OSCILLATIONS, QUASI-OSCILLATIONS AND JOINT CONTINUITY

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Abstract. Parallel to the concept of quasi-separate continuity, we define a notion for quasi-oscillation of a mapping $f : X \times Y \to \mathbb{R}$. We also introduce a topological game on $X$ to approximate the oscillation of $f$. It follows that under suitable conditions, every quasi-separately continuous mapping $f : X \times Y \to \mathbb{R}$ has the Namioka property. An illuminating example is also given.

1. Introduction

Throughout this paper, unless explicitly stated otherwise, we will assume that $X$ and $Y$ are topological spaces and $Y$ is compact. Let $f : X \times Y \to \mathbb{R}$ be a mapping. Following [7], $f$ is called quasi-separately continuous at $(x_0, y_0) \in X \times Y$ if the function $t \mapsto f(x_0, t)$ is continuous at $y_0$ and for every finite set $F$ of $Y$ and $\varepsilon > 0$, there is some open set $V \subset X$ such that $x_0 \in V$ and $|f(x, y) - f(x_0, y)| < \varepsilon$ whenever $x \in V$ and $y \in F$. The function $f$ is called quasi-separately continuous if $f$ is quasi-separately continuous at each point of $X \times Y$. We define the quasi-oscillation of a mapping $f : X \times Y \to \mathbb{R}$ at $x_0 \in X$ as follows:

$$Q(f, x_0) = \sup_{F \text{ finite}} \{\inf \{\sup_{(x, y) \in V \times F} |f(x, y) - f(x_0, y)| : V \text{ open}, x_0 \in V\}\}. $$

It is easy to see that $f : X \times Y \to \mathbb{R}$ is quasi-separately continuous at $(x_0, y_0)$ if and only if $f$ is continuous with respect to second variable in $y_0$ and $Q(f, x_0) = 0$.

Following [6], a mapping $f : X \times Y \to \mathbb{R}$ is said to have the Namioka property if there exists a dense in $G_\delta$ subset $D$ of $X$ such that $f$ is jointly continuous at each point of $D \times Y$.

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In this paper, we are interested to the following problem: Suppose that \( f : X \times Y \to \mathbb{R} \) is a mapping. Under what conditions on \( X \), there are constants \( c_1 \) and \( c_2 \) such that

\[
\mathcal{O}(f; (x, y)) \leq c_1 \sup_{t \in X} Q(f, t) + c_2 \sup_{(t, s) \in X \times Y} \mathcal{O}(f(t, \cdot), s)
\]

for each point \((x, y) \in D \times Y\), where

\[
\mathcal{O}(f(t, \cdot), s) = \inf\{\text{diam}(f(\{t\} \times U)): U \text{ is open in } Y \text{ and } s \in U\}
\]

denotes the oscillation of \( y \mapsto f(t, y) \) in \( s \) and \( D \) is a dense \( G_\delta \) subset of \( X \)?

Problems of this type are considered by some authors (see e.g. [1, 2, 10, 11] and the references therein).

In this paper, inspired by [1, 5] and [9], we will introduce a topological game \( \mathcal{G}(X) \) on \( X \). Then we will show that for each mapping \( f : X \times Y \to \mathbb{R} \), there exists a dense \( G_\delta \) subset \( D \) of \( X \) such that the oscillation of \( f \) at each point of \( D \times Y \) is less than \( 10 \sup_{x \in X} Q(f, x) + 6 \sup_{(x, y) \in X \times Y} \mathcal{O}(f(x, \cdot), y) \) provided that the first player has no winning strategy in \( \mathcal{G}(X) \).

It follows that under the above condition on \( X \), every quasi-separately continuous mapping \( f : X \times Y \to \mathbb{R} \) has the Namioka property. This can be considered as a generalization of the main result in [12].

2. Main results

The story of topological games goes back to Baire [4]. Since then several topological games were invented and applied by some authors [5, 8, 9, 12]. Here, we introduce a topological game as follows.

