Generalizations of Buzano inequality for n-tuples of vectors in inner product spaces with applications

S. S. Dragomir[1,2]

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia
[2]DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa
E-mail: sever.dragomir@vu.edu.au

Abstract

In this paper some generalizations of Buzano inequality for n-tuples of vectors in inner product spaces are given. Applications for norm and numerical radius inequalities for n-tuples of bounded linear operators and for functions of normal operators defined by power series with nonnegative coefficients are also provided.

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1 Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the *Schwarz inequality*

$$\|x\| \|y\| \geq |\langle x, y \rangle| \text{ for any } x, y \in H. \tag{1.1}$$

The equality case holds in (1.1) if and only if there exists a constant $\lambda \in \mathbb{K}$ such that $x = \lambda y$.

In 1985 the author [5] (see also [18]) established the following refinement of (1.1):

$$\|x\| \|y\| \geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| + |\langle x, e \rangle \langle e, y \rangle| \geq |\langle x, y \rangle| \tag{1.2}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Using the triangle inequality for modulus we have

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \geq |\langle x, e \rangle \langle e, y \rangle| - |\langle x, y \rangle|$$

and by (1.2) we get

$$\begin{aligned} \|x\| \|y\| &\geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| + |\langle x, e \rangle \langle e, y \rangle| \\ &\geq 2 |\langle x, e \rangle \langle e, y \rangle| - |\langle x, y \rangle|, \end{aligned}$$

which implies the Buzano inequality [3]

$$\frac{1}{2} [\|x\| \|y\| + |\langle x, y \rangle|] \geq |\langle x, e \rangle \langle e, y \rangle| \tag{1.3}$$

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that holds for any $x, y, e \in H$ with $\|e\| = 1$.

For other Schwarz related inequalities in inner product spaces, see [1], [6]-[10], [16], [17], [21], [22], [23], [24], [25], [26], [27], [28] and the monographs [13] and [14].

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [20, p. 1]:

$$W(T) = \{\langle Tx, x\rangle, \ x \in H, \ \|x\| = 1\}. \tag{1.4}$$

It is well known (see for instance [20]) that:

(i) The numerical range of an operator is convex (the Toeplitz-Hausdorff theorem);

(ii) The spectrum of an operator is contained in the closure of its numerical range;

(iii) T is self-adjoint if and only if W is real.

The *numerical radius* $w(T)$ of an operator T on H is defined by [20, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.5}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ and the following inequality holds true

$$w(T) \leq \|T\| \leq 2w(T), \text{ for any } T \in B(H). \tag{1.6}$$

Utilising Buzano's inequality (1.3) we obtained the following inequality for the numerical radius [11] or [12]:

Theorem 1.1. Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space and $T : H \to H$ a bounded linear operator on H. Then

$$w^2(T) \leq \frac{1}{2}\left[w\left(T^2\right) + \|T\|^2\right]. \tag{1.7}$$

The constant $\frac{1}{2}$ is best possible in (1.7).

From the above result (1.7) we obviously have

$$w(T) \leq \left\{\frac{1}{2}\left[w\left(T^2\right) + \|T\|^2\right]\right\}^{1/2} \leq \left\{\frac{1}{2}\left(\left\|T^2\right\| + \|T\|^2\right)\right\}^{1/2} \leq \|T\| \tag{1.8}$$

and

$$w(T) \leq \left\{\frac{1}{2}\left[w\left(T^2\right) + \|T\|^2\right]\right\}^{1/2} \leq \left\{\frac{1}{2}\left(w^2(T) + \|T\|^2\right)\right\}^{1/2} \leq \|T\|, \tag{1.9}$$

that provide refinements for the first inequality in (1.6).

For numerical radius inequalities see the recent monograph [15] and the references therein.

Motivated by the above results, we establish some generalizations of Buzano inequality for n-tuples of vectors in inner product spaces. Applications for norm and numerical radius inequalities for n-tuples of bounded linear operators and for functions of normal operators defined by power series with nonnegative coefficients are also provided.

2 Main results

We have the following generalization of Buzano's inequality:

Theorem 2.1. Let $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_n$ be a probability distribution, i.e. we recall that $p_i > 0$ for any $i \in \{1, ..., n\}$ and $\sum_{i=1}^n p_i = 1$. For any $\mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n) \in H^n$ we have

$$\left(\sum_{i=1}^n p_i \|x_i\|^2\right)^{1/2} \left(\sum_{i=1}^n p_i \|y_i\|^2\right)^{1/2} \tag{2.1}$$
$$\geq \left|\sum_{i=1}^n p_i \langle x_i, y_i\rangle - \left\langle \sum_{i=1}^n p_i x_i, e\right\rangle \left\langle e, \sum_{i=1}^n p_i y_i\right\rangle\right| + \left|\left\langle \sum_{i=1}^n p_i x_i, e\right\rangle \left\langle e, \sum_{i=1}^n p_i y_i\right\rangle\right|$$
$$\geq \left|\sum_{i=1}^n p_i \langle x_i, y_i\rangle\right|$$

and

$$\frac{1}{2}\left[\left(\sum_{i=1}^n p_i \|x_i\|^2\right)^{1/2} \left(\sum_{i=1}^n p_i \|y_i\|^2\right)^{1/2} + \left|\sum_{i=1}^n p_i \langle x_i, y_i\rangle\right|\right] \tag{2.2}$$
$$\geq \left|\left\langle \sum_{i=1}^n p_i x_i, e\right\rangle \left\langle e, \sum_{i=1}^n p_i y_i\right\rangle\right|,$$

for any $e \in H$ with $\|e\| = 1$.

