ON QUASI *-PARANORMAL OPERATORS

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Abstract. An operator $T \in B(H)$ is called quasi *-paranormal if $||T^*Tx||^2 \leq ||T^3x||||Tx||$ for all $x \in H$. If $\mu$ is an isolated point of the spectrum of $T$, then the Riesz idempotent $E$ of $T$ with respect to $\mu$ is defined by

$$E := \frac{1}{2\pi i} \int_{\partial D} (\mu I - T)^{-1} d\mu,$$

where $D$ is a closed disk centered at $\mu$ which contains no other points of the spectrum of $T$. Stampfli [Trans. Amer. Math. Soc., 117 (1965), 469–476], showed that if $T$ satisfies the growth condition $G_1$, then $E$ is self-adjoint and $E(H) = N(T - \mu)$. Recently, Uchiyama and Tanahashi [Integral Equations and Operator Theory, 55 (2006), 145–151] obtained Stampfli’s result for paranormal operators. In general even though $T$ is a paranormal operator, the Riesz idempotent $E$ of $T$ with respect to $\mu \in \text{isoc}(T)$ is not necessary self-adjoint. In this paper $2 \times 2$ matrix representation of a quasi *-paranormal operator is given. Using this representation we show that if $E$ is the Riesz idempotent for a nonzero isolated point $\lambda_0$ of the spectrum of a quasi *-paranormal operator $T$, then $E$ is self-adjoint if and only if the null space of $T - \lambda_0$ satisfies $N(T - \lambda_0) \subseteq N(T^* - \lambda_0)$. Other related results are also given.

1. Introduction and Preliminaries

Let $B(H)$ be the algebra of all bounded linear operators acting on infinite dimensional separable complex Hilbert space $H$. Let $T$ be an operator in $B(H)$. The operator $T$ is said to be positive (denoted $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. The operator $T$ is said to be a $p$-hyponormal operator if and only if $(T^*T)^p \geq (TT^*)^p$ for a positive number $p$. In [16], the class of log-hyponormal operators is defined as follows: $T$ is a log-hyponormal operator if it is invertible.
and satisfies the following relation \( \log T^*T \geq \log TT^* \). Class of \( p \)-hyponormal operators and class of log-hyponormal operators were defined as extension class of hyponormal operators, i.e., \( T^*T \geq TT^* \). It is well known that every \( p \)-hyponormal operator is a \( q \)-hyponormal operator for \( p \geq q > 0 \), by the Löwner-Heinz theorem "\( A \geq B \geq 0 \) ensures \( A^\alpha \geq B^\alpha \) for any \( \alpha \in [0, 1] \)", and every invertible \( p \)-hyponormal operator is a log-hyponormal operator since \( \log \) is an operator monotone function. An operator \( T \) is paranormal if
\[
\|T^*Tx\|^2 \leq \|T^3x\|\|Tx\|
\]
for all \( x \in H \). It is also well known that there exists a hyponormal operator \( T \) such that \( T^2 \) is not a hyponormal operator (see [8]). In [6] authors, Furuta, Ito and Yamazaki introduced the class \( A \) operators, respectively class \( A(k) \) operators defined as follows: for each \( k > 0 \), an operator \( T \) is \( A(k) \) class operator if
\[
\left( T^*|T|^2^{-k} \right)^{\frac{1}{1+k}} \geq |T|^2,
\]
(1.1)
(for \( k = 1 \) it defines the class \( A \) operators) which includes the class of log-hyponormal operators (see Theorem 2, in [6]) and is included in the class of paranormal operators, in case where \( k = 1 \) (see Theorem 1 in [6]). An operator \( T \in B(H) \) is called \((p,k)\)-quasihyponormal for a positive number \( 0 < p \leq 1 \) and a positive integer \( k \), if
\[
T^{*k}(T^{*T}p - (TT^{*})^p)T^k \geq 0.
\]
I.H. Kim [11] introduced \((p,k)\)-quasihyponormal operators and proved many interesting properties of \((p,k)\)-quasihyponormal operators. It is shown [3] that \( T \) is paranormal if and only if
\[
T^{*2}T^2 - 2\lambda T^{*}T + \lambda^2 \geq 0, \text{ for all } \lambda > 0.
\]
An operator \( T \in B(H) \) is said to be \(*\)-paranormal if \( \|T^*x\|^2 \leq \|T^2x\| \) for all unit vector \( x \) in \( H \).

Hyponormal operators are paranormal and \(*\)-paranormal. An operator \( T \in B(H) \) is said to be normoloid if \( \|T\| = r(T) \) (the spectral radius of \( T \)). Paranormal operators are normaloid and \(*\)-paranormal operators are normaloid ([1, 7, 9, 14]). The class of paranormal operators was defined by Istrătescu, Saitô and Yoshino [9] as class \((N)\). Furuta [4] renamed this class from class \((N)\) to paranormal. The class of \(*\)-paranormal operators was defined by S.M. Patel [14]. The class of \( k\)-\(*\)-paranormal operators was defined by M.Y. Lee, S.H. Lee and C.S. Ryoo [12]. In order to extend the class of paranormal and \(*\)-paranormal operators we introduce the class of quasi-\(*\) paranormal operators defined as follows:

**Definition 1.1.** An operator \( T \) is called quasi \(*\)-paranormal if it satisfies the following inequality:
\[
\|T^*Tx\|^2 \leq \|T^3x\|\|Tx\|
\]
for all \( x \in H \).