\( \mathcal{G}(X) \) is played by two players \( \beta \) and \( \alpha \) as follows: \( \beta \) starts a game by choosing a non-empty open set \( U_1 \subset X \). \( \alpha \) answers by selecting a couple \((V_1, x_1)\), where \( V_1 \subset U_1 \) and \( x_1 \in X \). In step \( n \), \( \beta \)'s move is a non-empty open \( U_n \subset V_{n-1} \). Then \( \alpha \)'s \( n \)-th move is a pair \((V_n, x_n)\) where \( V_n \) is a non-empty open subset of \( U_n \) and \( x_n \in X \). The player \( \alpha \) wins the game \( \mathcal{G}(X) \) if there is some \( z \in \bigcap_{n=1}^{\infty} V_n \) such that for every open subset \( G \) in \( X \) with \( z \in \overline{G} \),

\[
G \cap \{x_1, x_2, \ldots\} \neq \emptyset.
\]

A strategy \( s \) for \( \alpha \) in the game \( \mathcal{G}(X) \) is a rule which determines \( \alpha \)'s move at each stage. \( X \) is called \( \beta \)-favorable for the play \( \mathcal{G}(X) \) if \( \beta \) has a winning strategy in this play, otherwise \( X \) is said to be \( \beta \)-unfavorable for this play. Clearly every separable Baire space \( X \) is \( \beta \)-unfavorable for the game \( \mathcal{G}(X) \).

A similar topological game, with a different winning rule, was introduced in [5].

Let \( Z \) be a metric space and \( r > 0 \), a family \( \mathfrak{F} \subset Z^X \) is said to be \( r \)-equicontinuous if there is an open neighborhood \( W \) of \( \Delta \), the diagonal of \( X \times X \), such that

\[
d(f(x), f(x')) < r \quad \text{for all } f \in \mathfrak{F} \text{ and } (x, x') \in W.
\]
Theorem 2.1. Let $X$ be a $\beta$-unfavorable space and $f : X \times Y \to \mathbb{R}$ be a mapping. Then there is a dense $G_δ$ subset $D$ of $X$ such that
\[ O(f, (x, y)) \leq 10 \sup_{t \in X} Q(f, t) + 6 \sup_{(s, t) \in X \times Y} O(f(t, \cdot), s) \] for all $(x, y) \in D \times Y$.

In particular, if $f : X \times Y \to \mathbb{R}$ is quasi-separately continuous, then it has the Namioka property.

Let \[ a = \sup_{x \in X} Q(f, x), \quad b = \sup_{(x, y) \in X \times Y} O(f(x, \cdot), y). \]

In order to prove the above theorem, we need to some auxiliary results.

Lemma 2.2. Suppose that $\{f(x, \cdot) : x \in U\}$ is $r$-equicontinuous for some $r > 0$ and a non-empty open subset $U$ of $X$. Then for each $\varepsilon > 0$, there exist a non-empty open subset $U'$ of $U$ and a finite open cover $\{V_1, \ldots, V_n\}$ of $Y$ such that \[ \text{diam}(f(U' \times V_i)) \leq 2(r + a) + \varepsilon \] for each $1 \leq i \leq n$.

Proof. Since $\{f(x, \cdot) : x \in U\}$ is $r$-equicontinuous, there is a neighborhood $W$ of $\Delta$ such that \[ |f(x, y) - f(x, y')| < r \quad x \in U, (y, y') \in W. \]

For each $y \in Y$, put $W_y = \{y' : (y, y') \in W\}$. Then $\{W_y : y \in Y\}$ is an open cover for $Y$. Since $Y$ is compact, there are points $y_1, \ldots, y_n \in Y$ such that $Y = \bigcup_{i=1}^n W_{y_i}$. Write $V_i = W_{y_i}$ for each $1 \leq i \leq n$. Fix some $x_1 \in U$. Since $Q(f, x_1) < a + \varepsilon/2$, there is some non-empty open subset $U_1 \subset U$ such that
\[ |f(x_1, y_1) - f(x, y_1)| < a + \varepsilon/2 \quad (x \in U_1). \]

Suppose that for $1 \leq k < n$ points $x_1, \ldots, x_k$ and open subsets $U_1, \ldots, U_k$ of $U$ have been selected. Then choose some arbitrary point $x_{k+1} \in U_k$. By our assumption, $Q(f, x_k) < a + \varepsilon/2$, therefore we can find some non-empty open subset $U_{k+1} \subset U_k$ such that
\[ |f(x_k, y_k) - f(x_k, y_k)| < a + \varepsilon/2 \quad (x \in U_{k+1}). \]

