On the generalized orthogonal stability of mixed type additive-cubic functional equations in modular spaces

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Abstract

In this paper, we establish the Hyers-Ulam-Rassias stability of the mixed type additive-cubic functional equation

$$f(2x+y)+f(2x-y)-f(4x)=2[f(x+y)+f(x-y)]-8f(2x)+10f(x)-2f(-x),$$

with $x \perp y$, where $\perp$ is the orthogonality in the sense of Rätz in modular spaces.

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1 Introduction

The study of stability problems for functional equations is related to a question of Ulam [21] in 1940, concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [10]. In 1950, a generalized version of Hyers' theorem for approximate additive mapping was given by Aoki [2]. In 1978, Rassias [17] provided a generalization of Hyers Theorem which allows the Cauchy difference to be unbounded. This stability phenomenon is called the Hyers-Ulam-Rassias stability.

Stability problems for some functional equations have been extensively investigated by several authors, and in particular one of the most important functional equation in this topic is

$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{1.1}$$

which is studied by Adam [1], P. Găvruta [7], M. Eshaghi [6], and A. Najati [13].

Recently, Gh. Sadeghi [19] proved the Hyers-Ulam stability of the generalized Jensen functional equation $f(rx+sy)=rg(x)+sh(x)$ in modular spaces, using the fixed point method, also Iz. EL-Fassi and S. Kabbaj in [5] investigated the Hyers-Ulam-Rassias stability of (1.1) in modular spaces. The theory of modulars on linear spaces and the corresponding theory of modular linear spaces were founded by H. Nakano [14]. In the present time the theory of modulars and modular spaces is extensively applied, in particular, in the study of various Orlicz spaces [15] and interpolation theory [12]. The importance for applications consists in the richness of the structure of modular spaces, that-besides being Banach spaces (or $F-$spaces in more general setting)- are equipped with modular equivalent of norm or metric notions. Numerous papers on the stability of some functional equations have been published by different authors, we refer, for example, to [3], [4], [11] and [20].

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There are several orthogonality notions on a real normed spaces as Birkhoff-James, Carlsson, Singer, Roberts, Pythagorean, isosceles and Diminnie. Let us recall the orthogonality space in the sense of Rätz; cf. [18].

Suppose E is a real vector space with $\dim E \geq 2$ and $\perp$ is a binary relation on E with the following properties:

(O1) totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in E$;

(O2) independence: if $x, y \in E - \{0\}$, $x \perp y$, then, x, y are linearly independent;

(O3) homogeneity: if $x, y \in E$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

(O4) the Thalesian property: if P is a 2-dimensional subspace of E, $x \in P$ and $\lambda \in \mathbb{R}^+$, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(E, \perp)$ is called an orthogonality space. By an orthogonality normed space, we mean an orthogonality space having a normed structure. Some interesting examples of orthogonality spaces are:

(i) The trivial orthogonality on a vector space E defined by (O1), and for nonzero elements $x, y \in E$, $x \perp y$ if and only if x, y are linearly independent.

(ii) The ordinary orthogonality on an inner product space $(E, \langle . \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.

(iii) The Birkhoff-James orthogonality on a normed space $(E, \|.\|)$ defined by $x \perp y$ if and only if $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in E$. Clearly examples (i) and (ii) are symmetric but example (iii) is not. However, it is remarkable to note, that a real normed space of dimension greater than or equal to 3 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric.

The Orthogonal Cauchy functional equation

$$f(x+y) = f(x) + f(y) \qquad (x, y \in E, \ \ x \perp y) \tag{1.2}$$

in which $\perp$ is an abstract orthogonally was first investigated by S. Gudder and D. Strawther [9]. R. Ger and J. Sikorska discussed the orthogonal stability of the equation (1.2) in [8]. S. Ostadbashi and J. Kazemzadeh [16] investigated the problem of the Orthogonal stability of the mixed additive-cubic functional equation

$$f(2x+y) + f(2x-y) - f(4x) = 2[f(x+y) + f(x-y)] - 8f(2x) + 10f(x) - 2f(-x) \ (x \perp y), \tag{1.3}$$

in Banach space.

In the present paper, we establish the Hyers-Ulam-Rassias stability of orthogonally mixed additive-cubic functional equation (1.3) in modular spaces. Therefore, we generalized the main results of [16].

2 Preliminary

In this section, we give the definitions that are important in the following.

Definition 2.1. Let X be an arbitrary vector space.
(a) A functional $\rho : X \to [0, \infty]$ is called a modular if for arbitrary $x, y \in X$,
(i) $\rho(x) = 0$ if and only if $x = 0$,
(ii) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$,
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,
(b) if (iii) is replaced by
(iii)' $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,
then we say that ρ is a convex modular.