Proof. For a probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_n$, we define the inner product

$$\langle \mathbf{x}, \mathbf{y}\rangle_p := \sum_{i=1}^n p_i \langle x_i, y_i\rangle$$

for n-tuples $\mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n) \in H^n$.

The attached norm is given by

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^n p_i \|x_i\|^2\right)^{1/2}$$

for $\mathbf{x} = (x_1, ..., x_n) \in H^n$.

Let $\mathbf{e} = (e_1, ..., e_n) \in H^n$ with $\sum_{i=1}^n p_i \|e_i\|^2 = 1$. Making use of (1.2) and (1.3) for the inner product $\langle \cdot, \cdot\rangle_p$ we have the inequalities

$$\left(\sum_{i=1}^n p_i \|x_i\|^2\right)^{1/2} \left(\sum_{i=1}^n p_i \|y_i\|^2\right)^{1/2} \tag{2.3}$$
$$\geq \left|\sum_{i=1}^n p_i \langle x_i, y_i\rangle - \sum_{i=1}^n p_i \langle x_i, e_i\rangle \sum_{i=1}^n p_i \langle e_i, y_i\rangle\right| + \left|\sum_{i=1}^n p_i \langle x_i, e_i\rangle \sum_{i=1}^n p_i \langle e_i, y_i\rangle\right|$$
$$\geq \left|\sum_{i=1}^n p_i \langle x_i, y_i\rangle\right|$$

and

$$\frac{1}{2}\left[\left(\sum_{i=1}^{n} p_i \|x_i\|^2\right)^{1/2}\left(\sum_{i=1}^{n} p_i \|y_i\|^2\right)^{1/2} + \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle\right|\right] \tag{2.4}$$
$$\geq \left|\sum_{i=1}^{n} p_i \langle x_i, e_i\rangle \sum_{i=1}^{n} p_i \langle e_i, y_i\rangle\right|,$$

for any $\mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n) \in H^n$.

If we take $\mathbf{e} = (e, ..., e) \in H^n$ with $\|e\| = 1$, then we get from (2.3) and (2.4) the desired inequalities (2.1) and (2.2). ∎

Remark 2.2. If we take in (2.1) and (2.2) $n = 1$ and $p_1 = 1$, then we get the inequalities (1.2) and (1.3).

We observe that, if we take $H = \mathbb{C}$ with the inner product $\langle z, w\rangle = z\overline{w}$ then by taking above $x_i = a_i \in \mathbb{C}$, $y_i = \overline{b}_i, i \in \{1, ..., n\}$ and $e = 1$ then from (2.1) and (2.2) we get the inequalities

$$\left(\sum_{i=1}^{n} p_i |a_i|^2\right)^{1/2}\left(\sum_{i=1}^{n} p_i |b_i|^2\right)^{1/2} \tag{2.5}$$
$$\geq \left|\sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i\right| + \left|\sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i\right|$$
$$\geq \left|\sum_{i=1}^{n} p_i a_i b_i\right|$$

and

$$\frac{1}{2}\left[\left(\sum_{i=1}^{n} p_i |a_i|^2\right)^{1/2}\left(\sum_{i=1}^{n} p_i |b_i|^2\right)^{1/2} + \left|\sum_{i=1}^{n} p_i a_i b_i\right|\right] \tag{2.6}$$
$$\geq \left|\sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i\right|.$$

We have the following norm inequality:

Theorem 2.3. Let $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_n$ be a probability distribution and $(A_1, ..., A_n)$, $(B_1, ..., B_n)$ two n-tuples of bounded linear operators on H. Then we have

$$\frac{1}{2}\left[\left\|\sum_{i=1}^{n} p_i |A_i|^2\right\|^{1/2}\left\|\sum_{i=1}^{n} p_i |B_i|^2\right\|^{1/2} + \left\|\sum_{i=1}^{n} p_i B_i^* A_i\right\|\right] \tag{2.7}$$
$$\geq \left\|\sum_{i=1}^{n} p_i A_i e\right\|\left\|\sum_{i=1}^{n} p_i B_i e\right\| \geq \left|\left\langle\left(\sum_{i=1}^{n} p_i B_i^*\right)\sum_{i=1}^{n} p_i A_i e, e\right\rangle\right|$$

or any $e \in H$ with $\|e\| = 1$.

