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# PARALLEL ITERATIVE METHODS FOR SOLVING THE COMMON NULL POINT PROBLEM IN BANACH SPACES

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ABSTRACT. We consider the common null point problem in Banach spaces. Then, using the hybrid projection method and the $\varepsilon$-enlargement of maximal monotone operators, we prove two strong convergence theorems for finding a solution of this problem.

## 1. INTRODUCTION

Let $H$ be a real Hilbert space, and let $f: H \longrightarrow (-\infty, \infty]$ be a proper, lower semicontinuous, and convex function. In order to find a minimum point of $f$, Martinet [11] proposed the iterative method as follows: $x_1 \in H$ and

$$x_{n+1} = \arg\min_{y \in H} \left\{ f(y) + \frac{1}{2}\|y - x_n\|^2 \right\}$$

for all $n \geq 1$. He proved that the sequence $\{x_n\}$ converges weakly to a minimum point of $f$. Note that, the above sequence $\{x_n\}$ can be rewritten in the form

$$\partial f(x_{n+1}) + x_{n+1} \ni x_n \qquad \forall n \geq 1.$$

We know that the subdifferential operator $\partial f$ of $f$ is a maximal monotone operator [14]. So, the problem of finding a null point of a maximal monotone operator plays an important role in optimization theory. One popular method of solving equation $0 \in A(x)$ where $A$ is a maximal monotone operator in Hilbert space $H$,


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is the proximal point algorithm. The proximal point algorithm generates, for any starting point $x_0 = x \in E$, a sequence $\{x_n\}$ by the rule

$$x_{n+1} = J^A_{r_n}(x_n), \qquad \forall n \in \mathbb{N},$$

where $\{r_n\}$ is a sequence of positive real numbers and $J^A_{r_n} = (I + r_n A)^{-1}$ is the resolvent of $A$. Moreover, Rockafellar [15] has given a more practical method which is an inexact variant of the method

$$x_n + e_n \in x_{n+1} + r_n A x_{n+1}, \qquad \forall n \in \mathbb{N}, \tag{1.1}$$

where $\{e_n\}$ is regarded as an error sequence and $\{r_n\}$ is a sequence of positive regularization parameters. Note that the algorithm (1.1) can be rewritten as

$$x_{n+1} = J^A_{r_n}(x_n + e_n) \qquad \forall n \in \mathbb{N}.$$

This method is called inexact proximal point algorithm. It was shown in Rockafellar [15] that if $\sum_{n=1}^{\infty} \|e_n\| < \infty$, then $x_n \rightharpoonup z \in H$ with $0 \in Az$.

There are many authors replaced the operator $A$ in the equation (1.1) by the $\varepsilon$-enlargement $A^{\varepsilon}$, see, for instance, Burachick, Iusem, and Svaiter [3], Solodov and Svaitere [17], Moudafi and Elisabeth [13], and others. In [3], Burachick and others used the enlargement $A^{\varepsilon}$ to devise an approximate generalized proximal point algorithm. The exact version of this algorithm can be stated as follows: Having $x_n$, the next element $x_{n+1}$ is the solution of

$$0 \in r_n A(x) + \triangledown f(x) - \triangledown f(x_n), \tag{1.2}$$

where $f$ is a suitable regularization function. Note that, if $f(x) = \dfrac{1}{2}\|x\|^2$, then the above algorithm becomes the classical proximal point algorithm. Approximate solutions of (1.2) are treated in [3] via $A^{\varepsilon}$. Specifically, an approximate solution of (1.2) can be regarded as an exact solution of

$$0 \in r_n A^{\varepsilon_n}(x) + \triangledown f(x) - \triangledown f(x_n),$$

for an appropriate value of $\varepsilon_n$. Note that, if $f(x) = \dfrac{1}{2}\|x\|^2$, then the above relation is equivalent to the problem of finding an element $x_{n+1} \in H$ and $v_{n+1} \in A^{\varepsilon_n}(x_{n+1})$ with $\varepsilon_n \geq 0$ such that

$$0 = r_n v_{n+1} + (x_{n+1} - x_n). \tag{1.3}$$

They proved that if $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, then the sequence $\{x_n\}$ converges weakly to a null point of $A$.

The problem of finding a common null point of a finite family of maximal monotone operators in Banach or Hilbert spaces is the interesting topic of nonlinear analysis. This problem has been investigated by many researchers, see, for instance, Sabach [16], Timnak, Naraghirad, and Hussain [19], Tuyen [20], Kim and Tuyen [10], and others.

Let $E$ be a reflexive Banach space, and let $A_i : E \longrightarrow 2^{E^*}$, $i = 1, 2, \ldots, N$, be $N$ maximal monotone operators such that $S = \cap_{i=1}^{N} A_i^{-1} 0 \neq \emptyset$. Let $g : E \longrightarrow \mathbb{R}$ be a Legendre function that is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of $E$. In 2011, Sabach [16] introduced two iterative methods for finding an element $x^* \in S$. He proved the strong convergence of sequence $\{x_n\}$ which is defined by

$$\begin{aligned}
&x_0 \in E \text{ chosen arbitrarily},\\
&y_n = \mathrm{Res}^g_{\lambda_n^N A_N} \dots \mathrm{Res}^g_{\lambda_n^1 A_1}(x_n + e_n),\\
&C_n = \{z \in E:\ D_g(z, y_n) \le D_g(z, x_n + e_n)\},\\
&Q_n = \{z \in E:\ \langle z - x_n, \bigtriangledown g(x_0) - \bigtriangledown g(x_n)\rangle \le 0\},\\
&x_{n+1} = \mathrm{proj}^g_{C_n \cap Q_n} x_0,\ n \ge 0,
\end{aligned}$$

or

$$\begin{aligned}
&x_0 \in E \text{ chosen arbitrarily},\\
&H_0 = E,\\
&y_n = \mathrm{Res}^g_{\lambda_n^N A_N} \dots \mathrm{Res}^g_{\lambda_n^1 A_1}(x_n + e_n),\\
&H_{n+1} = \{z \in H_n:\ D_g(z, y_n) \le D_g(z, x_n + e_n)\},\\
&x_{n+1} = \mathrm{proj}^g_{H_{n+1}} x_0,\ n \ge 0,
\end{aligned}$$

where, for each $i = 1, 2, \dots, N$, $\liminf_{n\to\infty} \lambda_n^i > 0$, the sequence of errors $\{e_n\}$ satisfies $\liminf_{n\to\infty} e_n = 0$, and $\mathrm{Res}^g_{\lambda_n^i A_i} = (\bigtriangledown g + \lambda_n^i A_i)^{-1} \bigtriangledown g$.