A modular ρ defines a corresponding modular space, i.e., the vector space X_ρ given by

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

Let ρ be a convex modular, the modular space X_ρ can be equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho = \inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

A function modular is said to satisfy the Δ_2-condition if there exists $\kappa > 0$ such that $\rho(2x) \leq \kappa\rho(x)$ for all $x \in X_\rho$.

Definition 2.2. Let $\{x_n\}$ and x be in X_ρ. Then
(i) we say $\{x_n\}$ is $\rho-$convergent to x and write $x_n \xrightarrow{\rho} x$ if and only if $\rho(x_n - x) \to 0$ as $n \to \infty$,
(ii) the sequence $\{x_n\}$, with $x_n \in X_\rho$, is called $\rho-$Cauchy if $\rho(x_n - x_m) \to 0$ as $m, n \to \infty$,
(iii) a subset S of X_ρ is called $\rho-$complete if and only if any $\rho-$Cauchy sequence is $\rho-$convergent to an element of S.

The modular ρ has the Fatou property if and only if $\rho(x) \leq \lim_{n\to\infty} \inf \rho(x_n)$ whenever the sequence $\{x_n\}$ is $\rho-$convergent to x.

Remark 2.3. If $x \in X_\rho$ then $\rho(ax)$ is a nondecreasing function of $a \geq 0$. Suppose that $0 < a < b$, then property (iii) of definition 2.1 with $y = 0$ shows that

$$\rho(ax) = \rho\left(\frac{a}{b} bx\right) \leq \rho(bx).$$

Moreover, if ρ is convex modular on X and $|\alpha| \leq 1$ then, $\rho(\alpha x) \leq |\alpha|\rho(x)$ and also $\rho(x) \leq \frac{1}{2}\rho(2x) \leq \frac{\kappa}{2}\rho(x)$ if ρ satisfy the Δ_2- condition for all $x \in X$.

Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ denote the sets of all positive integers and all real numbers, respectively.

3 Orthogonal Stability of Eq (1.3) in Modular Spaces

In this section we assume that the convex modular ρ has the Fatou property such that satisfies the Δ_2-condition with $0 < \kappa \leq 2$. In addition, we assume that $(E, \perp)$ denotes an orthogonality space and we define

$$Df(x, y) = f(2x + y) + f(2x - y) - f(4x) - 2[f(x + y) + f(x - y)] + 8f(2x) - 10f(x) + 2f(-x),$$

for all $x, y \in E$ with $x \perp y$, on the other hand, we give the Hyers-Ulam-Rassias stability of the equation (1.3) in modular spaces.

Proposition 3.1. Let $(E, \|.\|)$ with $\dim E \geq 2$ be a real normed linear space and X_ρ is a $\rho-$complete modular space. Let $f : E \to X_\rho$ be an odd mapping satisfying

$$\rho(Df(x,y)) \leq \varepsilon(\|x\|^p + \|y\|^p), \tag{3.1}$$

for all $x, y \in E$ with $x \perp y$, $\varepsilon \geq 0$ and $0 < p < 1$. Then there exists a unique orthogonally cubic-additive mapping $A_c : E \to X_\rho$ such that

$$\rho(f(2x) - 8f(x) - A_c(x)) \leq \frac{\varepsilon}{2 - \kappa 2^{p-1}} \|x\|^p \tag{3.2}$$

for all $x \in E$. Moreover

$$A_c(x) = \lim_{n\to\infty} \frac{f(2^{n+1}) - 8f(2^n x)}{2^n}$$

Proof. Letting $(x, y) = (0, 0)$ in (3.1), we get $f(0) = 0$. Put $y = 0$ in (3.1). We can do this because of (O1). Then

$$\rho(10f(2x) - f(4x) - 16f(x)) \leq \varepsilon \|x\|^p$$

for all $x \in E$. Hence

$$\rho(f(4x) - 8f(2x) - 2(f(2x) - 8f(x))) \leq \varepsilon \|x\|^p \tag{3.3}$$

for all $x \in E$. By letting $F(x) = f(2x) - 8f(x)$ in (3.3), we obtain

$$\rho(F(2x) - 2F(x)) \leq \varepsilon \|x\|^p \tag{3.4}$$

for all $x \in E$. We have

$$\rho\left(\frac{F(2x)}{2} - F(x)\right) = \rho\left(\frac{1}{2}(F(2x) - 2F(x))\right) \leq \frac{\varepsilon}{2} \|x\|^p \tag{3.5}$$

for all $x \in E$. Replacing x by $2x$ in (3.5), we arrive to

$$\rho\left(\frac{F(2^2x)}{2} - F(2x)\right) \leq \varepsilon 2^{p-1} \|x\|^p \tag{3.6}$$

