Homotopy classification theorem in algebraic geometry

Dedicated to Professor Yoshie Katurada on her sixtieth birthday

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Introduction. Let X be *a finite CW complex.* We denote by $K(X)$ the Grothendieck group of the classes of complex vector bundles over X. We further write Z, B_U for the integers with the discrete topology, the classifying space of the infinite unitary group respectively. Then the K-theoretic version of the homotopy classification theorem is given by the statement of the existence of a natural bijection:

$$K(X) \cong [X, B_U \times Z]$$

where $[X, B_U \times Z]$ denotes the set of homotopy classes of maps of X into $B_U \times Z$.

The objective of this paper is to present an algebro-geometric analogue to the above-mentioned theorem. We consider *a non-singular reduced affine k-scheme for an algebraically closed field k*, instead of a finite CW complex. Let X be a k-scheme of this kind. We write $K(X)$ for the Grothendieck group of the classes of coherent O_X-Modules. Let $G_{n,n}$ be the Grassmannian k-scheme of n-planes in affine $2n$-space A_k^{2n} where n ranges over the positive integers. Then there are natural closed immersions: $G_{n,n} \longrightarrow G_{l,l}$ for $l>n$. We denote by B_k the direct limit of $G_{n,n}$ in the category of geometrical k-spaces. Consider morphisms f, g: $X \longrightarrow B_k \times Z$. We define $f \sim g$ if and only if f is connected with g by a finite chain of rational homotopies. A class by the equivalence relation $\sim$ will be called *a rational homotopy class.* We write $[X, B_k \times Z]_{\mathrm{rat}}$ for the set of rational homotopy classes of k-morphisms: $X \longrightarrow B_k \times Z$. With these notations we have

Main Theorem. *There is a natural bijection*

$$K(X) \cong [X, B_k \times Z]_{\mathrm{rat}}.$$

Let X be an irreducible algebraic prescheme over an algebraically closed field k. Let Υ_n^m be the universal scheme vector bundle over $G_{n,m}$, i.e. the Grassmannian k-scheme of n-planes in affine $(m-n)$-space. We denote by p the natural projection: $\Upsilon_n^m \longrightarrow G_{n,m}$. We now state two theorems below which are used for the proof of the Main Theorem, because of their own interest.

THEOREM A. *Let E be a quasi-coherent O_X-Module which is a direct summand of a free O_X-Module of finite rank and m a sufficiently large integer. Then we can find a morphism G: $X \longrightarrow G_{n,m}$ such that there is a pull-back diagram*:

$$\begin{array}{ccc} V(\check{E}) & \longrightarrow & \Upsilon_n^m \\ \Big\downarrow & & \Big\downarrow p \\ X & \xrightarrow{\;G\;} & G_{n,m} \end{array}$$

in other words

$$V(\check{E}) = X \times_{G_{n,m}} \Upsilon_n^m .$$

THEOREM B. *Suppose two morphisms having the pull-back diagram in Theorem A. Then they are rationally homotopic in $G_{n,m'}$ for sufficiently large m'.*

1. Grassmannian schemes and universal scheme vector bundles. First we define the Grassmannian k-schemes for an arbitrary field k. Let Λ be the set of subsets λ of $\{1, \cdots, m\}$ with card.$\lambda = n$ where m and n are fixed positive integers. Let U_λ be $m!/(n! \times (m-n)!)$ copies of affine $n(m-n)$-space $\mathbf{A}_k^{m(n-n)}$ which are indexed by Λ. For convenience we introduce variables $X_{ij}^{(\lambda)}$ where i (resp. j) runs through $1, \cdots, n$ (resp. $1, \cdots, m-n$). We write R_λ for the polynomial ring $k[X_{ij}^{(\lambda)}]$ in $n(m-n)$ variables $X_{ij}^{(\lambda)}$ and consider U_λ as Spec R_λ. We wish to glue together U_λ $(\lambda \in \Lambda)$ and construct a k-scheme. Let us explain how U_λ and U_μ are glued for λ, $\mu \in \Lambda$. For that it suffices to take the example of $\lambda = \{1, \cdots, n\}$ and $\mu = \{1, \cdots, n-1, n+1\}$. Let:

