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INTRODUCTION
A Personal Tribute to Peter Freyd and Bill Lawvere

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The idea of a dedication to my advisors Peter Freyd [1] and Bill Lawvere [2] on the occasion of their 80th birthdays, having both of them taken place recently and just a year apart, has as main purpose to emphasize how influential they have been and how it is important to help to acquaint junior researchers with their work. The unanimous response that was received from those who were invited to serve as guest editors in this enterprise indicated that there was ample support for it. That this idea was met with approval from Hvedri Inassaridze, the chief editor of the Tbilisi Mathematical Journal, was not too surprising either, taking into account that category theory is well represented in Georgia.

Although Peter Freyd and Bill Lawvere shared Sammy Eilenberg as an ancestor, they each had different original interests. Whereas Peter had started out with a 1960 Princeton dissertation [1] on a general theory of categories and functors in addition to his contributions to stable homotopy theory [3], followed by his highly original book on abelian categories [2] influenced by Alexander Grothendieck and Peter Gabriel, Bill did so motivated instead by the work of Clifford Truesdell and Walter Noll on continuum mechanics, a subject which he postponed to deal with until later [22, 28, 30, 34] while coming up in the meantime with a novel concept of algebraic theory for his 1963 Columbia dissertation [20]. In both cases, their wide knowledge of mathematics became an important factor in their ability to grasp important concepts from various sources some of which had been previously unrelated, while doing so with rigour and precision not to speak of elegance.

In their current research programs, Lawvere and Freyd hardly overlap. Indeed, Bill Lawvere has of late been mostly pursuing his ambitious program on axiomatic cohesion [31, 32, 35, 37, 38], motivated by his initial interest on classical analysis and continuum physics. In his view, the latter requires more than just the partial invariants of locales and characteristic rings for its description. He thus proposed the study of different categories of space, all of which share a certain feature of cohesion. Envisaged, among other goals, are applications to the theory of distributions in a cohesive topos, needed for continuum physics. As for Peter Freyd, he has during the last few years dealt instead with matters of interest to computer science [13, 14, 15, 17] without however abandoning other projects, as his intriguing and ongoing program on algebraic real analysis [19] indicates. In his view, the existence of injective extensions in its category of models strongly limits an equational theory; in particular, it prohibits the possibility of a 'compactness theorem' for solving sets of equations and that prohibition has limited the usefulness of nonstandard analysis. He contends that by switching from Euclidean spaces to cubes one can avoid the problem and that a remarkably large part of traditional analysis can be made entirely equational so that, for instance, if one sticks

[1] http://www.genealogy.ams.org/id.php?id=23200
[2] http://www.genealogy.ams.org/id.php?id=18947

to maps between intervals one can characterize continuous derivatives without recourse to limits. Coming back to the diverse nature between the current research programs of Peter Freyd and Bill Lawvere, several mutual interactions and influences along the way have taken place, as I intend to point out.

Rather than continuing in this vein with information that the reader can easily obtain in the indicated sources, I shall instead concentrate on a few of the areas in which one or the other have substantially contributed, guided in so doing by the influence that they have had on others, not least on myself.

1 Abelian Categories and Beyond

In his 1964 book on abelian categories [2], Peter Freyd extended some of the contents of his thesis. As a graduate student at Penn at the time I was luckily exposed to it through his graduate course, quite a novelty in a department dominated by other areas.

The theory of abelian categories was acquiring increasing importance due to the work of David Buchsbaum in connection with homological algebra as presented by Henri Cartan and Sammy Eilenberg, and to that of Alexander Grothendieck in his revolutionary treatment of algebraic geometry. In so doing, Freyd had a double objective. One was to present the theory of abelian categories in an axiomatic way within the theory of categories. Another was to be able to use it effectively in order to recover the main results of the existing theory due to Peter Gabriel, a task that proved difficult due to the fact that within categories one could no longer argue as usual due to the absence of elements of the objects. The solution was to make official the increasing awareness of the idea that certain statements, true for the category of abelian groups, would also be true of any abelian category. This resulted in the Mitchell-Freyd-Lubkin representation theorem. Thus, the representation theorem became the target, although not the essence, of the book. Included in it was the adjoint functor theorem, one of Freyd's major achievements.