Proof. If we write the inequality (2.2) for $x_i = A_i x$, $y_i = B_i y$, then we get

$$\frac{1}{2}\left[\left(\sum_{i=1}^{n} p_i \left\|A_i x\right\|^2\right)^{1/2}\left(\sum_{i=1}^{n} p_i \left\|B_i y\right\|^2\right)^{1/2} + \left|\sum_{i=1}^{n} p_i \left\langle A_i x, B_i y\right\rangle\right|\right] \geq \left|\left\langle \sum_{i=1}^{n} p_i A_i x, e\right\rangle\left\langle e, \sum_{i=1}^{n} p_i B_i y\right\rangle\right|, \tag{2.8}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Observe that

$$\begin{aligned}\sum_{i=1}^{n} p_i \left\|A_i x\right\|^2 &= \sum_{i=1}^{n} p_i \left\langle A_i x, A_i x\right\rangle = \sum_{i=1}^{n} p_i \left\langle A_i x, A_i x\right\rangle \\ &= \sum_{i=1}^{n} p_i \left\langle A_i^* A_i x, x\right\rangle = \sum_{i=1}^{n} p_i \left\langle \left|A_i\right|^2 x, x\right\rangle \\ &= \left\langle \sum_{i=1}^{n} p_i \left|A_i\right|^2 x, x\right\rangle,\end{aligned}$$

$$\sum_{i=1}^{n} p_i \left\|B_i y\right\|^2 = \left\langle \sum_{i=1}^{n} p_i \left|B_i\right|^2 y, y\right\rangle$$

and

$$\sum_{i=1}^{n} p_i \left\langle A_i x, B_i y\right\rangle = \left\langle \sum_{i=1}^{n} p_i B_i^* A_i x, y\right\rangle$$

for any $x, y \in H$.

Then by (2.8) we get the inequality

$$\frac{1}{2}\left[\left\langle \sum_{i=1}^{n} p_i \left|A_i\right|^2 x, x\right\rangle^{1/2}\left\langle \sum_{i=1}^{n} p_i \left|B_i\right|^2 y, y\right\rangle^{1/2} + \left|\left\langle \sum_{i=1}^{n} p_i B_i^* A_i x, y\right\rangle\right|\right] \geq \left|\left\langle \sum_{i=1}^{n} p_i A_i e, x\right\rangle\left\langle \sum_{i=1}^{n} p_i B_i e, y\right\rangle\right|, \tag{2.9}$$

for any $x, y, e \in H$ with $\|e\| = 1$, which is an inequality of interest in itself.

Taking the supremum over $\|x\| = \|y\| = 1$ in (2.9) we get

$$\begin{aligned}
&\sup_{\|x\|=1}\left|\left\langle \sum_{i=1}^{n} p_i A_i e, x\right\rangle\right| \sup_{\|y\|=1}\left|\left\langle \sum_{i=1}^{n} p_i B_i e, y\right\rangle\right| \quad (2.10)\\
&= \sup_{\|x\|=\|y\|=1}\left|\left\langle \sum_{i=1}^{n} p_i A_i e, x\right\rangle \left\langle \sum_{i=1}^{n} p_i B_i e, y\right\rangle\right|\\
&\le \frac{1}{2}\sup_{\|x\|=\|y\|=1}\left[\left\langle \sum_{i=1}^{n} p_i \left|A_i\right|^2 x, x\right\rangle^{1/2} \left\langle \sum_{i=1}^{n} p_i \left|B_i\right|^2 y, y\right\rangle^{1/2}\right.\\
&\left.+\left|\left\langle \sum_{i=1}^{n} p_i B_i^* A_i x, y\right\rangle\right|\right]\\
&\le \frac{1}{2}\left[\sup_{\|x\|=1}\left\langle \sum_{i=1}^{n} p_i \left|A_i\right|^2 x, x\right\rangle^{1/2} \sup_{\|y\|=1}\left\langle \sum_{i=1}^{n} p_i \left|B_i\right|^2 y, y\right\rangle^{1/2}\right.\\
&\left.+\sup_{\|x\|=\|y\|=1}\left|\left\langle \sum_{i=1}^{n} p_i B_i^* A_i x, y\right\rangle\right|\right]
\end{aligned}$$

for any $e \in H$ with $\|e\| = 1$.

Since

$$\begin{aligned}
\sup_{\|x\|=1}\left|\left\langle \sum_{i=1}^{n} p_i A_i e, x\right\rangle\right| &= \left\|\sum_{i=1}^{n} p_i A_i e\right\|,\\
\sup_{\|y\|=1}\left|\left\langle \sum_{i=1}^{n} p_i B_i e, y\right\rangle\right| &= \left\|\sum_{i=1}^{n} p_i B_i e\right\|,\\
\sup_{\|x\|=1}\left\langle \sum_{i=1}^{n} p_i \left|A_i\right|^2 x, x\right\rangle^{1/2} &= \left\|\sum_{i=1}^{n} p_i \left|A_i\right|^2\right\|^{1/2},\\
\sup_{\|y\|=1}\left\langle \sum_{i=1}^{n} p_i \left|B_i\right|^2 y, y\right\rangle^{1/2} &= \left\|\sum_{i=1}^{n} p_i \left|B_i\right|^2\right\|^{1/2}
\end{aligned}$$

and

$$\sup_{\|x\|=\|y\|=1}\left|\left\langle \sum_{i=1}^{n} p_i B_i^* A_i x, y\right\rangle\right| = \left\|\sum_{i=1}^{n} p_i B_i^* A_i\right\|.$$

Making use of (2.10) we get the first inequality in (2.7).