for all $x \in E$. By (3.5) and (3.6), we have

$$\begin{aligned}\rho\left(\frac{F(2^2x)}{2^2} - F(x)\right) &= \rho\left(\frac{F(2^2x)}{2^2} - \frac{F(2x)}{2} + \frac{F(2x)}{2} - F(x)\right)\\ &\leq \frac{\kappa}{2}\rho\left(\frac{F(2x)}{2} - F(x)\right) + \frac{\kappa}{2^2}\rho\left(\frac{F(2^2x)}{2} - F(2x)\right)\\ &\leq \frac{\varepsilon}{2}(1 + \kappa 2^{p-2}) \|x\|^p \end{aligned} \tag{3.7}$$

for all $x \in E$. By mathematical induction, we can easily see that

$$\rho\left(\frac{F(2^n x)}{2^n} - F(x)\right) \leq \frac{\varepsilon}{2}\sum_{i=0}^{n-1} \kappa^i 2^{i(p-2)} \|x\|^p \tag{3.8}$$

for all $x \in E$. Indeed, for $n = 1$ the relation (3.8) is true. Assume that the relation (3.8) is true for n, and we show this relation rest true for $n+1$, thus we have

$$\begin{aligned}
\rho\left(\frac{F(2^{n+1}x)}{2^{n+1}} - F(x)\right) &= \rho\left(\frac{F(2^{n+1}x)}{2^{n+1}} - \frac{F(2x)}{2} + \frac{F(2x)}{2} - F(x)\right)\\
&\leq \frac{\kappa}{2}\rho\left(\frac{F(2x)}{2} - F(x)\right) + \frac{\kappa}{2^2}\rho\left(\frac{F(2^{n+1}x)}{2^n} - F(2x)\right)\\
&\leq \frac{\kappa}{2}\frac{\varepsilon}{2}\left\|x\right\|^p + \frac{\varepsilon}{2}\sum_{i=0}^{n-1}\kappa^{i+1}2^{(i+1)(p-2)}\left\|x\right\|^p\\
&\leq \frac{\varepsilon}{2}\sum_{i=0}^{n}\kappa^i 2^{i(p-2)}\left\|x\right\|^p,
\end{aligned}$$

hence the relation (3.8) is true for all $x \in E$ and $n \in \mathbb{N}^*$ ($\in \mathbb{N}^*$: the set of positive integers). Then (3.8) become

$$\rho\left(\frac{F(2^n x)}{2^n} - F(x)\right) \leq \frac{\varepsilon}{2}\frac{1-(\kappa 2^{p-2})^n}{1-\kappa 2^{p-2}}\left\|x\right\|^p \tag{3.9}$$

for all $x \in E$. Replacing x by $2^m x$ (with $m \in \mathbb{N}^*$) in (3.9), we obtain

$$\rho\left(\frac{F(2^{n+m}x)}{2^n} - F(2^m x)\right) \leq \frac{\varepsilon}{2}\frac{1-(\kappa 2^{p-2})^n}{1-\kappa 2^{p-2}}2^{mp}\left\|x\right\|^p \tag{3.10}$$

for all $x \in E$. Whence

$$\begin{aligned}
\rho\left(\frac{F(2^{n+m}x)}{2^{n+m}} - \frac{F(2^m x)}{2^m}\right) &\leq \frac{1}{2^m}\rho\left(\frac{F(2^{n+m}x)}{2^n} - F(2^m x)\right)\\
&\leq \frac{\varepsilon}{2}\frac{1-(\kappa 2^{p-2})^n}{1-\kappa 2^{p-2}}2^{m(p-1)}\left\|x\right\|^p \qquad (3.11)
\end{aligned}$$

for all $x \in E$. If $m, n \to \infty$ we get, the sequence $\left\{\frac{F(2^n x)}{2^n}\right\}$ is $\rho-$Cauchy sequence in the $\rho-$complete modular space X_ρ. Hence $\left\{\frac{F(2^n x)}{2^n}\right\}$ is $\rho-$convergent in X_ρ, and we well define the mapping $A_c = \lim_{n\to\infty}\frac{F(2^n x)}{2^n}$ from E into X_ρ satisfying

$$\rho(A_c(x) - F(x)) \leq \frac{\varepsilon}{2-\kappa 2^{p-1}}\left\|x\right\|^p, \tag{3.12}$$

for all $x \in E$, since ρ has Fatou property. To prove A_c satisfies $Df(x,y) = 0$, replace (x,y) by $(2^{n+1}x, 2^{n+1}y)$ in (3.1), it follows that