Abelian categories soon led to further developments in at least two directions. One of them, suggested by Lawvere, was to replace *Ab* by the category *Set* of sets and functions and to try to obtain, parallel to the characterization of categories of R-modules on the basis of abelian categories, one of categories of *Set*-valued functors on a small category and which led to my 1966 Penn dissertation. Prior to it, Lawvere had described the category *Set* axiomatically as a first-order theory [21], presented by himself at the 1964 International Conference on Logic and the Philosophy of Science in Jerusalem, which I was lucky to have attended. The theory of (closed) monoidal categories, and of enriched category theory, including that of autonomous and $\star$-autonomous categories, had in the meantime become of considerable importance within category theory, The basic sources for it were the 1965 paper by Sammy Eilenberg and Max Kelly in the La Jolla Conference and the 1984 comprehensive book 'Basic Concepts of Enriched Category Theory' by Max Kelly. Another development stemming from abelian categories, also due to Lawvere, was to replace the category Ab of abelian groups and homomorphisms by an arbitrary closed monoidal category V, leading through a specification of V as the closed interval $[0, \infty]$ to a novel theory of metric spaces [27].

2 Algebraic Theories and Monads

In his 1963 Columbia dissertation [20], Lawvere had introduced elegant definitions of notions of algebraic theory (now known as 'Lawvere theory') and of algebraic category. Lawvere theories are categories of a particular kind (have finite products), the algebras for a Lawvere theory L in a category A are particular kinds of functors (product preserving) from L to A, and the homomorphisms between the algebras are just natural transformations between them. The concepts of (algebraic) structure and semantics in the context of categories were introduced therein as a pair of adjoint functors.

That the theory of categories was still too new at the time was remarked by Freyd [4] where, while referring to the definitions and theories of Lawvere as remarkable, dealt with (equational) algebraic theories in a classical manner, albeit with models in categories other than *Set*. However, and as pointed out by Lawvere himself, the constructive definition of the tensor product $T_1 \otimes T_2$ of two equational theories T_1 and T_2 given by Freyd in that paper could be stated more generally for Lawvere theories as the algebraic structure of the category of product preserving functors from the second to the category of product preserving functors from the first to *Set*. In [23], Lawvere revisited the categorical approach to universal algebra initiated in his thesis along with the various extensions of it, arguing that, 'the categorical approach to universal algebra has been extended in various ways $(\cdots)$ but (that) despite the generality achieved there is still enough information in the machinery of algebraic categories, algebraic functors, adjoints to algebraic functors, the semantics and structure superfunctors etc., to allow consideration of specific problems analogous to those arising in group theory, ring theory, and other parts of universal algebra'.

Monads (triples, standard constructions), although a major breakthrough due to Eilenberg and Moore and fundamentally exploited by Jon Beck in his 1967 Columbia thesis and in the Barr-Beck (co)homology theory, temporarily obscured the importance of Lawvere theories and their algebras. I recall in this respect the thought-provoking lectures given by Lawvere himself in Zürich during the period 1965-66 which, in addition to describing the Eilenberg-Moore reversal to the Godement-Huber process whereby adjoint pairs give rise to monads, proceeded to exhibit the connection with algebraic theories through a notion of algebraic monad, essentially meaning that there are no infinitary operations, a 'defect' that would then be repaired in a paper by Fred Linton. The connections between Lawvere theories and monads, being nevertheless of considerable interest, were dealt by Lawvere [24] with a sweeping generalization from *Set* to the category **Cat** of small categories and functors, in such a way that not just the carrier but also the 'arities' were themselves taken to be categories. Notwithstanding the importance and ubiquity of monads in category theory today, Lawvere theories as originally conceived have shown since to be better suited for providing certain well-adapted models of synthetic differential geometry and topology as well as in programming languages.