Using Schwarz inequality in $(H; \langle\cdot,\cdot\rangle)$ we have

$$\begin{aligned}
\left\|\sum_{i=1}^{n} p_i A_i e\right\| \left\|\sum_{i=1}^{n} p_i B_i e\right\| &\ge \left|\left\langle \sum_{i=1}^{n} p_i A_i e, \sum_{i=1}^{n} p_i B_i e\right\rangle\right|\\
&= \left|\left\langle \left(\sum_{i=1}^{n} p_i B_i^*\right) \sum_{i=1}^{n} p_i A_i e, e\right\rangle\right|
\end{aligned}$$

for any $e \in H$ with $\|e\| = 1$, and the second inequality in (2.7) is proved. ∎

Corollary 2.4. Let $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_n$ be a probability distribution and $(A_1, ..., A_n)$ an n-tuple of bounded linear operators on H. Then we have

$$\frac{1}{2}\left[\left\|\sum_{i=1}^{n} p_i |A_i|^2\right\|^{1/2}\left\|\sum_{i=1}^{n} p_i |A_i^*|^2\right\|^{1/2} + \left\|\sum_{i=1}^{n} p_i A_i^2\right\|\right] \tag{2.11}$$
$$\geq \left\|\sum_{i=1}^{n} p_i A_i e\right\|\left\|\sum_{i=1}^{n} p_i A_i^* e\right\| \geq \left|\left\langle\left(\sum_{i=1}^{n} p_i A_i\right)^2 e, e\right\rangle\right|$$

for any $e \in H$ with $\|e\| = 1$.

It follows from (2.7) by taking $B_i = A_i^*$, $i \in \{1, ..., n\}$.

Remark 2.5. Taking the supremum over $\|e\| = 1$ in (2.7) and (2.11), then we get the numerical radius inequalities

$$w\left(\sum_{i=1}^{n} p_i B_i^* \sum_{i=1}^{n} p_i A_i\right) \tag{2.12}$$
$$\leq \sup_{\|e\|=1} \left\|\sum_{i=1}^{n} p_i A_i e\right\|\left\|\sum_{i=1}^{n} p_i B_i e\right\|$$
$$\leq \frac{1}{2}\left[\left\|\sum_{i=1}^{n} p_i |A_i|^2\right\|^{1/2}\left\|\sum_{i=1}^{n} p_i |B_i|^2\right\|^{1/2} + \left\|\sum_{i=1}^{n} p_i B_i^* A_i\right\|\right]$$

and

$$w\left(\left(\sum_{i=1}^{n} p_i A_i\right)^2\right) \tag{2.13}$$
$$\leq \sup_{\|e\|=1} \left(\left\|\sum_{i=1}^{n} p_i A_i e\right\|\left\|\sum_{i=1}^{n} p_i A_i^* e\right\|\right)$$
$$\leq \frac{1}{2}\left[\left\|\sum_{i=1}^{n} p_i |A_i|^2\right\|^{1/2}\left\|\sum_{i=1}^{n} p_i |A_i^*|^2\right\|^{1/2} + \left\|\sum_{i=1}^{n} p_i A_i^2\right\|\right],$$

where $(A_1, ..., A_n)$, $(B_1, ..., B_n)$ are two n-tuples of bounded linear operators on H.

We recall that a bounded linear operator T is *normal* if $TT^* = T^*T$. This is equivalent to the fact that $\|Tx\| = \|T^*x\|$ for any $x \in H$.

If $(A_1, ..., A_n)$ is an n-tuple of normal operators on H, then from the first inequality in (2.11) we get

$$\frac{1}{2}\left[\left\|\sum_{i=1}^{n} p_i |A_i|^2\right\| + \left\|\sum_{i=1}^{n} p_i A_i^2\right\|\right] \geq \left\|\sum_{i=1}^{n} p_i A_i e\right\|^2 \tag{2.14}$$

for any $e \in H$ with $\|e\| = 1$.

Taking the supremum over $\|e\| = 1$ in (2.14) we also have

$$\frac{1}{2}\left[\left\|\sum_{i=1}^{n} p_i |A_i|^2\right\| + \left\|\sum_{i=1}^{n} p_i A_i^2\right\|\right] \geq \left\|\sum_{i=1}^{n} p_i A_i\right\|^2, \tag{2.15}$$

where $(A_1, ..., A_n)$ is an n-tuple of normal operators on H.

If we take $p_i = \frac{q_i}{\sum_{k=1}^{n} q_k}$ with $q_i \geq 0, i \in \{1, ..., n\}$ and $\sum_{k=1}^{n} q_k > 0$, then we get from (2.15)

$$\frac{1}{2}\left[\left\|\sum_{i=1}^{n} q_i |A_i|^2\right\| + \left\|\sum_{i=1}^{n} q_i A_i^2\right\|\right] \sum_{k=1}^{n} q_k \geq \left\|\sum_{i=1}^{n} q_i A_i\right\|^2, \tag{2.16}$$

for any $(A_1, ..., A_n)$ an n-tuple of normal operators on H.