In 2017, Timnak and others [19] proposed a new Halpern-type iterative scheme for finding an element $x^* \in S$. They proved strong convergence of the sequence $\{x_n\}$ which is defined by

$$\begin{aligned}
&u \in E,\ x_1 \in E \text{ chose arbitrarily},\\
&y_n = \bigtriangledown g^*[\beta_n \bigtriangledown g(x_n) + (1 - \beta_n) \bigtriangledown g(\mathrm{Res}^g_{r_N A_N} \dots \mathrm{Res}^g_{r_1 A_1}(x_n))],\\
&x_{n+1} = \bigtriangledown g^*[\alpha_n \bigtriangledown g(u) + (1 - \alpha_n) \bigtriangledown g(y_n)],\ n \ge 1,
\end{aligned}$$

where $r_i > 0$, for each $i = 1, 2, \dots, N$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$ satisfying the following conditions:

i) $\lim_{n\to\infty} \alpha_n = 0$;
ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

In 2016, Ibaraki [7] studied the shrinking projection method [18] with error for finding a null point of a monotone operator in a Banach space. Let $A:\ E \longrightarrow 2^{E^*}$ be a monotone operator such that $A^{-1}0 \neq \emptyset$ and $D(A) \subset C \subset J_E^{-1} R(J_E + r_n A)$, where $C$ is a nonempty, closed, and convex subset of $E$, and $\{r_n\}$ is a sequence of positive real numbers. He considered the sequence $\{x_n\}$ generated by $x_1 = u \in C$, $C_1 = C$, and

$$\begin{aligned}
&y_n = J_{r_n}(x_n),\\
&C_{n+1} = \{z \in C:\ \langle y_n - z, J_E(x_n) - J_E(y_n)\rangle \ge 0\} \cap C_n,\\
&x_{n+1} \in \{z \in C:\ \|u - z\|^2 \le d(u, C_{n+1})^2 + \delta_{n+1}\} \cap C_{n+1},
\end{aligned}$$

where $\{\delta_n\}$ is a sequence of non-negative numbers and $d(u, C_{n+1})$ is the distance from $u$ to $C_{n+1}$. He proved that if $\limsup_{n\to\infty} \delta_n = 0$, then $\{x_n\}$ converges strongly to $P_{A^{-1}0}u$ as $n \to \infty$. The result of Ibaraki is the extension the results of Ibaraki and Kimura [6] and Kimura [9].

Thus, there are some open questions which are posed as follows:

1) Can we extend the above iterative method for finding an element $x^* \in S = \cap_{i=1}^{N} A_i^{-1}0 \neq \emptyset$, where $A_i$, $i = 1, 2, \ldots, N$, are maximal monotone operators on the Banach spaces $E$?
2) Can we replace the equation $y_n = J_{r_n}(x_n)$ by the following inclusion equation
$$r_n A^{\varepsilon_n}(y_n) + J_E(y_n) \ni J_E(x_n),$$
where $A^{\varepsilon_n}$ is the $\varepsilon_n$-enlargement of $A$ with $\varepsilon_n \geq 0$?

In this paper, by using the tools of $\varepsilon$-enlargement of maximal monotone operators and the shrinking projection method, we introduce two strong convergence theorems to answer two above open questions. This results are the extension of Ibaraki's result [7]. Moreover, we also give an application of the main results for solving the problem of finding a common minimum point of convex functions.

## 2. Preliminaries

Let $E$ be a real Banach space with norm $\|\cdot\|$, and let $E^*$ be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f\rangle$. When $\{x_n\}$ is a sequence in $E$, then $x_n \to x$ (resp. $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp. weak, weak$^*$) convergence of the sequence $\{x_n\}$ to $x$. Let $J_E$ denote the normalized duality mapping from $E$ into $2^{E^*}$ given by
$$J_E x = \left\{ f \in E^* : \ \langle x, f\rangle = \|x\|^2 = \|f\|^2 \right\} \qquad \forall x \in E.$$

We always use $S_E$ to denote the unit sphere $S_E = \{x \in E : \ \|x\| = 1\}$. A Banach space $E$ is said to be strictly convex if $x, y \in S_E$ with $x \neq y$ and, for all $t \in (0,1)$,
$$\|(1-t)x + ty\| < 1.$$
A Banach space $E$ is said to be uniformly convex if for any $\varepsilon \in (0, 2]$ and the inequalities $\|x\| \leq 1$, $\|y\| \leq 1$, $\|x - y\| \geq \varepsilon$, there exists a $\delta = \delta(\varepsilon) > 0$ such that
$$\frac{\|x+y\|}{2} \leq 1 - \delta.$$

A Banach space $E$ is said to be smooth provided the limit
$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$
exists for each $x$ and $y$ in $S_E$. In this case, the norm of $E$ is said to be Gâteaux differentiable. It is said to be uniformly Gâteaux differentiable if for each $y \in S_E$, this limit attained uniformly for $x \in S_E$.

Let $E$ be a reflexive Banach space; we know that $E$ is uniformly convex if and only if $E^*$ is uniformly smooth.

We have following properties of the normalized duality mapping $J_E$:

(i) $E$ is reflexive if and only if $J_E$ is surjective;
(ii) If $E^*$ is strictly convex, then $J_E$ is single-valued;
(iii) If $E$ is a smooth, strictly convex and reflexive Banach space, then $J_E$ is single-valued bijection;

(iv) If $E^*$ is uniformly convex, then $J_E$ is uniformly continuous on each bounded set of $E$.