$$\begin{aligned}
\rho\left(\frac{Df(2^{n+1}x, 2^{n+1}y)}{2^n}\right) &\leq \frac{1}{2^n}\rho(Df(2^{n+1}x, 2^{n+1}y))\\
&\leq \varepsilon 2^{n(p-1)+p}(\left\|x\right\|^p + \left\|y\right\|^p), \qquad (3.13)
\end{aligned}$$

for all $x, y \in E$. Again replace (x, y) by $(2^n x, 2^n y)$ in (3.1), it follows that

$$\begin{aligned}\rho\left(\frac{Df(2^n x, 2^n y)}{2^n}\right) &\leq \frac{1}{2^n}\rho(Df(2^{n+1}x, 2^{n+1}y))\\ &\leq \varepsilon 2^{n(p-1)}(\|x\|^p + \|y\|^p),\end{aligned} \tag{3.14}$$

for all $x, y \in E$. By (3.13) and (3.14), we get

$$\begin{aligned}\rho\left(\frac{Df(2^{n+1}x, 2^{n+1}y) - 8Df(2^n x, 2^n y)}{2^n}\right) &\leq \frac{\kappa}{2}\rho\left(\frac{Df(2^{n+1}x, 2^{n+1}y)}{2^n}\right) + \frac{\kappa^4}{2}\rho\left(\frac{Df(2^n x, 2^n y)}{2^n}\right)\\ &\leq \frac{\kappa}{2}\varepsilon 2^{n(p-1)+p}(\|x\|^p + \|y\|^p) + \frac{\kappa^4}{2}\varepsilon 2^{n(p-1)}(\|x\|^p + \|y\|^p)\end{aligned}$$

If $n \to \infty$ then, we conclude that $DA_c(x, y) = 0$, for all $x, y \in E$ with $x \perp y$. Therefore $A_c : E \to X_\rho$ is an orthogonally cubic-additive mapping satisfying (1.3). To prove the uniqueness, assume $A'_c : E \to X_\rho$ to be another orthogonally cubic-additive mapping satisfying (3.12). Then, for each $x, y \in E$ and for all $m \in \mathbb{N}$ on has

$$\begin{aligned}\rho(A_c(x) - A'_c(x)) &= \rho\left(\frac{A_c(2^m x)}{2^m} - \frac{A'_c(2^m x)}{2^m}\right)\\ &\leq \frac{\kappa}{2^{m+1}}[\rho(A_c(2^m x) - F(2^m x)) + \rho(A'_c(2^m x) - F(2^m x))]\\ &\leq \frac{\kappa\varepsilon 2^{m(p-1)}}{2 - \kappa 2^{p-1}}\|x\|^p\end{aligned}$$

If $m \to \infty$, we obtain $A_c = A'_c$. Q.E.D.

Proposition 3.2. Let $(E, \|.\|)$ with $\dim E \geq 2$ be a real normed linear space and X_ρ is a $\rho-$complete modular space. Let $f : E \to X_\rho$ be an odd mapping satisfying

$$\rho(Df(x, y)) \leq \varepsilon(\|x\|^p + \|y\|^p), \tag{3.15}$$

for all $x, y \in E$ with $x \perp y$, $\varepsilon \geq 0$ and $0 < p < 3$. Then there exists a unique orthogonally cubic-additive mapping $C_a : E \to X_\rho$ such that

$$\rho(f(2x) - 2f(x) - C_a(x)) \leq \frac{\varepsilon}{8 - \kappa 2^{p-1}}\|x\|^p \tag{3.16}$$

for all $x \in E$. Moreover

$$C_a(x) = \lim_{n\to\infty}\frac{f(2^{n+1}) - 2f(2^n x)}{8^n}$$

Proof. By (3.3), we have

$$\rho(f(4x) - 2f(2x) - 8(f(2x) - 2f(x))) \leq \varepsilon\|x\|^p \tag{3.17}$$

for all $x \in E$. By letting $G(x) = f(2x) - 2f(x)$ in (3.17), we get

$$\rho\left(\frac{G(2x)}{8} - G(x)\right) \leq \frac{\varepsilon}{8}\|x\|^p \tag{3.18}$$

for all $x \in E$. Now replacing x by $2x$ in (3.18), we find

$$\rho\left(\frac{G(2^2x)}{8} - G(2x)\right) \le \frac{\varepsilon 2^p}{8} \|x\|^p \tag{3.19}$$

for all $x \in E$. Then

$$\rho\left(\frac{G(2^2x)}{8^2} - \frac{G(2x)}{8}\right) \le \frac{\varepsilon 2^{p-3}}{8} \|x\|^p \tag{3.20}$$