In this way by (finite) induction on $k$, points $x_1, \ldots, x_n \in U$ and $U_1 \supset \cdots \supset U_n$ are determined. Put $U' = U_n$, then for each $1 \leq i \leq k$, $y \in V_i$ and $x \in U'$ we have
\[
|f(x, y) - f(x, y_i)| \leq |f(x, y) - f(x, y_i)| + |f(x, y_i) - f(x, y_i)|
\leq r + a + \varepsilon/2.
\]

It follows that for each $1 \leq i \leq k$, \[ \text{diam}(f(U' \times V_i)) \leq 2(r + a) + \varepsilon. \]

Lemma 2.3. For each non-empty open subset $U$ of $X$ and $\varepsilon > 0$, there is a non-empty open subset $U'$ of $U$ such that $\{f(t, \cdot) : t \in U'\}$ is $(4a + 3b + \varepsilon)$-equicontinuous.
Proof. Suppose that for some $\varepsilon > 0$, there is a non-empty open subset $U$ of $X$ such that $\{f(x, \cdot) : x \in U\}$ is not $(4a + 3b + \varepsilon)$-equicontinuous for each non-empty open subset $U'$ of $U$. We will define inductively a strategy for the player $\beta$ in $G(X)$. Put $U_1 = U$ as the first move of $\beta$. Let $n > 1$ and $(V_1, x_1), \ldots, (V_n, x_n)$ be selected by $\alpha$ and $\delta = \varepsilon/20$. Since for each $x \in X$, $\sup_{y \in Y} O(f(x, \cdot), y) \leq b$, by [3, Proposition 1.18], we can find some $g_x \in C(Y)$ such that $|g_x(y) - f(x, y)| < b/2 + \delta$ for all $y \in Y$. Let

$$W_n = \{(y, y') \in Y \times Y : |g_{x_i}(y) - g_{x_i}(y')| < \frac{1}{n}, 1 \leq i \leq n\}.$$ 

Thanks to continuity of $g_{x_i}$'s, $W_n$ is an open neighborhood of $\Delta$. Let $r = 4a + 3b + \varepsilon$. Since $\{f(x, \cdot) : x \in V_n\}$ is not $r$-equicontinuous, we can find some $t_n \in V_n$ and $(y_n, y'_n) \in W_n$ such that $|f(t_n, y_n) - f(t_n, y'_n)| \geq r$. Since $Q(f, t_n) \leq a$, there is a non-empty subset $U_{n+1} \subset V_n$ such that for each $t \in U_{n+1}$,

$$|f(t_n, y_n) - f(t, y_n)| < a + \delta \quad \text{and} \quad |f(t_n, y'_n) - f(t, y'_n)| < a + \delta.$$ 

Let $U_{n+1}$ be the answer of $\beta$ to $((V_1, x_1), \ldots, (V_n, x_n))$. Therefore a strategy for the player $\beta$ is inductively defined. Since this strategy is not winning for $\beta$, some play $\{(U_n, (V_n, x_n))\}$ is won by $\alpha$. Therefore, there is some $z \in \bigcap_{n \geq 1} V_n$ such that for each open subset $G$ of $X$ with $z \in \overline{G}, G \cap \{x_1, x_2, \ldots\} \neq \emptyset$. Let $(y_\infty, y'_\infty)$ be a cluster point of $\{(y_n, y'_n)\}$ in $Y \times Y$. Then for each $n \geq i \geq 1$, we have $|g_{x_i}(y_n) - g_{x_i}(y'_n)| < \frac{1}{n}$. Since $g_{x_i}$ is continuous, it follows that $g_{x_i}(y_\infty) = g_{x_i}(y'_\infty)$. Moreover, for each $n$ we have

$$r \leq |f(t_n, y_n) - f(t_n, y'_n)| \leq |f(t_n, y_n) - f(z, y_n)| + |f(z, y_n) - f(z, y'_n)| + |f(z, y'_n) - f(t_n, y'_n)|$$

$$< 2a + 2\delta + |f(z, y_n) - g_z(y_n)| + |g_z(y_n) - g_z(y'_n)| + |g_z(y'_n) - f(z, y'_n)|$$