We know that, if $E$ is a smooth, strictly convex, and reflexive Banach space and $C$ is a nonempty, closed, and convex subset of $E$, then, for each $x \in E$, there exists unique $z \in C$ such that

$$\|x - z\| = \inf_{y \in C} \|x - y\|.$$

The mapping $P_C : E \longrightarrow C$ defines by $P_C x = z$ is called metric projection from $E$ on to $C$, and we denote by $d(x, C) = \|x - z\|$.

Let $E$ be a smooth Banach space. Define a function $\phi : E \times E \longrightarrow \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, J_E y\rangle + \|y\|^2$$

for all $x, y \in E$. From the definition of $\phi$, it is easy to see that the function $\phi$ has the following properties:

(i) $(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2$ for all $x, y \in E$;
(ii) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, J_E z - J_E y\rangle$ for all $x, y, z \in E$;
(iii) If $E$ is strictly convex, then $\phi(x, y) = 0$ if and only if $x = y$.

Let $A : E \longrightarrow 2^{E^*}$ be an operator. The effective domain of $A$ is denoted by $D(A)$; that is, $D(A) = \{x \in E : Ax \neq \emptyset\}$. Recall that $A$ is called monotone operator if $\langle x - y, u - v\rangle \ge 0$ for all $x, y \in D(A)$ and for all $u \in Ax$ and $v \in A(y)$. A monotone operator $A$ on $E$ is called maximal monotone if its graph is not properly contained in the graph of any other monotone operator on $E$. We know that if $A$ is maximal monotone operator on $E$ and $E$ is a uniformly convex and smooth Banach space, then $R(J_E + rA) = E^*$, for all $r > 0$, where $R(J_E + rA)$ is the range of $J_E + rA$ [2]; if additionally $E$ is strictly convex then, for each $x \in E$ and $r > 0$, there exists unique $x_r \in E$ such that

$$J_E x \in J_E x_r + rAx_r.$$

Hence, in this case we can define a mapping $J_r : E \longrightarrow E$ by $J_r x = x_r$, and $J_r$ is called the generalized resolvent of $A$.

The set of null point of $A$ is defined by $A^{-1}0 = \{z \in E : 0 \in Az\}$, and we know that $A^{-1}0$ is a closed and convex subset of $E$.

Let $A : E \longrightarrow 2^{E^*}$ be a maximal monotone operator. In [4], for each $\varepsilon \ge 0$, Burachik and Svaiter defined $A^\varepsilon(x)$, an $\varepsilon$-enlargement of $A$, as follows:

$$A^\varepsilon x = \{u \in E^* : \langle y - x, v - u\rangle \ge -\varepsilon, \ \forall y \in E, \ v \in Ay\}.$$

It is easy to see that $A^0 x = Ax$, and if $0 \le \varepsilon_1 \le \varepsilon_2$, then $A^{\varepsilon_1} x \subseteq A^{\varepsilon_2} x$ for any $x \in E$. The using of element in $A^\varepsilon$ instead of $A$ allows an extra degree freedom which is very useful in various applications.

Let $\{C_n\}$ be the sequence of closed, convex, and nonempty subsets of a reflexive Banach space $E$. We define the subsets s-$\mathrm{Li}_n C_n$ and w-$\mathrm{Ls}_n C_n$ of $E$ as follows: $x \in$ s-$\mathrm{Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ converges strongly to $x$ and that $x_n \in C_n$ for all $n \ge 1$; $x \in$ w-$\mathrm{Ls}_n C_n$ if and only if there exists a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ and the sequence $\{y_k\} \subset E$ such that $y_k \rightharpoonup x$ and $y_k \in C_{n_k}$ for all $k \ge 1$. If s-$\mathrm{Li}_n C_n$ = w-$\mathrm{Ls}_n C_n = \Omega_0$, then $\Omega_0$ is called the limits of $\{C_n\}$ in the sense of Mosco [12], and it is denoted by $\Omega_0 = \text{M-}\lim_{n\to\infty} C_n$.

The following lemmas will be needed in what follows for the proof of main theorems.

**Lemma 2.1.** [21] *Let $E$ be a Banach space, $r \in (0,\infty)$, and $B_r = \{x \in E : \|x\| \le r\}$. If $E$ is uniformly convex, then there exists a continuous, strictly increasing, and convex function $g: [0, 2r] \longrightarrow [0,\infty)$ with $g(0) = 0$ such that*

$$\|\alpha x + (1-\alpha)y\|^2 \le \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)g(\|x-y\|)$$

*for all $x, y \in B_r$ and $\alpha \in [0,1]$.*

**Lemma 2.2.** [8] *Let $E$ be a uniformly convex and smooth Banach space, and let $\{y_n\}$ and $\{z_n\}$ be two sequences of $E$. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

**Lemma 2.3.** [5] *Let $E$ be a smooth, reflexive, and strictly convex Banach space having the Kadec–Klee property. Let $\{C_n\}$ be a sequence of nonempty, closed, and convex subsets of $E$. If $C_0 = M - \lim_{n\to\infty} C_n$ exists and is nonempty, then $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x$ for each $x \in C$.*

**Lemma 2.4.** [4] *The graph of $A^\varepsilon: \mathbb{R}_+ \times E \longrightarrow 2^{E^*}$ is demiclosed; that is, the conditions below hold:*

(i) *If $\{x_n\} \subset E$ converges strongly to $x_0$, $\{u_n \in A^{\varepsilon_n}x_n\}$ converges weakly* to $u_0$ in $E^*$, and $\{\varepsilon_n\} \subset \mathbb{R}_+$ converges to $\varepsilon$, then $u_0 \in A^\varepsilon x_0$;*

(ii) *If $\{x_n\} \subset E$ converges weak to $x_0$, $\{u_n \in A^{\varepsilon_n}x_n\}$ converges strongly to $u_0$ in $E^*$, and $\{\varepsilon_n\} \subset \mathbb{R}_+$ converges to $\varepsilon$, then $u_0 \in A^\varepsilon x_0$.*

## 3. Main results

Let $E$ be a uniformly convex and smooth Banach space, and let $A_i: E \longrightarrow 2^{E^*}$, $i = 1, 2, \dots, N$, be maximal monotone operators of $E$ into $2^{E^*}$ such that $S = \cap_{i=1}^N A_i^{-1}0 \neq \emptyset$. Consider the following problem.

$$\text{Find an element } x^* \in S. \tag{3.1}$$

In order to solve the Problem (3.1), we propose two algorithms as follows: Let $\{\varepsilon_n\}$ and $\{\delta_n\}$ be non-negative real sequences, and let $\{r_{i,n}\}$, $i = 1, 2, \dots, N$, be positive real sequences such that $\min_i\{\inf_n\{r_{i,n}\}\} \ge r > 0$.