for all $x \in E$. From (3.18) and (3.20), we have

$$\begin{aligned}\rho\left(\frac{G(2^2x)}{8^2} - G(x)\right) &\le \frac{\kappa}{2}\rho\left(\frac{G(2^2x)}{8^2} - \frac{G(2x)}{8}\right) + \frac{\kappa}{2}\rho\left(\frac{G(2x)}{8} - G(2x)\right)\\ &\le \frac{\varepsilon}{8}(1 + \frac{\kappa}{2}2^{p-3}) \|x\|^p\end{aligned}$$

for all $x \in E$. In general, using induction on a positive integer n, we obtain

$$\begin{aligned}\rho\left(\frac{G(2^nx)}{8^n} - G(x)\right) &\le \frac{\varepsilon}{8}\sum_{i=0}^{n-1}\left(\frac{\kappa}{2}\right)^i 2^{i(p-3)} \|x\|^p\\ &= \frac{\varepsilon}{8}\frac{1-(\kappa 2^{p-4})^n}{1-\kappa 2^{p-4}} \|x\|^p\end{aligned} \tag{3.21}$$

for all $x \in E$. Replacing x by $2^m x$ (with $m \in \mathbb{N}^*$) in (3.21), we get

$$\rho\left(\frac{G(2^{n+m}x)}{8^n} - G(2^m x)\right) \le \frac{\varepsilon}{8}\frac{1-(\kappa 2^{p-4})^n}{1-\kappa 2^{p-4}} 2^{mp} \|x\|^p \tag{3.22}$$

for all $x \in E$. Whence

$$\rho\left(\frac{G(2^{n+m}x)}{8^{n+m}} - \frac{F(2^m x)}{8^m}\right) \le \frac{\varepsilon}{8}\frac{1-(\kappa 2^{p-4})^n}{1-\kappa 2^{p-4}} 2^{m(p-3)} \|x\|^p \tag{3.23}$$

for all $x \in E$. If $m, n \to \infty$ we get, the sequence $\left\{\frac{G(2^n x)}{8^n}\right\}$ is $\rho-$Cauchy sequence in the $\rho-$complete modular space X_ρ. Hence $\left\{\frac{G(2^n x)}{8^n}\right\}$ is $\rho-$convergent in X_ρ, and we well define the mapping $C_a = \lim_{n\to\infty} \frac{G(2^n x)}{8^n}$ from E into X_ρ satisfying

$$\rho(C_a(x) - G(x)) \le \frac{\varepsilon}{8 - \kappa 2^{p-1}} \|x\|^p, \tag{3.24}$$

for all $x \in E$, since ρ has Fatou property. The rest of the proof is similar to the proof of proposition 3.1. Q.E.D.

Theorem 3.3. Let $(E, \|.\|)$ with $\dim E \ge 2$ be a real normed linear space and X_ρ is a $\rho-$complete modular space. Let $f : E \to X_\rho$ be an odd mapping satisfying

$$\rho(Df(x,y)) \le \varepsilon(\|x\|^p + \|y\|^p), \tag{3.25}$$

for all $x, y \in E$ with $x \perp y$, $\varepsilon \geq 0$ and $0 < p < 1$. Then there exists a unique orthogonally cubic-additive mapping $AC : E \to X_\rho$ such that

$$\rho(f(x) - AC(x)) \leq \frac{\kappa\varepsilon}{12}\left\{\frac{1}{2-\kappa 2^{p-1}} + \frac{1}{8-\kappa 2^{p-1}}\right\}\|x\|^p \tag{3.26}$$

for all $x \in E$. Moreover

$$AC(x) = \frac{-1}{6}A_c(x) + \frac{1}{6}C_a(x)$$

for all $x \in E$.

Proof. By proposition 3.1 and proposition 3.2, we have

$$\begin{aligned}
\rho(f(x) - AC(x)) &= \rho\left(f(x) + \frac{1}{6}A_c(x) - \frac{1}{6}C_a(x)\right)\\
&= \rho\left(\frac{-1}{6}[f(2x) - 8f(x) - A_c(x)] + \frac{1}{6}[f(2x) - 2f(x) - C_a(x)]\right)\\
&\leq \frac{\kappa}{12}\left\{\rho([f(2x) - 8f(x) - A_c(x)]) + \rho([f(2x) - 2f(x) - C_a(x)])\right\}\\
&\leq \frac{\kappa\varepsilon}{12}\left\{\frac{1}{2-\kappa 2^{p-1}} + \frac{1}{8-\kappa 2^{p-1}}\right\}\|x\|^p
\end{aligned}$$

for all $x \in E$. Q.E.D.

Remark 3.4. [16] Let $f : E \to X_\rho$ be an even mapping satisfying (1.3) (with $x \perp y$), then $f = 0$ on E.