$$< 2a + b + 4\delta + |g_z(y_n) - g_z(y'_n)|.$$ 

Thanks to continuity of $g_z$,

$$r \leq 2a + b + 4\delta + |g_z(y_\infty) - g_z(y'_\infty)|. \quad (2.1)$$ 

Since $Q(f, z) \leq a$, there is an open subset $G$ of $X$ such that $z \in \overline{G}$ and

$$|f(z, y_\infty) - f(t, y_\infty)| < a + \delta \quad \text{and} \quad |f(z, y'_\infty) - f(t, y'_\infty)| < a + \delta$$

for each $t \in G$. Take some $i \geq 1$ such that $x_i \in G$, then we have

$$|g_z(y_\infty) - g_z(y'_\infty)| \leq |g_z(y_\infty) - g_{x_i}(y_\infty)| + |g_{x_i}(y_\infty) - g_{x_i}(y'_\infty)| + |g_{x_i}(y'_\infty) - g_z(y'_\infty)|$$

$$\leq |g_z(y_\infty) - f(z, y_\infty)| + |f(z, y_\infty) - f(x_i, y_\infty)| + |f(x_i, y_\infty) - g_{x_i}(y_\infty)|$$

$$+ |g_{x_i}(y_\infty) - f(x_i, y'_\infty)| + 0 + |g_{x_i}(y'_\infty) - f(x_i, y'_\infty)|$$

$$+ |f(x_i, y'_\infty) - f(z, y'_\infty)| + |f(z, y'_\infty) - g_z(y'_\infty)|$$

$$\leq 2b + 4\delta + 2a + 2\delta = 2a + 2b + 6\delta.$$ 

It follows from the above inequality and (2.1) that

$$r \leq 2a + b + 4\delta + 2a + 2b + 6\delta = 4a + 3b + 10\delta = r - \varepsilon/2.$$ 

This contradiction proves our result. \qed
Proof of Theorem 2.1. Let $r = 10a + 6b$ and

$$A_n = \left\{ x \in X : \mathcal{O}(f, (x, y)) < r + \frac{1}{n} \text{ for all } y \in Y \right\} \quad (n \in \mathbb{N}).$$

Since $Y$ is compact and oscillation is upper semi-continuous, $A_n$ is open for each $n \in \mathbb{N}$. We will show that $A_n$ is dense in $X$ for each $n \in \mathbb{N}$. Let $U$ be an arbitrary non-empty open subset of $X$. By Lemma 2.3, there is a non-empty open subset $U'$ of $U$ such that $\{ f(t, \cdot) : t \in U' \}$ is $(4a + 3b + \frac{1}{8n})$-equicontinuous. According to Lemma 2.2, there exists a non-empty open subset $U''$ of $U'$ and a finite cover $\{ V_1, \ldots, V_m \}$ such that

$$\text{diam}(U'' \times V_i) \leq 2((4a + 3b + \frac{1}{8n}) + a) + \frac{1}{4n} < r + \frac{1}{n}. \quad (i = 1, 2, \ldots, m)$$

This means that $U'' \subset A_n \cap U$. Therefore $A_n$ is dense in $X$ for each $n \in \mathbb{N}$. Define $D = \bigcap_{n \geq 1} A_n$. Then for each $(x, y) \in D \times Y$, we have $\mathcal{O}(f, (x, y)) \leq 10a + 6b$. This completes the proof of the Theorem. □

Remark 2.4. (1) Saint-Raymond [12] proved that every separately continuous mapping $f : X \times Y \to \mathbb{R}$, where $X$ is a separable Baire space has the Namioka property. Since every separable Baire space is $\alpha$-favorable for the game $\mathcal{G}(X)$, by Theorem 2.1 this result is also true when $f$ is quasi-separately continuous.

(2) Let $X$ be a $\beta$-unfavorable space and $g : X \to \mathbb{R}$ be a quasi-continuous mapping which is not continuous. For example, let $g(x) = [x]$ for each $x \in \mathbb{R}$. Define $f : X \times Y \to \mathbb{R}$ by $f(x, y) = g(x)$. Since $f$ is not separately continuous, the results on joint continuity of separate continuous mappings can not be applied. However, $f$ is quasi-separately continuous. Therefore, by Theorem 2.1, $f$ has the Namioka property.

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References


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