**Algorithm 3.1.** *For a given point $u \in E$, we define the sequence $\{x_n\}$ by $x_1 = x \in E$, $C_1 = E$, and*

$$\begin{aligned}
&\text{Find } y_{i,n} \in E \text{ such that } J_E(y_{i,n}) - J_E(x_n) + r_{i,n}A_i^{\varepsilon_n}y_{i,n} \ni 0,\ i = 1,\dots,N\\
&\text{Choose } i_n \text{ such that } \|y_{i_n,n} - x_n\| = \max_{i=1,\dots,N}\{\|y_{i,n} - x_n\|\},\ \text{let } y_n = y_{i_n,n},\\
&C_{n+1} = \{z \in C_n :\ \langle y_n - z, J_E(x_n) - J_E(y_n)\rangle \ge -\varepsilon_n r_{i_n,n}\},\\
&\text{Find } x_{n+1} \in \{z \in C_{n+1} :\ \|u - z\|^2 \le d^2(u, C_{n+1}) + \delta_{n+1}\},\ n \ge 1.
\end{aligned} \tag{3.2}$$

**Algorithm 3.2.** *For a given point $u \in E$, we define the sequence $\{x_n\}$ by $x_1 = x \in E$, $C_1 = E$, and*

$$\begin{aligned}
&\text{Find } y_{i,n} \in E \text{ such that } J_E(y_{i,n}) - J_E(x_n) + r_{i,n}A_i^{\varepsilon_n} y_{i,n} \ni 0,\ i = 1, \dots, N;\\
&C_{n+1}^i = \{z \in C_n : \ \langle y_{i,n} - z, J_E(x_n) - J_E(y_{i,n})\rangle \geq -\varepsilon_n r_{i,n}\},\ i = 1, 2, \dots, N\\
&C_{n+1} = \cap_{i=1}^N C_{n+1}^i;\\
&\text{Find } x_{n+1} \in \{z \in C_{n+1} : \ \|u - z\|^2 \leq d^2(u, C_{n+1}) + \delta_{n+1}\},\ n \geq 1.
\end{aligned} \tag{3.3}$$

We will prove the strong convergence of Algorithms 3.1 and 3.2 under the following conditions:

C1) $\lim_{n\to\infty} \varepsilon_n r_{i,n} = 0$ for all $i = 1, 2, \dots, N$;
C2) $\lim_{n\to\infty} \delta_n = 0$.

*Remark* 3.3.

i) In Algorithm 3.2, in order to define the element $x_{n+1}$, we have to find the projection of $u$ onto the intersection of $n \times N$ half-spaces. But in Algorithm 3.1, we only find the projection of $u$ onto the intersection of $n$ half-spaces. So, the algorithm to define $x_{n+1}$ in Algorithm 3.1 is simpler than the algorithm in Algorithm 3.2. However, in the both cases, we can find the element $x_{n+1}$ by the approximation solution of the following minimization problem: Find a minimum point of the convex function $f(x) = \dfrac{1}{2}\|x - u\|^2$ over the intersection of a finite family of half-spaces $C_i$. In particular, if $E = \mathbb{R}^m$, then we can find $x_{n+1}$ easily by using the "Quadratic Programming Algorithms" package in MATLAB software.
ii) In Algorithms 3.1 and 3.2, if $N = 1$ and $\varepsilon_n = 0$, for all $n \geq 1$, then we obtain the Ibaraki's result [7, Theorem 4.2].

First, we need the following lemma.

**Lemma 3.4.** *If $\{C_n\}$ is a decreasing sequence of closed and convex subsets of a reflexive Banach space $E$ and $\Omega_0 = \cap_{n=1}^\infty C_n \neq \emptyset$, then $\Omega_0 = M\text{-}\lim_{n\to\infty} C_n$.*

*Proof.* Indeed, it is clear that if $x \in \Omega_0$, then $x \in \text{s-Li}_n C_n$ and $x \in \text{w-Ls}_n C_n$, because the sequence $\{x_n\}$ with $x_n = x$, for all $n \geq 1$, converges strongly to $x$. Thus, we have $\Omega_0 \subset \text{s-Li}_n C_n$ and $\Omega_0 \subset \text{w-Ls}_n C_n$.

Now we will show that $\Omega_0 \supseteq \text{s-Li}_n C_n$ and $\Omega_0 \supseteq \text{w-Ls}_n C_n$. Let $x \in \text{s-Li}_n C_n$, from the definition of $\text{s-Li}_n C_n$, there exists a sequence $\{x_n\} \subset E$ with $x_n \in C_n$, for all $n \geq 1$, such that $x_n \to x$, as $n \to \infty$. Since $\{C_n\}$ is a decreasing sequence, $x_{n+k} \in C_n$ for all $n \geq 1$ and $k \geq 0$. So, letting $k \to \infty$ and by the closedness of $C_n$, we get that $x \in C_n$ for all $n \geq 1$. Thus, $x \in \Omega_0$, and hence $\Omega_0 \supseteq \text{s-Li}_n C_n$. Next, let $y \in \text{w-Ls}_n C_n$, from the definition of $\text{w-Ls}_n C_n$, there exists a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ and the sequence $\{y_k\} \subset E$ such that $y_k \rightharpoonup x$ and $y_k \in C_{n_k}$ for all $k \geq 1$. From $\{C_n\}$ is a decreasing sequence, we have

$$y_{k+p} \in C_{n_k} \tag{3.4}$$

for all $k \geq 1$ and $p \geq 0$. Since $C_{n_k}$ is closed and convex, $C_{n_k}$ is weakly closed in $E$ for all $k \geq 1$. So, in (3.4) letting $p \to \infty$, we get that $y \in C_{n_k}$ for all $k \geq 1$. Since $C_k \supseteq C_{n_k}$, $y \in C_k$ for all $k \geq 1$. So, $y \in \Omega_0$, and hence $\Omega_0 \supseteq$ w-Ls$_n C_n$.