If we take $q_i = r_i^2$, $A_i = \frac{1}{r_i} T_i$, where r_i are nonzero real numbers, $i \in \{1, ..., n\}$ and $(T_1, ..., T_n)$ is an n-tuple of normal operators on H, then from (2.16) we get

$$\frac{1}{2}\left[\left\|\sum_{i=1}^{n} |T_i|^2\right\| + \left\|\sum_{i=1}^{n} T_i^2\right\|\right] \sum_{k=1}^{n} r_k^2 \geq \left\|\sum_{i=1}^{n} r_i T_i\right\|^2, \tag{2.17}$$

which for $T_i = z_i 1_H$, $i \in \{1, ..., n\}$ produces the *de Bruijn inequality*

$$\frac{1}{2}\left[\sum_{i=1}^{n} |z_i|^2 + \left|\sum_{i=1}^{n} z_i^2\right|\right] \sum_{k=1}^{n} r_k^2 \geq \left|\sum_{i=1}^{n} r_i z_i\right|^2. \tag{2.18}$$

We have the following numerical radius inequality:

Theorem 2.6. Let $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_n$ be a probability distribution and $(A_1, ..., A_n)$ an$n$-tuples of bounded linear operators on H. Then we have

$$\begin{aligned} & w^2\left(\sum_{i=1}^{n} p_i A_i\right) \\ & \leq \frac{1}{2}\left[\left\|\sum_{i=1}^{n} p_i |A_i|^2\right\|^{1/2} \left\|\sum_{i=1}^{n} p_i |A_i^*|^2\right\|^{1/2} + w\left(\sum_{i=1}^{n} p_i A_i^2\right)\right]. \end{aligned} \tag{2.19}$$

Proof. If in (2.9) we take $B_i = A_i^*, i \in \{1, ..., n\}$ and $x = y = e$, then we get the inequality

$$\begin{aligned} & \frac{1}{2}\left[\left\langle \sum_{i=1}^{n} p_i |A_i|^2 e, e\right\rangle^{1/2} \left\langle \sum_{i=1}^{n} p_i |A_i^*|^2 e, e\right\rangle^{1/2} + \left|\left\langle \sum_{i=1}^{n} p_i A_i^2 e, e\right\rangle\right|\right] \\ & \geq \left|\left\langle \sum_{i=1}^{n} p_i A_i e, e\right\rangle \left\langle \sum_{i=1}^{n} p_i A_i^* e, e\right\rangle\right| = \left|\left\langle \sum_{i=1}^{n} p_i A_i e, e\right\rangle\right|^2, \end{aligned} \tag{2.20}$$

for any $e \in H$ with $\|e\| = 1$.

By taking the supremum over $\|e\| = 1$ in (2.20), we get

$$\begin{aligned}
& w^2\left(\sum_{i=1}^{n} p_i A_i\right) \\
& = \sup_{\|e\|=1}\left|\left\langle\sum_{i=1}^{n} p_i A_i e, e\right\rangle\right|^2 \\
& \leq \frac{1}{2}\sup_{\|e\|=1}\left[\left\langle\sum_{i=1}^{n} p_i \left|A_i\right|^2 e, e\right\rangle^{1/2}\left\langle\sum_{i=1}^{n} p_i \left|A_i^*\right|^2 e, e\right\rangle^{1/2} + \left|\left\langle\sum_{i=1}^{n} p_i A_i^2 e, e\right\rangle\right|\right] \\
& \leq \frac{1}{2}\left[\sup_{\|e\|=1}\left(\left\langle\sum_{i=1}^{n} p_i \left|A_i\right|^2 e, e\right\rangle^{1/2}\left\langle\sum_{i=1}^{n} p_i \left|A_i^*\right|^2 e, e\right\rangle^{1/2}\right)\right. \\
& \left. + \sup_{\|e\|=1}\left|\left\langle\sum_{i=1}^{n} p_i A_i^2 e, e\right\rangle\right|\right] \\
& \leq \frac{1}{2}\left[\sup_{\|e\|=1}\left(\left\langle\sum_{i=1}^{n} p_i \left|A_i\right|^2 e, e\right\rangle^{1/2}\sup_{\|e\|=1}\left\langle\sum_{i=1}^{n} p_i \left|A_i^*\right|^2 e, e\right\rangle^{1/2}\right)\right. \\
& \left. + \sup_{\|e\|=1}\left|\left\langle\sum_{i=1}^{n} p_i A_i^2 e, e\right\rangle\right|\right. \\
& = \frac{1}{2}\left[\left\|\sum_{i=1}^{n} p_i \left|A_i\right|^2\right\|^{1/2}\left\|\sum_{i=1}^{n} p_i \left|A_i^*\right|^2\right\|^{1/2} + w\left(\sum_{i=1}^{n} p_i A_i^2\right)\right]
\end{aligned}$$

and the inequality (2.19) is proved. ∎

Remark 2.7. If we take in (2.19) $n = 1$ and $A_1 = T$, then we recapture the inequality (1.7), namely

$$w^2(T) \leq \frac{1}{2}\left[w\left(T^2\right) + \|T\|^2\right].$$

The case $n = 2$ is important since it allows to apply the above inequalities for the Cartesian decomposition of an operator.