3 Topos Theory and the Foundations of Mathematics

It is a widespread slogan that a topos may be viewed in various ways, to wit, as a universe of variable sets, as a generalized topological space, or as a semantical universe for higher-order logic. However, it is also widely acknowledged that neither of these views is enough in itself to describe, even informally, what a topos is or what it is good for. The notion of a topos stemmed from an original blending of previously unrelated areas, to wit, the theory of schemes due to Grothendieck, the theory of fibrations, and the so-called constructive algebra. As recently reported by Bill Lawvere in a gripping message to the categories community about the passing of his collaborator Myles Tierney, he and Myles had independently recognized the need for an axiomatic theory of sheaves and related matters, thus agreeing to collaborate on the construction of such a theory., first reported in [26]. Tierney emphasized that Grothendieck had made the category (rather than the space) the central aspect and so that this is what they should explicitly axiomatize. There was much in SGA4 that was relevant to such an enterprise, for instance the fact that in a presheaf topos, a covering specified by a ‘Grothendieck topology’ was no longer an infinite family of subobjects but a single subobject R, therefore making possible the formulation of notions in finitary terms, indeed in terms of a single operator whose properties were then made precise. Another unique feature of SGA4 is that it contains no definition of ‘topos’, as indeed every rigorous mention is of some ’U-topos’. This parameter U was essentially a model of set theory and previous work on the category of sets by Lawvere himself showed clearly that, for mathematical purposes, the use of composition of mappings is more effective than towers of membership. Following this line of thought, Lawvere and Tierney replaced U itself by an arbitrary topos $\mathscr{S}$ (in their determination of that term), and a general $\mathscr{S}$-topos $\mathscr{E}$ could therefore be seen as structured by a (geometric) morphism $\mathscr{E} \longrightarrow \mathscr{S}$. This provided a suitable codomain for the 2-functor from internal $\mathscr{S}$-sites to $\mathscr{S}$-toposes, whose image consisted of those toposes $\mathscr{E}$ which contained a ‘bound’ in $\mathscr{S}$. The work of Radu Diaconescu in his 1973 Rutgers thesis would give substance to this idea.

Topos theory permits a conceptually richer view of classical mathematics and it is in this capacity that its strength lies. One of the earliest applications of it was a new proof given by Myles Tierney of the independence of the continuum hypothesis from the axioms of set theory, proved originally by Paul Cohen by the forcing method. The set theory on which this and other independence proofs were based was that of a topos model of set theory, by which it was meant a Boolean topos with a natural numbers object, satisfying the axiom of choice (which implies Booleanness), and two-valuedness, that is, where the subobjects classifier Ω equals $2 = \{t, f\}$. This system of axioms has been shown by Peter Johnstone to be basically equivalent to a weak form (no replacement, only bounded comprehension) of Zermelo-Fraenkel set theory with the axiom of choice.

Just as abelian categories had provided a useful generalization of the category of abelian groups, toposes would also be one such for the category of sets, and therefore become relevant to the entire foundations of mathematics. This realization led Freyd [5] to explore a possible embedding theorem in this context, obtaining several metatheorems concerning the exactness theory of toposes. As an application, he showed that a topos has a natural numbers object if and only if it has an object A such that $1 + A \cong A$. Freyd followed up these investigations in [8] by giving a proof of the independence of the axiom of choice from the axioms of topos theory in an article that contains a wealth of new concepts and results, including that of a well-founded topos. In connection with this he also investigated in [12] the connection between choice (depending on how it is formulated) and well-ordering. To be contrasted is the contention by Lawvere [29], who argued that, although any $\mathscr{S}$-topos can be conceived as a generalized space, if interpreted to mean ‘étendue’ then the assertion that all such are localic is not the case, with Freyd’s contention [11] that all (Grothendieck) toposes are localic, in view of his having shown that there is a Boolean topos $\mathscr{B}$ such that for every (Grothendieck) topos $\mathscr{E}$ there is a locale L in $\mathscr{B}$ so that $\mathscr{E}$ appears as an exponential variety in the topos of L-sheaves over $\mathscr{B}$. It would not be an exaggeration to state that the Lawvere-Tierney theory of toposes has been (and still is) one of the main sources of research for a large number of category theorists. An update of the state of the theory was given by Lawvere in [33].