It is well known that for any operators \( A, B \) and \( C \),
\[
A^*A - 2\lambda B^*B + \lambda^2 C^*C \geq 0 \text{ for all } \lambda > 0 \Leftrightarrow \|Bx\|^2 \leq \|Ax\|\|Cx\| \text{ for all } x \in H.
\]
Thus we have. An operator \( T \in B(H) \) is quasi \(*\)-paranormal if and only if
\[
T^*(T^{*2}T^2 - 2\lambda TT^{*} + \lambda^2)T \geq 0, \text{ for all } \lambda > 0.
\]
It is well known that $T$ is $*$-paranormal if and only if 
\[ T^* T^2 - 2\lambda T T^* + \lambda^2 \geq 0, \] 
for all $\lambda > 0$.

Thus every $*$-paranormal operator is quasi $*$-paranormal and we have the following implications:

Hyponormal $\Rightarrow$ $*$-paranormal
\[ \Rightarrow \text{ quasi } *-\text{paranormal}. \]

If $T \in B(H)$, write $\sigma(T)$, $\sigma_p(T)$ for the spectrum of $T$ and for the approximate point spectrum of $T$, respectively. Let $T \in B(H)$. $N(T)$ denotes the null space of $T$ and $R(T)$ denotes the range of $T$. $T$ is called isoid if every isolated point of $\sigma(T)$ is an eigenvalue of $T$. Let $\mu \in \mathbb{C}$ be an isolated point of the spectrum of $T$.

Then the Riesz idempotent $E$ of $T$ with respect to $\mu$ is defined by
\[
E := \frac{1}{2\pi i} \int_{\partial D} (\mu I - T)^{-1} d\mu,
\]
where $D$ is a closed disk centered at $\mu$ which contains no other points of the spectrum of $T$. It is well known that the Riesz idempotent satisfies $E^2 = E$, $ET = TE$, $\sigma(T|_{E(H)}) = \{\mu\}$ and $N(T - \mu I) \subseteq E(H)$. In [17], Stampfli showed that if $T$ satisfies the growth condition $G_1$, then $E$ is self-adjoint and $E(H) = N(T - \mu)$.

Recently, Jeon and Kim [10] and A. Uchiyama [18] obtained Stampfli’s result for quasi-class $A$ operators and paranormal operators. In [13] the author obtained Jeon, Kim and Uchiyama results for $k$-quasi-paranormal operators. In general even though $T$ is a paranormal operator, the Riesz idempotent $E$ of $T$ with respect to $\mu \in \text{iso}(T)$ is not necessary self-adjoint.

In this paper $2 \times 2$ matrix representation of a quasi $*$-paranormal operator is given. Using this representation we show that if $E$ is the Riesz idempotent for a nonzero isolated point $\lambda_0$ of the spectrum of a quasi $*$-paranormal operator $T$, then $E$ is self-adjoint if and only if the null space of $T - \lambda_0$ satisfies $N(T - \lambda_0) \subseteq N(T^* - \lambda_0)$.

2. Main Results

**Lemma 2.1.** Let $T \in B(H)$ be quasi $*$-paranormal. If $R(T)$ is not dense, then 
\[
T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{R(T)} \oplus N(T^*) \text{ and } A = T|_{\overline{R(T)}} \text{ is } *-\text{paranormal.}
\]

**Proof.** Since $T$ is quasi $*$-paranormal and $T$ does not have dense range, we can represent $T$ as the upper triangular matrix
\[
T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{R(T)} \oplus N(T^*).
\]
We shall show that $A$ is $*$-paranormal. Since $T$ is quasi $*$-paranormal, we have 
\[
T^* (T^2 T^2 - 2\lambda T T^* + \lambda^2) T \geq 0 \text{ for all } \lambda > 0.
\]
Therefore
\[
\langle (T^2 T^2 - 2\lambda T T^* + \lambda^2) x, x \rangle = \langle (A^2 A^2 - 2\lambda AA^* - 2\lambda BB^* + \lambda^2) x, x \rangle \geq 0,
\]
for all $\lambda > 0$ and for all $x \in \overline{R(T)}$. Hence $\langle (A^2 A^2 - 2\lambda AA^* + \lambda^2)x, x \rangle \geq 2\langle \lambda BB^*x, x' \rangle \geq 0$ for all $\lambda > 0$. Hence $A$ is $*$-paranormal. \hfill \Box

It is easily seen that if $T$ is quasi $*$-paranormal and $R(T)$ is dense, then $T$ is $*$-paranormal. Thus we have the following proposition:

**Proposition 2.2.** Let $T \in B(H)$ be quasi $*$-paranormal. If $R(T)$ is dense, then $T$ is $*$-paranormal.

**Proposition 2.3.** Let $M$ be a closed $T$-invariant subspace of $H$. Then the restriction $T\big|_M$ of a quasi $*$-paranormal operator $T$ to $M$ is a quasi $*$-paranormal operator.