If we take in (2.19) $n = 2$ and $q_1 = q_2 = \frac{1}{2}$, then we have

$$w^2(A_1 + A_2) \leq \left\|\left|A_1\right|^2 + \left|A_2\right|^2\right\|^{1/2}\left\|\left|A_1^*\right|^2 + \left|A_2^*\right|^2\right\|^{1/2} + w\left(A_1^2 + A_2^2\right). \tag{2.21}$$

Assume that T is a bounded linear operator and consider the Cartesian decomposition

$$T = A + iB,$$

with the selfadjoint operators A, B given by

$$A = \frac{1}{2}\left(T^* + T\right),\ B = \frac{1}{2i}\left(T - T^*\right).$$

Take $A_1 = A$, $A_2 = iB$. Then $A_1 + A_2 = T$,

$$|A_1|^2 + |A_2|^2 = A^2 + B^2 = \frac{1}{2}\left(|T|^2 + |T^*|^2\right),$$

$$|A_1^*|^2 + |A_2^*|^2 = A^2 + B^2 = \frac{1}{2}\left(|T|^2 + |T^*|^2\right)$$

and

$$A_1^2 + A_2^2 = \frac{1}{4}\left(T^* + T\right)^2 + \frac{1}{4}\left(T - T^*\right)^2 = \frac{1}{2}\left(T^2 + \left(T^*\right)^2\right).$$

Using (2.21) we get

$$w^2\left(T\right) \le \frac{1}{2}\left[\left\||T|^2 + |T^*|^2\right\| + w\left(T^2 + \left(T^*\right)^2\right)\right] \tag{2.22}$$

for any T a bounded linear operator.

3 Applications for functions of normal operators

Recall some examples of power series with nonnegative coefficients

$$\frac{1}{1-\lambda} = \sum_{n=0}^{\infty} \lambda^n, \ \ \lambda \in D\left(0,1\right); \tag{3.1}$$

$$\ln \frac{1}{1-\lambda} = \sum_{n=1}^{\infty} \frac{1}{n}\lambda^n, \ \ \lambda \in D\left(0,1\right);$$

$$\exp\left(\lambda\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n, \ \ \lambda \in \mathbb{C};$$

$$\sinh \lambda = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\lambda^{2n+1}, \ \ \lambda \in \mathbb{C};$$

$$\cosh \lambda = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\lambda^{2n}, \ \ \lambda \in \mathbb{C}.$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \ \ \lambda \in D\left(0,1\right); \tag{3.2}$$

$$\sin^{-1}\left(\lambda\right) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\left(2n+1\right)n!}\lambda^{2n+1}, \ \ \lambda \in D\left(0,1\right);$$

$$\tanh^{-1}\left(\lambda\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \ \ \lambda \in D\left(0,1\right);$$

$${}_2F_1\left(\alpha,\beta,\gamma,\lambda\right) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\alpha\right)\Gamma\left(n+\beta\right)\Gamma\left(\gamma\right)}{n!\Gamma\left(\alpha\right)\Gamma\left(\beta\right)\Gamma\left(n+\gamma\right)}\lambda^n, \alpha,\beta,\gamma > 0,$$
$$\lambda \in D\left(0,1\right)$$

where Γ is *Gamma function*.

The following inequality for power series with nonenegative coefficients holds:

Theorem 3.1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients $a_n \geq 0$ for $n \in \mathbb{N}$ and having the radius of convergence $R > 0$ or $R = \infty$. If U, V are normal operators on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$, $U^*V = VU^*$ and $\alpha > 0$ such that $\alpha < R$ and $\|U\|, \|V\| \leq 1$, then

$$\begin{aligned} & |\langle f(\alpha V) e, x\rangle \langle f(\alpha U) e, y\rangle| \\ & \leq \frac{1}{2}\left[\left\langle f\left(\alpha |V|^2\right) x, x\right\rangle \left\langle f\left(\alpha |U|^2\right) y, y\right\rangle + |\langle f(U^*V) x, y\rangle|\right] f(\alpha) \end{aligned} \tag{3.3}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Proof. Using the inequality (2.9) we have for $n \geq 1$

$$\begin{aligned} & \frac{1}{2}\left[\left\langle \sum_{i=0}^{n} a_i \alpha^i \left|V^i\right|^2 x, x\right\rangle^{1/2} \left\langle \sum_{i=0}^{n} a_i \alpha^i \left|U^i\right|^2 y, y\right\rangle^{1/2} \right. \\ & \left. + \left|\left\langle \sum_{i=0}^{n} a_i \alpha^i (U^*)^i V^i x, y\right\rangle\right|\right] \sum_{i=0}^{n} a_i \alpha^i, \\ & \geq \left|\left\langle \sum_{i=0}^{n} a_i \alpha^i V^i e, x\right\rangle \left\langle \sum_{i=0}^{n} a_i \alpha^i U^i e, y\right\rangle\right| \end{aligned} \tag{3.4}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Since U, V are normal operators, then for $i \geq 1$

$$\left|V^i\right|^2 = \left(V^i\right)^* V^i = (V^*)^i V^i = (V^*V)^i = |V|^{2i}$$

and

$$\left|U^i\right|^2 = |U|^{2i}.$$

Also, since $U^*V = VU^*$, then

$$(U^*)^i V^i = (U^*V)^i$$

for any $i \geq 1$.