4 Category Theory Introductions

Although both the book by Peter Freyd and Andre Scedrov [16] and the book by Bill Lawvere and Steve Schanuel [36] are introductory books to category theory, there are several differences in their approach and objectives. Whereas the former, which is about general concepts and methods that occur throughout mathematics and increasingly so in computer science, emphasizes the geometric nature of the subject while explaining its connection to logic, the latter proposes to explore the concepts of a new and fundamental conception of mathematics that has led to better methods to understand and manipulate such concepts. There are also differences of style and level.

The Freyd-Scedrov book employs a graphic language designed by them to be used instead of ordinary language or even of a language formalized in first-order logic so as to better get their ideas across. This graphic language had been introduced by Freyd in a 1974 paper [6] dealing with equivalence of categories and its properties. Also the Lawvere-Schanuel book employs diagrams, not however in order to replace the usual language, but to illustrate ideas in an intuitive fashion. The Freyd-Scedrov book has two chapters. The first is a detailed exploration of categories with increasingly richer structure (cartesian, regular, prelogoi, logoi, topos). The second, labelled 'allegories', deals with the calculus of relations in the various types of categories introduced in the first chapter. While the first chapter may be said to be geometric, the second is purely algebraic and it is meant to be used as a computational tool. In particular, the bridge between categories and their logics is done via their associated allegories. A central theme therein is to obtain representation theorems from which metatheorems having the same applicability power as completeness theorems in logic are then derived. The Lawvere-Schanuel book intends to transmit a new way of thinking that is different from the usual in that it is based on a less rigid set theory that is better suited to capture dynamical concepts. The basic notion of a category is in it identified with one of a mathematical universe. In this vein, the notion of a morphism is presented as a process to pass from a given object to another object of the same universe. The rest of the book deals with categories which, although substantially different from the category of sets, are nevertheless constructed entirely from it. The concepts of category theory are introduced gradually and with plenty of examples. In this way they reach the notion of a topos which by then the reader will find easy to grasp.

In short, both introductory textbooks, though different in scope and style, are valuable tools for teaching the subject as it could not otherwise be expected, coming as they do from experts in the subject. The subject of equivalence is of fundamental importance in any theory of categories, particularly for categories in a topos not assumed to satisfy the axiom of choice. In a 1976 paper contained in a volume in honor of Sammy Eilenberg, edited by Myles Tierney and Alex Heller, Freyd [7] tackled the question of characterizing those properties of objects and morphisms invariant under equivalence of categories. Such properties he called 'diagrammatic'. Relevant to this question is the notion of a stack, first defined by Jean Giraud in terms of sites for Grothendieck toposes in connection with non-abelian cohomology, then given a simpler definition by Lawvere in his 1973 Perugia and 1974 Montreal lectures. In those lectures, Lawvere had proposed adding to the axioms for toposes one to the effect that categories in a topos be required to admit a stack completion within the topos. This led me to investigate with Bob Paré the subject of equivalences of categories in a topos in connection with the axiom of choice, and then to my construction of the stack completion of any category object in an $\mathscr{S}$-bounded topos. However, adding such an axiom to topos theory would eliminate interesting examples, so it is a matter to be decided depending on the context. In connection with this, and as I have more recently shown, the Morita equivalence for category objects in a topos not satisfying the axiom of choice involves not just the so-called Cauchy completion but also the stack completion.