Consequently, we obtain that s-Li$_n C_n$ = w-Ls$_n C_n$ = $\Omega_0$. Thus, $\Omega_0$ =M-$\lim_{n\to\infty} C_n$. □

The strong convergence of Algorithm 3.1 is given by the following theorem.

**Theorem 3.5.** *If the conditions* C1) *and* C2) *are satisfied, then the sequence* $\{x_n\}$ *generated by Algorithm 3.1 converges strongly to* $P_S u$*, as* $n \to \infty$.

*Proof.* First, we show that $S \subset C_n$, for all $n \geq 1$, by mathematical induction. Indeed, it is clear that $S \subset C_1 = E$. Suppose that $S \subset C_n$ for some $n \geq 1$. Take $v \in S$, we have

$$J_E(y_{i_n,n}) - J_E(x_n) + r_{i_n,n} A_{i_n}^{\varepsilon_n} y_{i_n,n} \ni 0, \qquad A_{i_n} v \ni 0.$$

From the definition of $A_{i_n}^{\varepsilon_n}$, we get

$$\langle y_n - v, J_E(x_n) - J_E(y_n)\rangle \geq -\varepsilon_n r_{i_n,n}.$$

Thus, $u \in C_{n+1}$. Since $v$ is arbitrary in $S$, $S \subset C_{n+1}$. So, by induction we obtain that $S \subset C_n$ for all $n \geq 1$.

Moreover, $C_n$ is a closed and convex subset of $E$ for all $n$. Hence, the sequence $\{x_n\}$ is well defined.

Now, for each $n$, denote by $p_n = P_{C_n} u$. Since, $\{C_n\}$ is the sequence of decreasing subsets of $E$ which contains $S$, and from Lemma 3.4, there exists the limit $\Omega_0$ =M-$\lim_{n\to\infty} C_n$. By Lemma 2.3, we have $p_n \to p_0 = P_{\Omega_0} u$, as $n \to \infty$.

Since $p_n = P_{C_n} u$, $d(u, C_n) = \|u - p_n\|$. From $x_n \in C_n$ and the definition of $C_n$, we have

$$\|u - x_n\|^2 \leq \|u - p_n\|^2 + \delta_n \qquad \forall n \geq 2. \tag{3.5}$$

From the convexity of $C_n$, we have $\alpha p_n + (1-\alpha)x_n \in C_n$ for all $\alpha \in (0,1)$. Thus, from the definition of $p_n = P_{C_n} u$ and Lemma 2.1, we get

$$\begin{aligned}\|p_n - u\|^2 &\leq \|\alpha p_n + (1-\alpha)x_n - u\|^2 \\ &\leq \alpha\|p_n - u\|^2 + (1-\alpha)\|x_n - u\|^2 - \alpha(1-\alpha)g(\|x_n - p_n\|),\end{aligned}$$

which implies that

$$\|p_n - u\|^2 \leq \|x_n - u\|^2 - \alpha g(\|x_n - p_n\|).$$

Thus, it follows from (3.5) that

$$\alpha g(\|x_n - p_n\|) \leq \delta_n \qquad \forall \alpha \in (0,1). \tag{3.6}$$

In (3.6), letting $\alpha \to 1^-$, we get

$$g(\|x_n - p_n\|) \leq \delta_n.$$

By the property of $g$ and $\delta_n \to 0$, we have

$$\|x_n - p_n\| \to 0.$$

From $p_{n+1} \in C_{n+1}$ and the definition of $C_{n+1}$, we have

$$\langle y_n - p_{n+1}, J_E(x_n) - J_E(y_n)\rangle \geq -\varepsilon_n r_{i_n,n}.$$

Thus, from the property of $\phi$, we obtain

$$\begin{aligned}-2\varepsilon_n r_{i_n,n} &\leq 2\langle p_{n+1} - y_n, J_E(y_n) - J_E(x_n)\rangle \\ &= \phi(p_{n+1}, x_n) - \phi(p_{n+1}, y_n) - \phi(y_n, x_n) \\ &\leq \phi(p_{n+1}, x_n) - \phi(y_n, x_n).\end{aligned}$$

Hence,

$$\phi(y_n, x_n) \leq \phi(p_{n+1}, x_n) + 2\varepsilon_n r_{i_n,n}.$$

From Lemma 2.2 and $p_n \to p_0$, $x_n \to p_0$, letting $n \to \infty$ we get that

$$\|x_n - y_n\| \to 0.$$

By the definition of $y_n$, we have

$$\|x_n - y_{i,n}\| \to 0 \qquad \forall i = 1, 2, \ldots, N.$$

This implies that $y_{i,n} \to p_0$ for all $i = 1, 2, \ldots, N$, as $n \to \infty$. Since $E$ is uniformly smooth, the duality mapping $J_E$ is uniformly norm-to-norm continuous on each bounded subset on $E$. Therefore, we obtain

$$\|J_E(x_n) - J_E(y_{i,n})\| \to 0, \qquad \forall i = 1, 2, \ldots, N. \tag{3.7}$$

Furthermore, from $\min_i\{\inf_n\{r_{i,n}\}\} \geq r > 0$ and (3.7), we have

$$0 \leftarrow \frac{J_E(x_n) - J_E(y_{i,n})}{r_{i,n}} \in A_i^{\varepsilon_n} y_{i,n}$$

for all $i = 1, 2, \ldots, N$, as $n \to \infty$. So, by Lemma 2.4, we obtain $p_0 \in A_i^{-1}0$ for all $i = 1, 2, \ldots, N$; that is, $p_0 \in S$.