Then from (3.4) we have

$$\begin{aligned} & \frac{1}{2}\left[\left\langle \sum_{i=0}^{n} a_i \alpha^i |V|^{2i} x, x\right\rangle^{1/2} \left\langle \sum_{i=0}^{n} a_i \alpha^i |U|^{2i} y, y\right\rangle^{1/2} \right. \\ & \left. + \left|\left\langle \sum_{i=0}^{n} a_i \alpha^i (U^*V)^i x, y\right\rangle\right|\right] \sum_{i=0}^{n} a_i \alpha^i \\ & \geq \left|\left\langle \sum_{i=0}^{n} a_i \alpha^i V^i e, x\right\rangle \left\langle \sum_{i=0}^{n} a_i \alpha^i U^i e, y\right\rangle\right|, \end{aligned} \tag{3.5}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Since $\left\|\alpha |V|^2\right\| = \alpha \|V\|^2 < R$, $\left\|\alpha |U|^2\right\| = \alpha \|U\|^2 < R$, $\|\alpha U^* V\| \le \alpha \|U\| \|V\| < R$, $\|\alpha U\| < R$ and $\|\alpha V\| < R$, then the series

$$\sum_{i=0}^{\infty} a_i \alpha^i U^i, \ \sum_{i=0}^{\infty} a_i \alpha^i V^i, \ \sum_{i=0}^{n} a_i \alpha^i |V|^{2i}, \ \sum_{i=0}^{n} a_i \alpha^i |U|^{2i}, \ \sum_{i=0}^{\infty} a_i \alpha^i (U^* V)^i$$

are convergent in $B(H)$ and $\sum_{i=0}^{\infty} a_i \alpha^i$ is convergent in $\mathbb{R}$.

Taking the limit over $n \to \infty$ in (3.5) we get the desired result (3.3). ∎

Remark 3.2. If we take the supremum over $\|x\| = \|y\| = 1$ in (3.3) then we get the norm inequality

$$\|f(\alpha V) e\| \|f(\alpha U) e\| \le \frac{1}{2} \left[\left\| f\left(\alpha |V|^2\right) \right\| \left\| f\left(\alpha |U|^2\right) \right\| + \|f(U^* V)\| \right] f(\alpha) \tag{3.6}$$

for any $e \in H$, $\|e\| = 1$.

By Schwarz inequality in H we have

$$\begin{aligned} \|f(\alpha V) e\| \|f(\alpha U) e\| &\ge |\langle f(\alpha V) e, f(\alpha U) e\rangle| \\ &= \left|\left\langle (f(\alpha U))^* f(\alpha V) e, e\right\rangle\right| \\ &= |\langle f(\alpha U^*) f(\alpha V) e, e\rangle| \end{aligned}$$

giving the inequality

$$|\langle f(\alpha U^*) f(\alpha V) e, e\rangle| \le \frac{1}{2} \left[\left\| f\left(\alpha |V|^2\right) \right\| \left\| f\left(\alpha |U|^2\right) \right\| + \|f(U^* V)\| \right] f(\alpha) \tag{3.7}$$

for any $e \in H$, $\|e\| = 1$.

Since U and V are normal and $U^* V = V U^*$, then $f(\alpha U^*)$ and $f(\alpha V)$ are normal and commute implying that $f(\alpha U^*) f(\alpha V)$ is normal. Taking the supremum over $\|e\| = 1$ we get

$$\|f(\alpha U^*) f(\alpha V)\| \le \frac{1}{2} \left[\left\| f\left(\alpha |V|^2\right) \right\| \left\| f\left(\alpha |U|^2\right) \right\| + \|f(U^* V)\| \right] f(\alpha). \tag{3.8}$$

Example 3.3. a. If U, V are normal operators on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$, $U^* V = V U^*$ and $\alpha > 0$ then

$$\begin{aligned} &|\langle \exp(\alpha V) e, x\rangle \langle \exp(\alpha U) e, y\rangle| \\ &\le \frac{1}{2} \left[\left\langle \exp\left(\alpha |V|^2\right) x, x \right\rangle \left\langle \exp\left(\alpha |U|^2\right) y, y \right\rangle + |\langle \exp(U^* V) x, y\rangle| \right] \exp(\alpha) \end{aligned} \tag{3.9}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

b. If U, V are normal operators on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$, $U^* V = V U^*$ and $\|U\|, \|V\| < 1$, $\alpha \in (0, 1)$ then

$$\begin{aligned} &\left| \left\langle (1_H - \alpha V)^{-1} e, x \right\rangle \left\langle (1_H - \alpha U)^{-1} e, y \right\rangle \right| \\ &\le \frac{1}{2} \left[\left\langle \left(1_H - \alpha |V|^2\right)^{-1} x, x \right\rangle \left\langle \left(1_H - \alpha |U|^2\right)^{-1} y, y \right\rangle \right. \\ &\left. + \left| \left\langle (1_H - U^* V)^{-1} x, y \right\rangle \right| \right] (1 - \alpha)^{-1} \end{aligned} \tag{3.10}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

References

[1] J. M. Aldaz, Strengthened Cauchy-Schwarz and Hölder inequalities. *J. Inequal. Pure Appl. Math.* **10** (2009), no. 4, Article 116, 6 pp.