$$M = \begin{pmatrix} & & X_{11}^{(\lambda)} \\ & 1_{n-1} & \vdots \\ 0 & \cdots & X_{n1}^{(\lambda)} \end{pmatrix}$$

$$M' = \begin{pmatrix} & & X_{11}^{(\mu)} \\ & 1_{n-1} & \vdots \\ 0 & \cdots & X_{n1}^{(\mu)} \end{pmatrix}$$

where 1_{n-1} denotes the unit matrix of order $n-1$. We note that the coefficients of M^{-1} (resp. M'^{-1}) belong to the ring $(R_\lambda)_{\det M}$ (resp. $(R_\mu)_{\det M'}$). Between the variables $X_{ij}^{(\lambda)}$, $X_{ij}^{(\mu)}$ we introduce the relation:

$$X^{(\mu)} = M^{-1} X^{(\lambda)}$$

where

$$X^{(\lambda)} = \left(\begin{array}{cccc} & & X_{11}{}^{(\lambda)} & \cdots & X_{1,m-n}{}^{(\lambda)} \\ & 1_n & & \cdots\cdots\cdots & \\ & & X_{n1}{}^{(\lambda)} & \cdots & X_{n,m-n}{}^{(\lambda)} \end{array} \right)$$

$$X^{(\mu)} = \left(\begin{array}{ccccccc} & & X_{11}{}^{(\mu)} & 0 & X_{12}{}^{(\mu)} & \cdots & X_{1,m-n}{}^{(\mu)} \\ & 1_{n-1} & & & & & \\ 0 \cdots & 0 & X_{n1}{}^{(\mu)} & 1 & X_{n2}{}^{(\mu)} & \cdots & X_{n,m-n}{}^{(\mu)} \end{array} \right).$$

From this it results that $X_{i'j'}{}^{(\mu)}$ $(i'=1,\cdots,n;\ j'=1,\cdots,m-n)$ are rational functions of $X_{ij}{}^{(\lambda)}$. We denote these rational functions by $r_{i'j'}$. Then $r_{i'j'}\in(R_\lambda)_{\det M}$. We clearly have $M'=M^{-1}$. Hence $\det M'=(\det M)^{-1}$. Therefore if we substitute $r_{i'j'}$ for $X_{i'j'}{}^{(\mu)}$ in $(P\in(R_\mu)_{\det M'})$, we have an element Q of $(R_\lambda)_{\det M}$. We define $T_{\lambda\mu}$: $(R_\mu)_{\det M'}\longrightarrow(R_\lambda)_{\det M}$ by setting $T_{\lambda\mu}(P)=Q$. This is an isomorphism and induces a scheme isomorphism ${}^aT_{\lambda\mu}$: Spec $(R_\lambda)_{\det M}\longrightarrow$ Spec $(R_\mu)_{\det M'}$. These isomorphisms satisfy the cocycle condition. Hence we can define a prescheme which is locally isomorphic to $\mathbf{A}_k{}^{n(m-n)}$. We denote it by $G_{n,m-n}$. Let i_λ be the natural inclusion: $k\subsetneq R_\lambda$. Then i_λ induces a morphism ${}^ai_\lambda$: $U_\lambda\longrightarrow$ Spec k. We can glue ${}^ai_\lambda$ into a morphism i: $G_{n,m-n}\longrightarrow$ Spec k. i is separated as easily seen. Hence $G_{n,m-n}$ can be considered as a k-scheme. We call this *the Grassmannian k-scheme of n-planes in affine space $\mathbf{A}_k{}^m$*.

Next we construct the universal scheme vector bundle over $G_{n,m-n}$. Let $\tilde{R}_\lambda$ be the polynomial rings which are obtained by adjunction of n new variables $X_h{}^{(\lambda)}$ $(h=1,\cdots,n)$ to R_λ. Then for each $\lambda\in\Lambda$ there is a natural injection: $R_\lambda\subsetneq\tilde{R}_\lambda$. It induces a k-morphism: Spec $\tilde{R}_\lambda\longrightarrow$ Spec R_λ. We denote it by p_λ. Between the variables let us introduce the relation:

$$(X_1{}^{(\mu)},\cdots,X_n{}^{(\mu)})=(X_1{}^{(\lambda)},\cdots,X_n{}^{(\lambda)})M.$$