5 Categorical Dynamics

Motivated by a desire to employ category theory in (elementary) physics, Lawvere outlined an entire program in his 1967 Chicago lectures on 'categorical dynamics' [22]. Following up on his 1966 Oberwolfach talk where he had also proposed a theory of distributions on categories of presheaves, in his 1983 Aarhus lectures he posed several questions concerning distributions on $\mathscr{S}$-toposes, with $\mathscr{S}$ a topos thought off to be *Set*. Following the lead of analysis, by a distribution on an $\mathscr{S}$-topos $\mathscr{E}$ it was meant an $\mathscr{S}$-cocontinuous functor from $\mathscr{E}$ to $\mathscr{S}$. Implicit in it is the idea of letting the object classifier $\mathscr{R}$ in Top$/\mathscr{S}$ play the role of 'the line', so that a distribution is in fact an instance of the double dualization that Lawvere referred to as 'the Riesz paradigm'.

An open question posed by Lawvere concerning the existence of a topos classifier for distributions seemed non-trivial since it could not be obtained as an instance of the known classifying toposes of coherent theories, but the connection between distributions and cosheaves, already implicit in the 1985 work of Andy Pitts and the 1976 forcing topologies introduced by Myles Tierney, led to my 1995 construction of the 'symmetric topos' $\Sigma(\mathscr{E})$, the $\mathscr{S}$-topos classifier of Lawvere distributions on an $\mathscr{S}$-topos $\mathscr{E}$. The idea a distribution as that of an extensive quantity modelled as a linear functional on an intensive quantity is traditional in analysis. The existence of the object classifier topos showed that the intuition of sheaves as intensive quantities was rigorously correct, while that of the distributions classifier topos showed likewise that the intuition of distributions as extensive quantities was an appropriate one. Since Lawvere distributions are generalized points, the symmetric topos construction could also be presented by analogy with that of the symmetric algebra, as was later shown in my collaboration with Aurelio Carboni, extending it to a construction of the symmetric monad, a completion Kock-Zöberlein monad on the 2-category of locally presentable categories whose Eilenberg-Moore algebras were identified with certain cocomplete objects.

The connection between distributions and cosheaves would then lead to one between distributions and complete spreads thanks to the 1957 work of the topologist R. H. Fox, extended by Jonathon Funk from topological spaces to locales and then, in collaboration with me, from locales in a topos $\mathscr{S}$ to bounded $\mathscr{S}$-toposes. The exploration of the interactions between distributions, the symmetric monad, and complete spreads culminated in our joint 2006 book 'Singular Coverings of Toposes'. The notion of a spread had been introduced by R. H. Fox in order to formulate the idea of a branched covering as the completion of an unbranched covering. This completion process appears in a simpler form if it is applied to what he called 'spreads', a wider class of objects that encompasses the branched and folded coverings of Tucker. The topological concept of a spread encompasses the combinatorial notions used until then in connection with the knot invariants defined by Seifert, which he then showed are invariants of the topological type of the knot. In connection with this, it is noted that Peter Freyd is known for the striking result in knot theory, obtained in collaboration with David Yetter and simultaneously with others [10, 18], which is that of a new polynomial isotopy invariant of knots and links, along with a coherence theorem via knot theory. This is possibly a hidden connection between the work of Peter Freyd and Bill Lawvere, and one that in my view is still largely unexplored. In [25] Lawvere already effectively set the scene for future developments on both the smooth and the recursive aspects analysis. I shall touch upon them in the next two sections, pointing out also the relevance to them by work from Freyd and his collaborators.

6 Synthetic Differential Geometry and Beyond

In the same 1966 Oberwolfach lectures on categorical dynamics [22], Lawvere had set down the basis of a new subject, a branch of (applied) category theory which came to be labelled 'synthetic differential geometry' and whose development still continues largely through the work of Anders Kock and others. With the introduction of toposes as a surrogate of set theory whose underlying logic is Heyting rather than Boolean, and where no appeal to an axiom of choice is permitted, it became possible to formally introduce infinitesimals in the same spirit as in the work of Charles Ehresmann and André Weil in the 50's. The data for a model of synthetic differential geometry in the form of the Kock-Lawvere axiom, is that of a pair $(\mathscr{E}, R)$ with $\mathscr{E}$ a (Grothendieck) topos and R a commutative ring with unit in it, of which it was assumed that jets of functions from R^n to R are representable and that the representing objects be tiny, or well supported atoms, where an object A of $\mathscr{E}$ is said to be an atom if the endofunctor $(-)^A : \mathscr{E} \to \mathscr{E}$ has a right adjoint.