[2] N. G. de Bruijn, Problem 12, *Wisk. Opgaven*, **21** (1960), 12-14.

[3] M. L. Buzano, Generalizzazione della diseguaglianza di Cauchy-Schwarz. (Italian), *Rend. Sem. Mat. Univ. e Politech. Torino*, **31** (1971/73), 405–409 (1974).

[4] A. De Rossi, A strengthened Cauchy-Schwarz inequality for biorthogonal wavelets. *Math. Inequal. Appl.* **2** (1999), no. 2, 263–282.

[5] S. S. Dragomir, Some refinements of Schwartz inequality, Simpozionul de Matematici şi Aplicaţii, Timişoara, Romania, 1-2 Noiembrie 1985, 13–16.

[6] S. S. Dragomir, Some inequalities for power series of selfadjoint operators in Hilbert spaces via reverses of the Schwarz inequality. *Integral Transforms Spec. Funct.* **20** (2009), no. 9-10, 757–767.

[7] S. S. Dragomir, A potpourri of Schwarz related inequalities in inner product spaces. I. *J. Inequal. Pure Appl. Math.* **6** (2005), no. 3, Article 59, 15 pp.

[8] S. S. Dragomir, A potpourri of Schwarz related inequalities in inner product spaces. II. *J. Inequal. Pure Appl. Math.* **7** (2006), no. 1, Article 14, 11 pp.

[9] S. S. Dragomir, Reverses of Schwarz, triangle and Bessel inequalities in inner product spaces. *J. Inequal. Pure Appl. Math.* **5** (2004), no. 3, Article 76, 18 pp.

[10] S. S. Dragomir, New reverses of Schwarz, triangle and Bessel inequalities in inner product spaces. *Aust. J. Math. Anal. Appl.* **1** (2004), no. 1, Art. 1, 18 pp. New York, 2007. xii+243 pp. ISBN: 978-1-59454-903-8; 1-59454-903-6.

[11] S. S. Dragomir, Inequalities for the norm and the numerical radius of linear operators in Hilbert spaces. *Demonstratio Math.* **40** (2007), no. 2, 411–417.

[12] S. S. Dragomir, Some inequalities for the norm and the numerical radius of linear operators in Hilbert spaces. *Tamkang J. Math.* **39** (2008), no. 1, 1–7.

[13] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*. Nova Science Publishers, Inc., Hauppauge, NY, 2005. viii+249 pp. ISBN: 1-59454-202-3.

[14] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Triangle and Heisenberg Type in Inner Product Spaces*. Nova Science Publishers, Inc.,

[15] S. S. Dragomir, *Inequalities for the Numerical Radius of Linear Operators in Hilbert Spaces*. Springer Briefs in Mathematics. Springer, 2013. x+120 pp. ISBN: 978-3-319-01447-0; 978-3-319-01448-7.

[16] S. S. Dragomir and Anca C. Goşa, Quasilinearity of some composite functionals associated to Schwarz's inequality for inner products. *Period. Math. Hungar.* **64** (2012), no. 1, 11–24.

[17] S. S. Dragomir and B. Mond, Some mappings associated with Cauchy-Buniakowski-Schwarz's inequality in inner product spaces. *Soochow J. Math.* **21** (1995), no. 4, 413–426.

[18] S. S. Dragomir and I. Sándor, Some inequalities in pre-Hilbertian spaces. *Studia Univ. Babeş-Bolyai Math.* **32** (1987), no. 1, 71–78.

[19] H. Gunawan, On n-inner products, n-norms, and the Cauchy-Schwarz inequality. *Sci. Math. Jpn.* **55** (2002), no. 1, 53–60

[20] K. E. Gustafson and D. K. M. Rao, *Numerical Range,* Springer-Verlag, New York, Inc., 1997.

[21] E. R. Lorch, The Cauchy-Schwarz inequality and self-adjoint spaces. *Ann. of Math.* (2) **46**, (1945). 468–473.

[22] C. Lupu and D. Schwarz, Another look at some new Cauchy-Schwarz type inner product inequalities. *Appl. Math. Comput.* **231** (2014), 463–477.

[23] M. Marcus, The Cauchy-Schwarz inequality in the exterior algebra. *Quart. J. Math. Oxford Ser.* (2) **17** 1966 61–63.

[24] P. R. Mercer, A refined Cauchy-Schwarz inequality. *Internat. J. Math. Ed. Sci. Tech.* **38** (2007), no. 6, 839–842.

[25] F. T. Metcalf, A Bessel-Schwarz inequality for Gramians and related bounds for determinants. *Ann. Mat. Pura Appl.* (4) **68** 1965 201–232.

[26] T. Precupanu, On a generalization of Cauchy-Buniakowski-Schwarz inequality. *An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat.* (N.S.) **22** (1976), no. 2, 173–175.

[27] K. Trenčevski and R. Malčeski, On a generalized n-inner product and the corresponding Cauchy-Schwarz inequality. *J. Inequal. Pure Appl. Math.* **7** (2006), no. 2, Article 53, 10 pp.

[28] G.-B. Wang and J.-P. Ma, Some results on reverses of Cauchy-Schwarz inequality in inner product spaces. *Northeast. Math. J.* **21** (2005), no. 2, 207–211.