Then $X_{h'}{}^{(\mu)}$ $(h'=1,\cdots,n)$ turn out to be rational functions of $X_h{}^{(\lambda)}$ which we denote by $r_{h'}$. Since $r_{h'}\in(\tilde{R}_\lambda)$, we can assign to each $\tilde{P}\in(\tilde{R}_\mu)_{\det M'}$ an element $\tilde{Q}\in(\tilde{R}_\lambda)_{\det M}$ which is obtained by the substitution of $r_{i'j'}$, $r_{h'}$ for $X_{i'j'}{}^{(\mu)}$, $X_{h'}{}^{(\mu)}$. The isomorphism $\tilde{T}_{\lambda\mu}$: $\tilde{P}\longmapsto\tilde{Q}$ induces an isomorphism ${}^a\tilde{T}_{\lambda\mu}$: Spec $(\tilde{R}_\lambda)_{\det M}\longrightarrow$ Spec $(\tilde{R}_\mu)_{\det M'}$. Since ${}^a\tilde{T}_{\lambda\mu}$ satisfy the cocycle condition, we get a prescheme Υ_n^m by gluing Spec $\tilde{R}_\lambda$ $(\lambda\in\Lambda)$. It is actually a k-scheme. Besides the k-morphisms p_λ $(\lambda\in\Lambda)$ can be glued into a k-morphism p: $\Upsilon_n^m\longrightarrow G_{n,m-n}$. This can be easily seen from the commutative diagrams:

$$\begin{array}{ccc} \operatorname{Spec} (\widetilde{R}_\lambda)_{\det M} & \xrightarrow{{}^a\widetilde{T}_{\lambda\mu}} & \operatorname{Spec} (\widetilde{R}_\mu)_{\det M'} \\ \downarrow {\scriptstyle p_\lambda} & & \downarrow {\scriptstyle p_\mu} \\ \operatorname{Spec} (R_\lambda)_{\det M} & \xrightarrow{{}^aT_{\lambda\mu}} & \operatorname{Spec} (R_\mu)_{\det M'} \end{array}$$

We call the $G_{n,m-n}$-prescheme Υ_n^m the *universal scheme vector bundle* because we have the following proposition.

Let E be the sheaf of germs of section of Υ_n^m. Then E can be viewed as a Module over the structure sheaf of $G_{n,m-n}$.

PROPOSITION 1. *E is a quasi-coherent Module and the $G_{n,m-n}$-scheme Υ_n^m is isomorphic to the scheme vector bundle $V(\check{E})$ associated to E.*

PROOF. Let us consider $\widetilde{R}_\lambda$ as a R_λ-algebra by the natural injection: $R_\lambda \subsetneq \widetilde{R}_\lambda$. Then there are natural isomorphisms:

$$(1) \qquad \Gamma(U_\lambda, E) \cong \operatorname{Hom}_{\mathrm{Alg}}(\widetilde{R}_\lambda, R_\lambda) = \operatorname{Hom}_{\mathrm{Mod}}(R_\lambda^n, R_\lambda)$$

where R_λ^n denotes the direct sum of n copies of R_λ. For $f \in R_\lambda$ we also have a natural isomorphism: $\Gamma((U_\lambda)_f, E) \cong \operatorname{Hom}_{\mathrm{Mod}}((R_\lambda)_f^n, (R_\lambda)_f)$. Hence we see $\Gamma(U_\lambda, E)_f = \Gamma((U_\lambda)_f, E)$. This shows that $E|U_\lambda$ is the sheaf associated to the R-module $\Gamma(U_\lambda, E)$. Hence E is quasi-coherent. From (1) we have

$$\Gamma(U_\lambda, \check{E}) = \operatorname{Hom}_{\mathrm{Mod}}(\Gamma(U_\lambda, E), R_\lambda) = R_\lambda^n.$$