Finally, we show that $p_0 = P_S u$. Indeed, let $x^* = P_S u$. Since $S \subset C_n$, $x^* \in C_n$. Thus, from $p_n = P_{C_n} u$, we have

$$\|p_n - u\| \leq \|u - x^*\| \qquad \forall n \geq 1.$$

Letting $n \to \infty$, we get that $\|u - p_0\| \leq \|u - x^*\|$. By the uniqueness of $x^*$, we obtain that $p_0 = x^* = P_S u$.

This completes the proof. □

Now, we will prove the strong convergence of Algorithm 3.2.

**Theorem 3.6.** *If the conditions* C1) *and* C2) *are satisfied, then the sequence* $\{x_n\}$ *generated by Algorithm 3.2 converges strongly to* $P_S u$*, as* $n \to \infty$.

*Proof.* First, we show that $S \subset C_n$, for all $n \geq 1$, by mathematical induction. Indeed, it is clear that $S \subset C_1 = E$. Suppose that $S \subset C_n$ for some $n \geq 1$. Take $v \in S$, we have

$$J_E(y_{i,n}) - J_E(x_n) + r_{i,n} A_i^{\varepsilon_n} y_{i,n} \ni 0 \qquad A_i v \ni 0.$$

From the definition of $A_i^{\varepsilon_n}$, we get

$$\langle y_{i,n} - v, J_E(x_n) - J(y_{i,n})\rangle \geq -\varepsilon_n r_{i,n}.$$

Thus, $v \in C_{n+1}^i$ for all $i = 1, 2, \ldots, N$. So, $v \in C_{n+1} = \cap_{i=1}^N C_{n+1}^i$. By induction we obtain that $S \subset C_n$ for all $n \geq 1$.

Now, for each $n$, putting $p_n = P_{C_n} u$. It is similar to the proof of Theorem 3.5, we obtain the following statements:

a) $p_n \to p_0 = P_{\Omega_0} u$ with $\Omega_0 = \cap_{n=1}^{\infty} C_n$;
b) $\|x_n - p_n\| \to 0$.

We have $p_{n+1} \in C_{n+1} = \cap_{i=1}^{N} C_{n+1}^i$. Hence, $p_{n+1} \in C_{n+1}^i$ for all $i = 1, 2, \ldots, N$. Thus, from the definition of $C_{n+1}^i$, we have

$$\langle y_{i,n} - p_{n+1}, J_E(x_n) - J_E(y_{i,n})\rangle \geq -\varepsilon_n r_{i,n}.$$

Thus, from the property of $\phi$, we obtain

$$\begin{aligned}-2\varepsilon_n r_{i,n} &\leq 2\langle p_{n+1} - y_{i,n}, J_E(y_{i,n}) - J_E(x_n)\rangle \\ &= \phi(p_{n+1}, x_n) - \phi(p_{n+1}, y_{i,n}) - \phi(y_{i,n}, x_n) \\ &\leq \phi(p_{n+1}, x_n) - \phi(y_{i,n}, x_n).\end{aligned}$$

Hence,

$$\phi(y_{i,n}, x_n) \leq \phi(p_{n+1}, x_n) + 2\varepsilon_n r_{i,n}$$

for all $i = 1, 2, \ldots, N$. From a), b), and Lemma 2.2, we obtain that

$$\|x_n - y_{i,n}\| \to 0$$

for all $i = 1, 2, \ldots, N$.
The rest of the proof follows the pattern of Theorem 3.5.
This completes the proof. □

Next, we have the following corollaries.

**Corollary 3.7.** *Let $E$ be a uniformly convex and smooth Banach space, and let $A_i : E \longrightarrow 2^{E^*}$, $i = 1, 2, \ldots, N$, be maximal monotone operators of $E$ into $2^{E^*}$ such that $S = \cap_{i=1}^{N} A_i^{-1} 0 \neq \emptyset$. Let $J_r^i$ be the generalized resolvent of $A_i$ for $r > 0$ with $i = 1, 2, \ldots, N$. Let $\{\delta_n\}$ be non-negative real sequence, and let $\{r_{i,n}\}$, $i = 1, 2, \ldots, N$, be positive real sequences such that $\min_i\{\inf_n\{r_{i,n}\}\} \geq r > 0$. For a given point $u \in E$, we define the sequence $\{x_n\}$ by $x_1 = x \in E$, $C_1 = E$, and*

i) $y_{i,n} = J_{r_{i,n}}^i x_n$, $i = 1, 2, \ldots, N$

ii) *Choose $i_n$ such that* $\|y_{i_n,n} - x_n\| = \max\limits_{i=1,\ldots,N}\{\|y_{i,n} - x_n\|\}$, *let* $y_n = y_{i_n,n}$,

$C_{n+1} = \{z \in C_n : \langle y_n - z, J_E(x_n) - J_E(y_n)\rangle \geq 0\}$, *or*

ii*) $C_{n+1}^i = \{z \in C_n : \langle y_{i,n} - z, J_E(x_n) - J_E(y_{i,n})\rangle \geq 0\}$, $i = 1, 2, \ldots, N$

$C_{n+1} = \cap_{i=1}^{N} C_{n+1}^i$,

iii) *Find* $x_{n+1} \in \{z \in C_{n+1} : \|u - z\|^2 \leq d^2(u, C_{n+1}) + \delta_{n+1}\}$, $n = 1, 2, \ldots$.

*If $\lim_{n\to\infty} \delta_n = 0$, then the sequence $\{x_n\}$ converges strongly to $P_S u$, as $n \to \infty$.*

*Proof.* In (3.2) and (3.3) if $\varepsilon_n = 0$, for all $n \geq 1$, then the elements $y_{i,n}$, $i = 1, 2, \ldots, N$, can be rewritten in the form

$$J_E(y_{i,n}) - J_E(x_n) + r_{i,n} A_i y_{i,n} \ni 0;$$

this is equivalent to

$$y_{i,n} = J_{r_{i,n}}^i x_n$$

for all $i = 1, 2, \ldots, N$.