Proposition 3.5. Let $(E, \|.\|)$ with $\dim E \geq 2$ be a real normed linear space and X_ρ is a $\rho-$complete modular space. Let $f : E \to X_\rho$ be an even mapping satisfying

$$\rho(Df(x, y)) \leq \varepsilon(\|x\|^p + \|y\|^p), \tag{3.27}$$

for all $x, y \in E$ with $x \perp y$, $\varepsilon \geq 0$ and $0 < p < 1$. Then

$$\rho(f(x)) \leq \frac{\varepsilon}{2}\frac{1+\kappa 2^{p-2}}{1-\kappa 2^{p-2}}\|x\|^p \tag{3.28}$$

for all $x \in E$.

Proof. Letting $(x, y) = (0, 0)$ in (3.27), we get $f(0) = 0$. Putting $(x, y) = (0, x)$ in (3.27), we obtain

$$\rho(f(x)) = \rho\left(\frac{1}{2}2f(x)\right) \leq \frac{1}{2}\rho(2f(x)) \leq \frac{\varepsilon}{2}\|x\|^p \tag{3.29}$$

for all $x \in E$. Replacing x by $2x$ in (3.29), we find

$$\rho(f(2x)) \leq \frac{\varepsilon 2^p}{2}\|x\|^p \tag{3.30}$$

for all $x \in E$. Thus

$$\begin{aligned}\rho\left(\frac{1}{2}f(2x)-f(x)\right) &\leq \frac{\kappa}{2^2}\rho(f(2x))+\frac{\kappa}{2}\rho(f(x)) \\ &\leq \frac{\varepsilon}{2}(1+\kappa 2^{p-2})\left\|x\right\|^p \end{aligned} \tag{3.31}$$

for all $x \in E$. Now replacing x by $2x$ in (3.31), we get

$$\begin{aligned}\rho\left(\frac{f(2^2x)}{2^2}-\frac{f(2x)}{2}\right) &\leq \frac{1}{2}\rho\left(\frac{1}{2}f(2^2x)-f(2x)\right) \\ &\leq \frac{\varepsilon}{2}(1+\kappa 2^{p-2})2^{p-1}\left\|x\right\|^p \end{aligned} \tag{3.32}$$

for all $x \in E$. It follows that

$$\begin{aligned}\rho\left(\frac{f(2^2x)}{2^2}-f(x)\right) &= \rho\left(\frac{f(2^2x)}{2^2}-\frac{f(2x)}{2}+\frac{f(2x)}{2}-f(x)\right) \\ &\leq \frac{\kappa}{2}\rho\left(\frac{f(2^2x)}{2^2}-\frac{f(2x)}{2}\right)+\frac{\kappa}{2}\rho\left(\frac{f(2x)}{2}-f(x)\right) \\ &\leq \frac{\varepsilon(1+\kappa 2^{p-2})}{2}\left(1+\frac{\kappa}{2}2^{p-1}\right)\left\|x\right\|^p \end{aligned} \tag{3.33}$$

for all $x \in E$. In general, using induction on a positive integer n, we obtain

$$\begin{aligned}\rho\left(\frac{f(2^nx)}{2^n}-f(x)\right) &\leq \frac{\varepsilon(1+\kappa 2^{p-2})}{2}\sum_{i=0}^{n-1}\left(\frac{\kappa}{2}\right)^i 2^{i(p-1)}\left\|x\right\|^p \\ &= \frac{\varepsilon(1+\kappa 2^{p-2})}{2}\frac{1-(\kappa 2^{p-2})^n}{1-\kappa 2^{p-2}}\left\|x\right\|^p \end{aligned} \tag{3.34}$$

for all $x \in E$. Since $\left\{\frac{f(2^nx)}{2^n}\right\}$ is $\rho-$Cauchy sequence in the $\rho-$complete modular space X_ρ (the proof is similar to that of proposition 3.1). Hence $\left\{\frac{f(2^nx)}{2^n}\right\}$ is $\rho-$convergent in X_ρ, and we well define the mapping $A_e = \lim_{n\to\infty}\frac{f(2^nx)}{2^n}$ from E into X_ρ satisfying

$$\rho(f(x)-A_e(x)) \leq \frac{\varepsilon}{8-\kappa 2^{p-1}}\left\|x\right\|^p, \tag{3.35}$$

for all $x \in E$, since ρ has Fatou property. The proof of $DA_e(x,y)=0$ (with $x \perp y$) is similar to the proof of proposition 3.1. A_e is even orthogonally cubic-additive mapping, by remark 3.4, $A_e(x)=0$ for all $x \in E$, and this completes the proof. Q.E.D.

