_{n+1}$. Then for any pair of positive definite matrices $A, B \in M_n$,*

$$\operatorname{Tr}(A^2)+\operatorname{Tr}(B^2)-\operatorname{Tr}(|A^2-B^2|) \le 2\operatorname{Tr}(Af(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}),$$

where $g(t)=\frac{t}{f(t)}$ for $t\in(0,\infty)$.

Corollary 1.10 ([5])**.** *Let A, B be positive definite matrices, then for all $0 \le s \le 1$*

$$\operatorname{Tr}(A+B-|A-B|) \le 2\operatorname{Tr}(A^{1-s}B^s).$$

Proof. By adding $\varepsilon > 0$ to A and B, we may assume that A and B are positive invertible matrices.

Firstly, we consider the case $s \in [\frac{1}{2}, 1]$. Let $f(t) = t^{1-2s}$. Then $tf(t) = t^{2-2s}$ is operator monotone on $(0,\infty)$. Substitute $X = A^{\frac{1}{2}}$ and $Y = B^{\frac{1}{2}}$ into the inequality (1.3) in Theorem 1.8, we get

$$\operatorname{Tr}(A+B-|A-B|) \le 2\operatorname{Tr}(A^{1-s}B^s).$$

The remaining case $0 \le s \le \frac{1}{2}$ obviously follows by interchanging the roles of A and B. □

Remark 1.11. In Lemma 1.6 and Lemma 1.7 operator monotonicity and $\mathcal{C}_{2n}$-property of inverse functions were considered. There exists counterexample that the inverse of a n-matrix function may not be n-matrix. Indeed, it is well-known that $f_s(t) = t^s (0 \le s \le 1)$ is operator monotone, but the inverse $f_s^{-1}(t) = t^{1/s}$ of f_s is not 2-monotone. A similar picture for $\mathcal{C}_n$-functions is still not clear.

Inequality (1.3) in Theorem 1.8 is different to generalized Powers-Srørmer inequality in [12]. The proof of (1.3) is based on the fact that $(tf)\circ g^{-1} \in \mathcal{C}_{2n}|_{g((0,\infty))}$. If we have the condition $f \circ g^{-1} \in \mathcal{C}_{2n}|_{g((0,\infty))}$, then by similar arguments above we can get the generalized Powers-Størmer inequality as in [12]. More precisely, we have the following theorem.

Theorem 1.12. *Let f be a function in $\mathcal{C}_{2n}$ such that $f\circ g^{-1} \in \mathcal{C}_{2n}|_{(g(0,\infty))}$, where $g(t) = \dfrac{t}{f(t)}$, $t \in (0,\infty)$. Then for any pair of positive definite matrices $A, B \in M_n$,*

$$\operatorname{Tr}(A)+\operatorname{Tr}(B)-\operatorname{Tr}(|A-B|) \le 2\operatorname{Tr}(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}).$$

Since the proof of this theorem is done by the same steps in Theorem 1.8, the detail is left to reader.

2. Matrix monotonicity inequality of indefinite type

Let J ($\neq I_n$ — unit in M_n) be a selfadjoint involution different to identity, that means, $J = J^*, J^2 = I_n$. For a matrix A its J-adjoint $A^\sharp$ is defined as follows: $A^\sharp = JA^*J$. A matrix A is said to be *J-selfadjoint* if $A = A^\sharp$, or, $JA = A^*J$. For a pair of J-selfadjoint matrices A, B, we define an indefinite order relation $A \leq^J B$ as follows:

$$A \leq^J B \quad \text{if} \quad JA \leq JB.$$

It is known as a result of Potapov-Ginzburg (see [6, Chapter 2, Section 4]) that $\sigma(JA^*JA) \subset [0, +\infty)$ for any A. If A is a J-selfadjoint operator with $\sigma(A) \subset (0, \infty)$, then for any function $f(t) \in \mathcal{C}_n$ the matrix $f(A)$ is well-defined by Corollary 1.2. Note that $f(A)$ is J-selfadjoint.

It is well-known that any operator monotone function on $(-1, 1)$ has an integral representation

$$f(t) = f(0) + \int_{-1}^{1} \frac{t}{1 - t\lambda} d\mu(\lambda),$$

where $d\mu(\cdot)$ is a positive measure on $[-1, 1]$. T. Ando [4] used this fact to study operator monotonicity inequality of indefinite type.

Theorem 2.1 ([4], Theorem 4)**.** *Let J be a selfadjoint involution, and A, B be J-selfadjoint matrices with spectra in (α, β). Then*

$$A \leq^J B \Longrightarrow f(A) \leq^J f(B)$$

for any operator monotone function $f(t)$ on (α, β).