So, apply Theorems 3.5 and 3.6 with $\varepsilon_n = 0$ for all $n \geq 1$, we obtain the proof of this corollary. $\square$

**Corollary 3.8.** *Let $E$ be a uniformly convex and smooth Banach space, and let $A_i: E \longrightarrow 2^{E^*}$, $i = 1, 2, \ldots, N$, be maximal monotone operators of $E$ into $2^{E^*}$ such that $S = \cap_{i=1}^N A_i^{-1}0 \neq \emptyset$. Let $\{\varepsilon_n\}$ be non-negative real sequence, and let $\{r_{i,n}\}$, $i = 1, 2, \ldots, N$, be positive real sequences such that $\min_i\{\inf_n\{r_{i,n}\}\} \geq r > 0$. For a given point $u \in E$, we define the sequence $\{x_n\}$ by $x_1 = x \in E$, $C_1 = E$, and*

i) *Find $y_{i,n} \in E$ such that $J_E(y_{i,n}) - J_E(x_n) + r_{i,n}A_i^{\varepsilon_n}y_{i,n} \ni 0,\ i = 1, 2, \ldots, N$*

ii) *Choose $i_n$ such that $\|y_{i_n,n} - x_n\| = \max_{i=1,\ldots,N}\{\|y_{i,n} - x_n\|\}$, let $y_n = y_{i_n,n}$,*

*$C_{n+1} = \{z \in C_n :\ \langle y_n - z, J_E(x_n - y_n)\rangle \geq -\varepsilon_n r_{i_n,n}\}$, or*

ii*) *$C^i_{n+1} = \{z \in C_n :\ \langle y_{i,n} - z, J_E(x_n) - J_E(y_{i,n})\rangle \geq -\varepsilon_n r_{i,n}\},\ i = 1, 2, \ldots, N$*

*$C_{n+1} = \cap_{i=1}^N C^i_{n+1}$,*

iii) *$x_{n+1} = P_{C_{n+1}}u,\ n = 1, 2, \ldots$.*

*If $\lim_{n\to\infty} \varepsilon_n r_{i,n} = 0$ for all $i = 1, 2, \ldots, N$, then the sequence $\{x_n\}$ converges strongly to $P_S u$, as $n \to \infty$.*

*Proof.* In (3.2) and (3.3), if $\delta_n = 0$, for all $n \geq 1$, then we have the element $x_{n+1}$ is defined by
$$x_{n+1} \in \{z \in C_{n+1} :\ \|u - z\| \leq d(u, C_{n+1})\};$$
that is, $x_{n+1} = P_{C_{n+1}}u$.

So, apply Theorem 3.5 with $\delta_n = 0$ for all $n \geq 1$, we obtain the proof of this corollary. $\square$

*Remark* 3.9. If $\varepsilon = \delta_n = 0$, for all $n \geq 1$, then in Corollaries 3.7 and 3.8 the sequence $\{x_n\}$ will be defined by $x_1 = x \in E$, $C_1 = E$, and

i) $y_{i,n} = J^i_{r_{i,n}}x_n,\ i = 1, 2, \ldots, N$

ii) Choose $i_n$ such that $\|y_{i_n,n} - x_n\| = \max_{i=1,\ldots,N}\{\|y_{i,n} - x_n\|\}$, let $y_n = y_{i_n,n}$,

$C_{n+1} = \{z \in C_n :\ \langle y_n - z, J_E(x_n) - J_E(y_n)\rangle \geq 0\}$, or

ii*) $C^i_{n+1} = \{z \in C_n :\ \langle y_{i,n} - z, J_E(x_n) - J_E(y_{i,n})\rangle \geq 0\},\ i = 1, 2, \ldots, N$

$C_{n+1} = \cap_{i=1}^N C^i_{n+1}$,

iii) $x_{n+1} = P_{C_{n+1}}u,\ n = 1, 2, \ldots$.

*Remark* 3.10. In Remark 3.9, if $E$ is a real Hilbert space and $N = 1$, then we obtain the result of of Takahashi, Takeuchi, and Kubota (see, [18, Theorem 4.5]). But, in this case we do not use the condition $r_n \to \infty$. So, the Corollaries 3.7 and 3.8 are more general than the result of Takahashi and others.

## 4. An application

Let $E$ be a Banach space, and let $f:\ E \longrightarrow (-\infty, \infty]$ be a proper, lower semicontinuous, and convex function. The subdifferential of $f$ is multi-valued mapping $\partial f: E \longrightarrow 2^{E^*}$ which is defined by

$$\partial f(x) = \{g \in E^* : \ f(y) - f(x) \geq \langle y - x, g\rangle, \ \forall y \in E\}$$

for all $x \in E$. We know that $\partial f$ is maximal monotone operator (see [14]) and $x_0 \in \arg\min_E f(x)$ if and only if $\partial f(x_0) \ni 0$.

The $\varepsilon$-subdiferential enlargement of $\partial f$, is given by

$$\partial_\varepsilon f(x) = \{u \in E^* : \ f(y) - f(x) \geq \langle y - x, u\rangle - \varepsilon, \ \forall y \in E\}$$

for each $\varepsilon \geq 0$. We know that $\partial_\varepsilon f(x) \subset (\partial f)^\varepsilon(x)$ for any $x \in E$. Moreover, in the some particular cases, we have that $\partial_\varepsilon f(x) \subsetneq (\partial f)^\varepsilon(x)$ (see, [3, Example 2 and Example 3]).

In [1] when $E$ is a real Hilbert space, Alvarez proposed the following approximate inertial proximal algorithm:

$$c_n \partial_{\varepsilon_n} f(x_{n+1}) + x_{n+1} - x_n - \alpha_n(x_n - x_{n-1}) \ni 0.$$

In [13], Moudafi and Elisabeth extended the above iterative method in the form

$$c_n(\partial f)^{\varepsilon_n}(x_{n+1}) + x_{n+1} - x_n - \alpha_n(x_n - x_{n-1}) \ni 0. \tag{4.1}$$

They proved that if there exists $c > 0$ such that $c_n \geq c$ for all $n \geq 1$, and there is $\alpha \in [0, 1)$ such that $\{\alpha_n\} \subset [0, \alpha]$, $\sum_{n=1}^\infty c_k \varepsilon_k < \infty$, and

$$\sum_{n=1}^\infty \alpha_n \|x_n - x_{n-1}\|^2 < \infty,$$

then the sequence $\{x_n\}$ converges weakly to a minimum point of $f$.