Theorem 3.6. Let $(E, \|.\|)$ with $\dim E \geq 2$ be a real normed linear space and X_ρ is a $\rho-$complete modular space. Let $f: E \to X_\rho$ be a mapping satisfying

$$\rho(Df(x,y)) \leq \varepsilon(\|x\|^p+\|y\|^p), \tag{3.36}$$

for all $x, y \in E$ with $x \perp y$, $\varepsilon \geq 0$ and $0 < p < 1$. Then there exists a unique orthogonally cubic-additive mapping $AC : E \to X_\rho$ such that

$$\rho(f(x) - AC(x)) \leq \frac{\kappa\varepsilon}{4}\left\{\frac{1+\kappa 2^{p-2}}{1-\kappa 2^{p-2}} + \frac{\kappa}{6}\left[\frac{1}{2-\kappa 2^{p-1}} + \frac{1}{8-\kappa 2^{p-1}}\right]\right\} \tag{3.37}$$

for all $x \in E$.

Proof. Let f^e and f^0 are even and odd part of f such that $f^e(x) = \frac{f(x)+f(-x)}{2}$, $f^o(x) = \frac{f(x)-f(-x)}{2}$. Then we have

$$\begin{aligned}\rho(Df^e(x,y)) &= \rho\left(\frac{Df(x,y)+Df(-x,-y)}{2}\right) \leq \frac{1}{2}\rho(Df(x,y)) + \frac{1}{2}\rho(Df(-x,-y))\\ &\leq \varepsilon(\|x\|^p + \|y\|^p).\end{aligned}$$

By proposition 3.5, we have

$$\rho(f^e(x)) \leq \frac{\varepsilon}{2}\frac{1+\kappa 2^{p-2}}{1-\kappa 2^{p-2}}\|x\|^p \tag{3.38}$$

for all $x \in E$. Similarly we obtain

$$\begin{aligned}\rho(Df^o(x,y)) &= \rho\left(\frac{Df(x,y)-Df(-x,-y)}{2}\right) \leq \frac{1}{2}\rho(Df(x,y)) + \frac{1}{2}\rho(Df(-x,-y))\\ &\leq \varepsilon(\|x\|^p + \|y\|^p).\end{aligned}$$

By theorem 3.3, there exists a unique orthogonally cubic-additive mapping $AC : E \to X_\rho$ such that

$$\rho(f^o(x) - AC(x)) \leq \frac{\kappa\varepsilon}{12}\left\{\frac{1}{2-\kappa 2^{p-1}} + \frac{1}{8-\kappa 2^{p-1}}\right\}\|x\|^p \tag{3.39}$$

for all $x \in E$. It follows from (3.38) and (3.39) that

$$\begin{aligned}\rho(f(x) - AC(x)) &= \rho(f^e(x) + f^o(x) - AC(x)) \leq \frac{\kappa}{2}\rho(f^e(x)) + \frac{\kappa}{2}\rho(f^o(x) - AC(x))\\ &\leq \frac{\kappa\varepsilon}{4}\frac{1+\kappa 2^{p-2}}{1-\kappa 2^{p-2}}\|x\|^p + \frac{\kappa^2\varepsilon}{24}\left\{\frac{1}{2-\kappa 2^{p-1}} + \frac{1}{8-\kappa 2^{p-1}}\right\}\|x\|^p\\ &= \frac{\kappa\varepsilon}{4}\left\{\frac{1+\kappa 2^{p-2}}{1-\kappa 2^{p-2}} + \frac{\kappa}{6}\left[\frac{1}{2-\kappa 2^{p-1}} + \frac{1}{8-\kappa 2^{p-1}}\right]\right\}\end{aligned}$$

for all $x \in E$. Q.E.D.

Corollary 3.6.1. [16] Let $(E, \|.\|)$ with $\dim E \geq 2$ be a real normed linear space and X is a Banach space. Let $f : E \to X$ be mappings satisfying

$$\|Df(x,y)\| \leq \varepsilon(\|x\|^p + \|y\|^p), \tag{3.40}$$

for all $x, y \in E$ with $x \perp y$, $\varepsilon \geq 0$ and $0 < p < 1$. Then there exists a unique orthogonally cubic-additive mapping $AC : E \to X$ such that

$$\|f(x) - AC(x)\| \leq \frac{\varepsilon}{2}\left\{\frac{1+2^{p-1}}{1-2^{p-1}} + \frac{1}{3}\left[\frac{1}{2-2^p} + \frac{1}{8-2^p}\right]\right\} \tag{3.41}$$

for all $x \in E$.

Proof. It is well known that every normed space is a modular space with the modular $\rho(x) = \|x\|$ and $\kappa = 2$. Q.E.D.