For n-monotone functions his proof is not applicable, since an integral representation of n-monotone functions is not clear in general. Fortunately, we can extend Ando's result to class $\mathcal{C}_{2n}$ with a help of Corollary 1.2 .

The assertions of the following lemma were obtained in [4]. But for convenience of readers we give a proof.

Lemma 2.2. *Let A, B be J-selfadjoint matrices in M_n such that $\sigma(A)$, $\sigma(B) \subset (0, +\infty)$. Then*

$$A \leq^J B \quad \Longrightarrow \quad B^{-1} \leq^J A^{-1}.$$

Proof. Mention that for any matrix $C \in M_n$,

$$JC^\sharp BC - JC^\sharp AC = C^*(JB - JA)C \geq 0, \quad \text{i.e.} \quad C^\sharp AC \leq^J C^\sharp BC.$$

Since $\sigma(A) \subset (0, +\infty)$ and the function $f(t) = t^{1/2}$ is operator monotone on $(0, \infty)$, the J-selfadjoint square root $A^{1/2}$ is well defined and its reverse $A^{-1/2}$ is also J-selfadjoint. In the case $B = I_n$, we have

$$A^{-1} - I_n = A^{-1/2}(I_n - A)A^{-1/2} \geq^J 0. \tag{2.1}$$

In general case,

$$I_n = B^{-1/2}BB^{-1/2} \geq^J B^{-1/2}AB^{-1/2} = [A^{1/2}B^{-1/2}]^\sharp A^{1/2}B^{-1/2}.$$

On account of a result of Potapov-Ginzburg mentioned, and since $B^{-1/2}AB^{-1/2}$ is invertible, the latter implies that $\sigma(B^{-1/2}AB^{-1/2}) \subset (0, +\infty)$. By (2.1), we obtain

$$I_n \leq^J (B^{-1/2}AB^{-1/2})^{-1} = B^{1/2}A^{-1}B^{1/2},$$

which equivalent to $A^{-1} \geq^J B^{-1}$. □

Theorem 2.3. *Let $f \in \mathcal{C}_{2n}$. Then for any pair of J-selfadjoint matrices $A \leq^J B$ in M_n such that $\sigma(A)$, $\sigma(B) \subset (0, \infty)$,*

$$f(A) \leq^J f(B). \tag{2.2}$$

Proof. Let λ_i $(1 \leq i \leq n)$ and μ_j $(1 \leq j \leq n)$ be the sets of eigenvalues of A and B, respectively.

Then there is an interpolation function $h \in \mathcal{C}_{2n}$ such that $f(\lambda) = h(\lambda)$ for $\lambda \in \{\lambda_i, \mu_j\}_{1 \leq i,j \leq n}$. By Corollary 1.4, there is a positive Radon measure ρ on $[0, \infty]$ such that

$$f(\alpha) = \int_{[0,\infty]} \frac{\alpha(1+s)}{s+\alpha} d\rho(s) \quad (\alpha \in \{\lambda_i, \mu_j\}_{1 \leq i,j \leq n}).$$

Then inequality (2.2) is equivalent to the following:

$$\int_{[0,\infty]} A(1+s)(s+A)^{-1} d\rho(s) \leq^J \int_{[0,\infty]} B(1+s)(s+B)^{-1} d\rho(s).$$

Therefore, it suffices to prove that

$$A(s+A)^{-1} \leq^J B(s+B)^{-1} \quad (s > 0),$$

or equivalently,

$$(s+A)^{-1} \geq^J (s+B)^{-1} \quad (s > 0). \tag{2.3}$$

From $A \leq^J B$ it follows that $s + A \leq^J s + B \quad (s > 0)$. On the other hand, $\sigma(s + A), \sigma(s + B) \subset (s, \infty) \subset (0, \infty)$. On account of Lemma 2.2 we obtain (2.3). □

Remark 2.4. A similar conclusion for matrix convex functions on $[0, \infty)$ is wrong. Indeed, it is well-known that the function $f(t) = t^2$ $(t \in (0, \infty))$ is operator convex. Let A be an arbitrary J-positive matrix (that means, JA is positive) with spectrum in $(2, \infty)$. Put $B = A + J$. It is clear that $A \leq^J B$ and $\sigma(B) \subset (0, \infty)$. We have

$$f(\frac{A}{2} + \frac{B}{2}) \not\leq^J \frac{1}{2} f(A) + \frac{1}{2} f(B),$$

that is,

$$\frac{1}{2}(A^2 + B^2) - (\frac{A+B}{2})^2 = \frac{1}{4}(B-A)^2 = \frac{1}{4}J^2 = \frac{I}{4} \not\geq^J 0.$$

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