Therefore we obtain a natural isomorphism of the symmetric algebra of $\Gamma(U_\lambda, E)$ onto the polynomial ring $\widetilde{R}_\lambda$. This gives rise to a natural isomorphism $\tilde{\imath}$: Spec $\widetilde{R}_\lambda \longrightarrow$ Spec $\Gamma(U_\lambda, S(E))$, where $S(E)$ is the symmetric Algebra of Module E. Let i'_λ be the restriction of $\tilde{\imath}_\lambda$ on Spec $(\widetilde{R}_\lambda)_{\det M}$. Then $i'^{-1}_\lambda i'_\mu$ is equal to ${}^a\widetilde{T}_{\mu\lambda}$. Hence we see that the isomorphism $\tilde{\imath}_\lambda$ $(\lambda \in \Lambda)$ can be glued into a global isomorphism of Υ_n^m onto $V(E)$. This completes the proof.

PROPOSITION 2. $G_{n,m-n}$ is isomorphic to $G_{m-n,n}$.

PROOF. For $\lambda \in \Lambda$ we set $\bar{\lambda} = \{1, \cdots, m\} - \lambda$. Then $G_{m-n,n}$ is covered by the affine open sets $U_{\bar{\lambda}}$ which can be identified with Spec $R_{\bar{\lambda}}$ where $R_{\bar{\lambda}} = k[X_{ji}^{(\bar{\lambda})}]$ $(i = 1, \cdots, n; j = 1, \cdots, m-n)$. We first construct an isomorphism: Spec $R_\lambda \longrightarrow$ Spec $R_{\bar{\lambda}}$ for each $\lambda \in \Lambda$ and then show that they can be glued together. We again take the example of $\lambda = \{1, \cdots, n\}$ and $\mu = \{1, \cdots, n-1, n+1\}$ for the convenience of writing. Let us denote by Y the $(m-n)$-by-m matrix with unknowns Y_{jk} as the (j, k)-element respectively where $j = 1, \cdots, m-n$ and $k = 1, \cdots, m$. Consider the matrix equation with the unknown Y:

$$X^{(\lambda)t}Y=0\,.$$

It has a unique solution $Y^{(\lambda)}$ if we impose the condition:

$$Y_{j,n-j'}=\delta_{jj'} \qquad (j,j'=1,\cdots,m-n)$$

on Y. Actually we have $Y_{ji}=-X_{ij}{}^{(\lambda)}$. Let $P\in R_{\bar\lambda}$. Substitutig $-X_{ij}{}^{(\lambda)}(=Y_{ji})$ for $X_{ji}{}^{(\bar\lambda)}$ in P, we get a polynomial in R_λ. This gives rise to an isomorphism of $R_{\bar\lambda}$ onto R_λ. It induces an isomorphism: Spec $R_\lambda \longrightarrow$ Spec $R_{\bar\lambda}$ which will be denoted by $\bar\iota_\lambda$. We write

$$\bar M = \begin{pmatrix} -X_{n1}{}^{(\lambda)} & 0 \cdots & 0 \\ \vdots & & 1_{n-1} \\ -X_{n,m-n}{}^{(\lambda)} & & \end{pmatrix}$$

Consider now the equation $X^{(\mu)t}Y=0$ and solve it on the condition:

$$Y_{ln}=1\,,\quad Y_{jn}=0\,,\quad Y_{j',n+j}=\delta_{j'j}$$
$$j=2,\cdots,m-n\,,\quad j'=1,\cdots,m-n\,.$$

We denote the solution by $Y^{(\mu)}$. As for μ, we have a natural isomorphism $\bar\iota_\mu$: Spec $R_\mu \longrightarrow$ Spec $R_{\bar\mu}$. Since the solution is unique, $Y^{(\mu)}=\bar M^{-1}Y^{(\lambda)}$ up to $T_{\lambda\mu}$. Hence $\bar\iota_\lambda=\bar\iota_\mu$ in $U_\lambda\cap U_\mu$. We can therefore glue these isomorphisms and obtain a natural isomorphism

$$\bar\iota:\quad G_{n,m-n}\longrightarrow G_{m-n,n}\,.$$

This completes the proof.