A convex function φ defined on the interval $[0, \infty)$, non-decreasing and continuous for $\alpha \geq 0$ and such that $\varphi(0) = 0$, $\varphi(\alpha) > 0$ for $\alpha > 0$, $\varphi(\alpha) \to \infty$ as $\alpha \to \infty$, is called an Orlicz function. The Orlicz function φ satisfies the Δ_2-condition if there exist $k > 0$ such that $\varphi(2\alpha) \leq k\varphi(\alpha)$ for all $\alpha > 0$. Let (Ω, Σ, μ) be a measure space. Let us consider the space L^0_μ consisting of all measurable real-valued (or complex-valued) function on Ω. Define for every $f \in L^0_\mu$ the Orlicz modular $\rho_\varphi(f)$ by the formula

$$\rho_\varphi(f) = \int_\Omega \varphi(|f|)d\mu$$

The associated modular function space with respect to this modular is called an Orlicz space, and will be denoted by $L^\varphi_\mu(\Omega)$ or briefly L^φ. In other words

$$L^\varphi = \left\{ f \in L^0_\mu : \rho_\varphi(\lambda f) \to 0 \,\text{as}\, \lambda \to 0 \right\}$$

or equivalently as

$$L^\varphi = \left\{ f \in L^0_\mu : \rho_\varphi(\lambda f) < \infty \,\text{for some}\, \lambda > 0 \right\}.$$

It is known that the Orlicz space L^φ is ρ_φ-complete. Moreover, $(L^\varphi, \|.\|_{\rho_\varphi})$ is a Banach space, where the Luxemburg norm $\|.\|_{\rho_\varphi}$ is defined as follows

$$\|f\|_{\rho_\varphi} = \inf \left\{ \lambda > 0 : \int_\Omega \varphi \left(\frac{|f|}{\lambda} \right) d\mu \leq 1 \right\}.$$

Moreover, if ℓ is the space of sequences $x = (x_i)_{i=1}^\infty$ with real or complex terms x_i, $\varphi = (\varphi_i)_{i=1}^\infty$, φ_i are Orlicz functions and $\pi_\varphi(x) = \Sigma_{i=1}^\infty \varphi_i(|x_i|)$, we shall write ℓ^φ in place of L^φ. The space ℓ^φ is called the generalized Orlicz sequence space. The motivation for the study of modular spaces (and Orlicz spaces) and many examples are detailed in [14, 15]. Now, we give a following examples.

Example 3.7. Let $(E, \|.\|)$ with $\dim E \geq 2$ be a real normed linear space, φ is an Orlicz function and satisfy the Δ_2-condition with $0 < \kappa \leq 2$. Let $f : E \to L^\varphi$ be a mapping satisfying

$$\int_\Omega \varphi(|Df(x, y)|)d\mu \leq \varepsilon(\|x\|^p + \|y\|^p), \tag{3.42}$$

for all $x, y \in E$ with $x \perp y$, $\varepsilon \geq 0$ and $0 \leq p < 1$. Then there exists a unique orthogonally cubic-additive mapping $AC : E \to L^\varphi$ such that

$$\int_\Omega \varphi(|f(x) - AC(x)|)d\mu \leq \frac{\kappa\varepsilon}{4} \left\{ \frac{1 + \kappa 2^{p-2}}{1 - \kappa 2^{p-2}} + \frac{\kappa}{6} \left[\frac{1}{2 - \kappa 2^{p-1}} + \frac{1}{8 - \kappa 2^{p-1}} \right] \right\}$$

for all $x \in E$.

Example 3.8. Let $(E, \|.\|)$ with $\dim E \geq 2$ be a real normed linear space, $\widehat{\varphi} = (\varphi_i)$ be sequence of Orlicz functions satisfying the Δ_2-condition with $0 < \kappa \leq 2$ and let $(\ell^{\widehat{\varphi}}, \pi_{\widehat{\varphi}})$ be generalized Orlicz sequence space associated to $\widehat{\varphi} = (\varphi_i)$. Let $f : E \to \ell^{\widehat{\varphi}}$ be a mapping satisfying

$$\pi_{\widehat{\varphi}}(Df(x, y)) \leq \varepsilon(\|x\|^p + \|y\|^p),$$

for all $x, y \in E$ with $x \perp y$, $\varepsilon \geq 0$ and $0 \leq p < 1$. Then there exists a unique orthogonally cubic-additive mapping $AC : E \to \ell^{\widehat{\varphi}}$ such that

$$\pi_{\widehat{\varphi}}(f(x) - AC(x)) \leq \frac{\kappa\varepsilon}{4}\left\{\frac{1+\kappa 2^{p-2}}{1-\kappa 2^{p-2}} + \frac{\kappa}{6}\left[\frac{1}{2-\kappa 2^{p-1}} + \frac{1}{8-\kappa 2^{p-1}}\right]\right\}$$

for all $x \in E$.

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