Note that, if $\alpha_n = 0$ for all $n \geq 1$, then (4.1) becomes

$$c_n(\partial f)^{\varepsilon_n}(x_{n+1}) + x_{n+1} - x_n \ni 0.$$

From Theorems 3.5 and 3.6, we have the following theorem.

**Theorem 4.1.** *Let $E$ be a uniformly convex and smooth Banach space, and let $f_i$, $i = 1, 2, \ldots, N$, be proper, lower semicontinuous, and convex functions of $E$ into $(-\infty, \infty]$ such that $S = \cap_{i=1}^N \arg\min_{x\in E} f_i(x) \neq \emptyset$. Let $\{\varepsilon_n\}$ and $\{\delta_n\}$ be non-negative real sequences, and let $\{r_{i,n}\}$, $i = 1, 2, \ldots, N$, be positive real sequences such that $\min_i\{\inf_n\{r_{i,n}\}\} \geq r > 0$. For a given point $u \in E$, we define the sequence $\{x_n\}$ by $x_1 = x \in E$, $C_1 = E$, and*

i) *Find $y_{i,n} \in E$ such that $J_E(y_{i,n}) - J_E(x_n) + r_{i,n}(\partial f_i)^{\varepsilon_n}(y_{i,n}) \ni 0, \ i = 1, 2, \ldots, N$*

ii) *Choose $i_n$ such that $\|y_{i_n,n} - x_n\| = \max_{i=1,\ldots,N}\{\|y_{i,n} - x_n\|\}$, let $y_n = y_{i_n,n}$,*

*$C_{n+1} = \{z \in C_n : \ \langle y_n - z, J_E(x_n) - J_E(y_n)\rangle \geq -\varepsilon_n r_{i_n,n}\}$, or*

ii*) *$C^i_{n+1} = \{z \in C_n : \ \langle y_{i,n} - z, J_E(x_n) - J_E(y_{i,n})\rangle \geq -\varepsilon_n r_{i,n}\}, \ i = 1, 2, \ldots, N$*

*$C_{n+1} = \cap_{i=1}^N C^i_{n+1}$,*

iii) *Find $x_{n+1} \in \{z \in C_{n+1} : \ \|u - z\|^2 \leq d^2(u, C_{n+1}) + \delta_{n+1}\}, \ n = 1, 2, \ldots.$*

*If $\lim_{n\to\infty} \varepsilon_n r_{i,n} = \lim_{n\to\infty} \delta_n = 0$, for all $i = 1, 2, \ldots, N$, then the sequence $\{x_n\}$ converges strongly to $P_S u$, as $n \to \infty$.*

*Remark* 4.2. Since $\partial_\varepsilon f(x) \subset (\partial f)^\varepsilon(x)$, in Theorem 4.1, we can replace $(\partial f_i)^{\varepsilon_n}$ by $(\partial f_i)_{\varepsilon_n}$ for all $i = 1, 2, \ldots, N$.

In Theorem 4.1, if $\varepsilon_n = 0$ for all $n \geq 1$, then we have the following corollary.

**Corollary 4.3.** *Let $E$ be a uniformly convex and smooth Banach space, and let $f_i$, $i = 1, 2, \ldots, N$, be proper, lower semi-continuous, and convex functions of $E$ into $(-\infty, \infty]$ such that $S = \cap_{i=1}^N \arg\min_E f_i(x) \neq \emptyset$. Let $\{\delta_n\}$ be non-negative real sequence, and let $\{r_{i,n}\}$, $i = 1, 2, \ldots, N$, be positive real sequences such that $\min_i\{\inf_n\{r_{i,n}\}\} \geq r > 0$. For a given point $u \in E$, we define the sequence $\{x_n\}$ by $x_1 = x \in E$, $C_1 = E$, and*

i) $y_{i,n} = \arg\min_{y\in E}\left\{f_i(y) + \dfrac{1}{2r_{i,n}}\|y\|^2 - \dfrac{1}{r_{i,n}}\langle y, J_E(x_n)\rangle\right\},\ i = 1, 2, \ldots, N$

ii) *Choose $i_n$ such that $\|y_{i_n,n} - x_n\| = \max_{i=1,\ldots,N}\{\|y_{i,n} - x_n\|\}$, let $y_n = y_{i_n,n}$,*

$C_{n+1} = \{z \in C_n :\ \langle y_n - z, J_E(x_n) - J_E(y_n)\rangle \geq 0\}$, *or*

ii*) $C_{n+1}^i = \{z \in C_n :\ \langle y_{i,n} - z, J_E(x_n) - J_E(y_{i,n})\rangle \geq 0\},\ i = 1, 2, \ldots, N$

$C_{n+1} = \cap_{i=1}^N C_{n+1}^i$,

iii) *Find $x_{n+1} \in \{z \in C_{n+1} :\ \|u - z\|^2 \leq d^2(u, C_{n+1}) + \delta_{n+1}\},\ n = 1, 2, \ldots.$*

*If $\lim_{n\to\infty}\delta_n = 0$, then the sequence $\{x_n\}$ converges strongly to $P_S u$, as $n \to \infty$.*

*Proof.* We have

$$y_{i,n} = \arg\ \min_{y\in E}\left\{f_i(y) + \frac{1}{2r_{i,n}}\|y\|^2 - \frac{1}{r_{i,n}}\langle y, J_E(x_n)\rangle\right\}$$

if and only if

$$\partial f_i(y_{i,n}) + \frac{1}{r_{i,n}}\big(J_E(y_{i,n}) - J_E(x_n)\big) \ni 0,$$

which implies that

$$y_{i,n} = J_{r_{i,n}}^i(x_n),$$

where $J_{r_{i,n}}^i = (J_E + r_{i,n}\partial f_i)^{-1}$.
So, by using Theorems 3.5 and 3.6 we get the proof of this corollary. □

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