A simple yet far-reaching suggestion for best exploiting the richness of the context of synthetic differential geometry has been put forward by Lawvere as follows: 'Take equations of classical mechanics, do this synthetically, apply to function spaces, get equations of continuum mechanics'. In connection with the calculus of variations, this is precisely what we did in the paper by myself and Murray Heggie, which John W. Gray included in the Proceedings of the Special Session on Mathematical Applications of Category Theory 89th Annual Meeeting of the AMS, held in Denver, Colorado, January 5-9, 1983. Similar situations can be envisaged but have yet to be worked out.

If a topos is constructed from topology, then it is often the case that one or more topological structures can be put on its objects. What is remarkable is that, even if nothing at all is assumed of a topos, its intrinsic logic is enough to produce a topological structure on its objects, an interesting idea due to Jacques Penon, in turn leading to a theory called 'synthetic differential topology', developed by myself in 1986 in collaboration with Eduardo Dubuc, and later on to applications of it to the theory of stable smooth mappings, their unfoldings and their singularities, in work now contained in a 2018 forthcoming book 'Synthetic Differential Topology' by myself with Felipe Gago and Ana María San Luis. In synthetic differential topology, not just jets but also germs of mappings from R^n to R are assumed to be representable by tiny objects. A model of the latter that is 'well adapted' is the Dubuc topos $\mathscr{G}$ of 'germ-determined' ideals, belonging to a class of models constructed using a notion of C^∞-ring which is due to Lawvere and goes back to the algebraic theories of his thesis. Whereas the tiny objects representing jets can be said to be algebraic, those representing germs are logical in flavor. In both cases, it is the fact that the logic of a topos model is Heyting rather than Boolean that is of crucial importance. As conjectured by Lawvere [28, 30] and then proved by Freyd [9], if A is a tiny object of a topos $\mathscr{E}$, then the full subcategory $\mathscr{T}_A$ of A-discrete objects of $\mathscr{E}$ is both reflective and coreflective in $\mathscr{E}$, hence it is itself a topos. Further work along these lines is contained in the 1987 Penn thesis of David Yetter.

7 Fix Point Theorems, Recursiveness, and Programming Languages

Theoretical computer science constitutes another exciting area of category theory which, along with algebraic theories and toposes, is one where both Lawvere and Freyd have substantially concentrated on at one point or another. The subject of fix points is central to investigations related to theoretical computer science. In the context of cartesian closed categories, Lawvere [25] proved a fixed point theorem that would relate to recursion theory for computation and so also to the effective topos. The latter had been constructed by Martin Hyland in 1982 and it is a notable example of a topos that it is not a Grothendieck topos, so that general constructions such as that of the stack completion of an internal category in it are not guaranteed. Among other features that distinguishes the effective topos from Grothendieck toposes is the fact that the global section functor has a right adjoint and no left adjoint. It was Freyd's realization of its presentation via an allegory that then led Aurelio Carboni into devising its exact completion, and it was that and all the work that came after it, ending with the characterization of those exact completions which are toposes due to Matias Menni, that explained that it was indeed a very interesting situation.

The 'pamphlet' by Freyd [6] would also prove influential in these matters. In a lecture given at the 1990 Como Category Theory Conference, Freyd [17] observed what others had also noticed, most importantly Dana Scott, namely that many programming languages are inconsistent with standard mathematical foundations, and that the task of finding sound interpretations for what computer scientists actually do constituted 'the highest form of applied mathematics'. In the same source, Freyd proposed to add conditions called 'algebraically complete' and 'algebraically compact'. In subsequent papers with several collaborators, Freyd pursued these and other ideas in connection with the semantics of programming languages.