**Proof.** Let

$$T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

on $H = M \oplus M^\perp$. Since $T$ is quasi $*$-paranormal, we have

$$T^3 T^3 - 2\lambda T^* TT^* T^* + \lambda^2 T^* T \geq 0 \quad \text{for all } \lambda > 0.$$

Hence

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^* \left\{ \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^2 \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^2 - 2\lambda \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \right\} + \lambda^2 \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \geq 0$$

for all $\lambda > 0$.

Therefore

$$\begin{pmatrix} A^* (A^2 A^2 - 2\lambda (AA^* + CC^*) + \lambda^2) A & E \\ F & G \end{pmatrix},$$

for some operators $E, F$ and $G$. Hence

$$A^* (A^2 A^2 - 2\lambda AA^* + \lambda^2) A \geq A^* (2\lambda CC^*) A \geq 0,$$

for all $\lambda > 0$. This implies that $A = T\big|_M$ is quasi $*$-paranormal. \hfill \Box

We will denote the ascent of $T$ by $p(T)$ and the descent of $T$ by $q(T)$. In the following theorem we will give a necessary and sufficient condition for the Riesz idempotent $E$ of a quasi $*$-paranormal operator to be self-adjoint. For this we need the following lemma.

**Theorem 2.4.** Let $T \in B(H)$ be quasi $*$-paranormal. If $\mu$ is a non-zero isolated point of $\sigma(T)$, then $\mu$ is a simple pole of the resolvent of $T$.

**Proof.** Assume that $R(T)$ is dense. Then $T$ is $*$-paranormal and $\mu$ is a simple pole of the resolvent of $T$ [19]. So we may assume that $T$ does not have dense range. Then by Lemma 2.1 the operator $T$ can be decomposed as follows:

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ on } H = R(T) \oplus N(T^*),$$
where $A$ is $\ast$-paranormal. Now if $\mu$ is a non-zero isolated point of $\sigma(T)$, then $\mu \in \mathrm{iso}\sigma(A)$ because $\sigma(T) = \sigma(A) \cup \{0\}$. Therefore $\mu$ is a simple pole of the resolvent of $A$ and the $\ast$-paranormal operator $A$ can be written as follows:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$
onumber

on $R(T) = N(A - \mu) \oplus R(A - \mu)$, where $\sigma(A_1) = \{\mu\}$. Therefore

$$T - \mu = \begin{pmatrix} 0 & 0 & B_1 \\ 0 & A_2 - \mu & B_2 \\ 0 & 0 & -\mu \end{pmatrix} = \begin{pmatrix} 0 & D \\ 0 & F \end{pmatrix}$$
onumber

on $H = N(A - \mu) \oplus R(A - \mu) \oplus N(T^*)$, where

$$F = \begin{pmatrix} A_2 - \mu & B_2 \\ 0 & -\mu \end{pmatrix}.$$

We claim that $F$ is an invertible operator on $R(A - \mu) \oplus N(T^*)$. Indeed,

1. $A_2 - \mu I$ is invertible. If not, then $\mu$ will be an isolated point in $\sigma(A_2)$. Since $A_2$ is $\ast$-paranormal, $\mu$ is an eigenvalue of $A_2$ and so $A_2x = \mu x$ for some non-zero vector $x$ in $R(A - \mu I)$. On the other hand, $Ax = A_2x$ implying $x$ is in $N(A - \mu I)$. Hence $x$ must be a zero vector. This contradicts leads to (1).

2. $F$ is invertible. Indeed, by (1) and [8, Problem 71], $(A_2 - \mu I)(-\mu I)$ is invertible. It is easy to show that $p(T - \mu) = q(T - \mu) = 1$. Hence $\mu$ is a simple pole of the resolvent of $T$.

\[\Box\]

**Theorem 2.5.** Let $T \in B(H)$ be quasi $\ast$-paranormal. Assume $0 \neq \mu \in \mathrm{iso}\sigma(T)$ and $E$ is the Riesz idempotent of $T$ with respect to $\mu$. Then $E$ is self-adjoint if and only if $N(T - \mu) \subseteq N(T^* - \overline{\mu})$.

**Proof.** Since $E$ is the Riesz idempotent of $T$ with respect to $\mu$ and $T$ is quasi $\ast$-paranormal, it results from Theorem 2.1 that

$$R(E) = N(T - \mu) \quad \mathrm{and} \quad N(E) = R(T - \mu).$$

Assume that $E$ is self-adjoint. Then $E$ is an orthogonal projection. Hence $R(E^\perp) = N(E)$. Therefore we get

$$N(T - \mu) \subseteq N(T^* - \overline{\mu})$$

by using the equality

$$R(T - \mu) = N(T^* - \overline{\mu})^\perp.$$ 

Conversely, assume that

$$N(T - \mu) \subseteq N(T^* - \overline{\mu}).$$

Then $N(T - \mu)$ and $R(T - \mu)$ are orthogonal. Hence $R(E)^\perp = N(E)$, and so $E$ is self-adjoint.

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References


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