As pointed out by Lawvere, the mathematical theory of recursion, recursive spaces, and recursively enumerable subspaces, can and should be treated like any other mathematical theory, just as Banach and Ersov foresaw. This point of view was made quite explicit by Phil Mulry in his 1980 Buffalo thesis in the form of a particular Grothendieck topos which was then compared with Hyland's effective topos in the 1986 Oxford thesis by Pino Rosolini. These and others ideas seem to call for a synthetic recursion theory. The work done by mathematicians in Holland, most notably in the 2008 realizability book by Jaap van Oosten, seems relevant and important in that direction.

Concluding Remarks

I conclude this necessarily sketchy review of some of the most important contributions by Bill Lawvere and Peter Freyd by pointing out what should by now be an obvious conclusion to make, namely, that both of them have worked in a variety of subjects within category theory and computer science yet deeply rooted in mathematical practice, and that their ideas have been influential for a simple reason, namely, their originality and vision.

Among the various aspects of their work on which I have neither commented or nor included in the references is their deep interest and contributions to the philosophy (and anti-philosophy) of mathematics. The list of references for each author is also far from complete.

As I believe to have already sufficiently indicated, there should be much to gain in studying and pursuing the work of these two remarkable mathematicians. As for the individual contributions to this special issue, I have come to the conclusion that it is better to let them stand on their own. The readers of this special issue will thus find a variety of topics among the latter, including but not exhausting all of those barely touched upon in this introduction.

References

[1] P. J. Freyd, *Functor Theory*, Dissertation, Princeton University, 1960.

[2] P. J. Freyd, *Abelian Categories*, Harper and Row, New York, Evanston, and London, 1964.

[3] P. J. Freyd, Stable Homotopy, in S. Eilenberg, D. K. Harrison, S. MacLane (Edts.), *Proceedings of the Conference on Categorical Algebra. La Jolla 1965*,121–172, Springer-Verag Berlin . Heidelberg. New York, 1966.

[4] P. J. Freyd, Algebra Valued Functors In General and Tensor Products in Particular, *Colloquium Mathematicum* **14** (1966), 89–106.

[5] P. J. Freyd, Aspects of Topoi, *Bull. Austral. Math. Soc.* **7** (1972), 1–76.

[6] P. J. Freyd, On Canonizing Category Theory, or on Functorializing Model Theory, a Pamphlet, Unpublished, University of Pennsylvania, 1974.

[7] P. J. Freyd, Properties Invariant with Equivalence Types of Categories, in: A. Heller and M. Tierney, *Algebra, Topology and Category Theory.* Academic Press, 1976, 55–60.

[8] P. J. Freyd, The Axiom of Choice, *J. Pure and Appl. Algebra* **19** (1980), 103–125.

[9] P. J. Freyd, Intrinsic Differential Geometry, Unpublished lecture, Philadelphia, 1983.

[10] P. J. Freyd, D. N. Yetter, J. Huste, W. B. R. Lickorishlock, K. Millet, A. Oceanu, A new polynomial invariant of knots and links, *Bull. Amer. Math. Soc.* **12-2** (1985), 239–246.

[11] P. J. Freyd, All topoi are localic; or, Why permutation models prevail, *J. Pure Appl. Algebra* **46** (1987), 49–58.

[12] P. J. Freyd, Choice and well-ordering, *Ann. Pure Appl. Logic* **35** (1987), 146–166.

[13] P. J. Freyd, H. Friedman, A. Scedrov, Lindenbaum algebras of intuitionistic theories and free categories, *Annals Pure Appl. Logic 35* (1987), 167–172.

[14] P. J. Freyd, P. S. Mulry, G. Rosolini, D. S. Scott, Extensional PERs, in: *Proc. Fifth IEEE Symp. on Logic in Computer Science* (1990), 346–354.

[15] P. J. Freyd, E. P. Robinson, G. Rosolini, Functorial parametricity, in: *Proc. Seventh IEEE Symp. on Logic in Computer Science* (1992), 444–452.

[16] P. J. Freyd and A. Scedrov, *Categories, Allegories*, North Holland, 1990.

[17] P. J. Freyd, Algebraically Complete Categories, in: A. Carboni, M. C. Pedicchio and G. Rosolini (Eds.), *Proceedings Como 1990*, Lecture Notes in Mathematics **1448** (1991), 95–104.

[18] P. J. Freyd and D N Yetter, Coherence theorems via knot theory, J. Pure Appl. Alg. **78** (1992), 49–76.

[19] P. J. Freyd, Algebraic Real Analysis, *Theory and Applications of Categories* **20-10** (2008), 215–306.

[20] F. W. Lawvere, Functorial Semantics of Algebraic Theories, Dissertation, Columbia University, 1963, *Proc. of the Nat. Acad. Sciences U. S. A.* **50** (1963), 869–872.

[21] F. W. Lawvere, An elementary theory of the category of sets, *Proc. Nat. Acad. Sci. U. S. A.* **52** (1964), 1506–1511.

[22] F. W. Lawvere, Categorical Dynamics, Lecture at the Oberwolfach meeting in Category Theory, 1966. in A. Kock (Ed.), *Topos Theoretic Methods in Geometry*, Various Publications Series **30**, Matematisk Institut, Aarhus Universitet, 1979, 1–28.

[23] F. W. Lawvere, Some Algebraic Problems in the Context of Functorial Semantics of Algebraic Theories, *Reports of the Midwest Categories Seminar II Lecture Notes in Mathematics* **61**, Springer-Verlag (1968), 41–61.

[24] F. W. Lawvere, Ordinal Sums and Equational Doctrines, *Lecture Notes in Mathematics* **80**, Springer-Verlag (1969), 141-155.

[25] F. W. Lawvere, Diagonal Arguments and Cartesian Closed Categories, Springer Lecture Notes in Mathematics **92**, Springer-Verlag (1969), 134-145.

[26] F. W. Lawvere, Quantifiers and Sheaves, *Actes du Congrès International des Mathématiciens* Nice 1970, 229–334.

[27] F. W. Lawvere, Metric Spaces, Generalized Logic, and Closed Categories, *Rendiconti del seminario matematico e fisico di Milano* **43** (1973), 133–166.

[28] F. W. Lawvere, Towards the Description in a Smooth Topos of the Dynamically Possible Motions and Deformations of a Continuous Body, Cahiers de Top. et Géo. Diff. Catégoriques **21** (1980), 337–392.

[29] F. W. Lawvere, Categories of spaces may not be generalized spaces, as exemplified by directed graphs, *Rev. Colomb. Mat.* **20** (1986), 179–186.

[30] F. W. Lawvere, Outline of Synthetic Differential Geometry, Unpublished notes, Buffalo 1988.

[31] F. W. Lawvere, Cohesive Toposes and Cantor's "Lauter Einsen", *Phil. Math.* **3-2** (1994), 5–15.

[32] F. W. Lawvere, Volterra's functionals and covariant cohesion of space, in: R. Betti, F. W. Lawvere (Eds), *Supplemento ai Rendiconti del Circolo Matematico di Palermo*, Serie J, **64** (2000), 201–214.

[33] F. W. Lawvere, Comments on the Development of Topos Theory, in: J. P. Pier, (ed.), *Development of Mathematics 1950–2000.* Birkhäuser, Basel-Boston-Berlin, 2000, 715–734.

[34] F. W Lawvere, Categorical Algebra for Continuum Micro Physics, *J. Pure Appl. Algebra* **175** (2002), 267–287.

[35] F. W. Lawvere, Axiomatic Cohesion, *Theory and Applications of Categories* **19-3** (2007), 41–49.

[36] F. W. Lawvere and S. Schanuel, *Conceptual Mathematics : A First Introduction to Categories*, Cambridge University Press (2nd edition) 2009.

[37] F. W. Lawvere and M. Menni, The Hopf Algebra of Möbius Intervals, *Theory and Applications of Categories* **24** (2010), 221-265.

[38] F. W. Lawvere and M. Menni, Internal choice holds for the discrete part of any cohesive topos satisfying stable connected codiscreteness, *Theory and Applications of Categories* **30-26** (2015), 909–932.