2. Construction of the classifying morphism. Let k be an arbitrary field. Let X be a k-prescheme. Then *a k-valued point* of X is a k-morphism f: Spec $k\longrightarrow X$. Spec k consists of a single point. We write x for the image of Spec k by f. f gives rise to a k-homomorphism of $O_{X,x}$ into k. We denote it by the same letter f. Let U be an affine open set in X which contains x. Let r_U be the restriction: $\Gamma(U,O_X)\longrightarrow O_{X,x}$. The kernel of $f\circ r_U$: $\Gamma(U,O)\longrightarrow k$ is denoted by I. We use the letter A for $\Gamma(U,O_X)$ from now on. Then we have a k-vector space isomorphism

$$A\cong k\oplus I\,.$$

Now let E be a quasi-coherent O_X-Module. Suppose there is an exact sequence:

$$(2) \qquad O\longrightarrow E\longrightarrow O_X^m\longrightarrow O_X^m/E\longrightarrow O$$

which splits locally, provided that m is some positive integer. We write

$\Gamma(U, E)$ as M and $\Gamma(U, O_X^m/E)$ as N. Let U be so small that the exact sequence (2) splits on U. Then we have an A-module isomorphism

$$g_U: \quad M \oplus N \cong A^m .$$

g_U induces an isomorphism: $I \cdot M \oplus I \cdot N \cong I^m$. We therefore have an isomorphism:

$$M/I \cdot M \oplus N/I \cdot N \cong (A/I)^m .$$

By restricting the coefficient ring to k, we get a k-vector space isomorphism. which gives rise to an injection

$$j: \quad M/I \cdot M \subsetneq k^m .$$

We denote by M_U the subspace $j(M/I \cdot M)$ of k^m.

LEMMA 1. *For sufficiently small U, M_U does not depend on the choice of U, but is determined uniquely by the k-valued point f.*

PROOF. Let U' be an affine open set such that $U' \subset U$ and $x \in U'$. Let r (resp. $\tilde{r}$) be the restriction homomorphism of A (resp. M) on $A' = \Gamma(U', O_X)$ (resp. $M' = \Gamma(U', E)$). Then the diagrams:

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[dr, "r_U"'] & A' \arrow[d, "r_{U'}"] \\ & O_{X,x} \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "g_U"] \arrow[d, "\tilde{r}"] & A^m \arrow[d, "r^m"] \\ M' \arrow[r, "g_{U'}"] & A'^m \end{tikzcd}$$

are commutative where $r^m: A^m \longrightarrow A'^m$ is defined by

$$r^m(a_1, \cdots, a_m) = \big(r(a_1), \cdots, r(a_m)\big).$$

The first diagram implies that r sends I in $I' = \operatorname{Ker} f \circ r_{U'}$. Hence we obtain the commutative diagram:

$$\begin{tikzcd} M_U \arrow[r] \arrow[d] & k^m \arrow[d, "i_d"] \\ M^{U'} \arrow[r] & k^m \end{tikzcd}$$

from the second diagram where the horizontal arrows are the inclusions. We therefore have $M_U \subsetneq M_{U'}$. This inclusion can be replaced by the equality if U is sufficiently small. This completes the proof.

Let us denote by $X(k)$ the set of k-valued points of k-prescheme X. By the injection: $f \longmapsto x$, we can identify $X(k)$ with a subset of X. Hence we can induce a topology on $X(k)$ from that of X. From now on we consider $X(k)$ as a topological space equipped with this induced topology.

Let q_M be the projection of $M\oplus N$ on the first factor M. We define $q\in\mathrm{End}_A A^m$ by $q=g_U\circ q_M\circ g_U^{-1}$. With respect to the canonical base of A^m there corresponds a matrix α to q. We set

$$\alpha=\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix}$$

Let f_U be the natural projection of A on A/I, i.e., $f\circ r_U$. Then the vectors $(f_U(a_{11}),\cdots,f_U(a_{m1})),\cdots,(f_U(a_{1m}),\cdots,f_U(a_{mm}))$ span the vector subspace M_U in k^m. Suppose U is sufficiently small. Then this subspace is uniquely determined by f, which is guaranteed by Lemma 1. We use the symbol $G_X(f)$ instead of M_U. $\dim G_X(f)$ equals the maximum order of square submatrix β of α such that $f_U(\det\beta)\neq 0$, or equivalently $\det\beta\notin I$. We write b for $\det\beta$. For fixed β the set of $g\in U\cap X(k)$ with $g_U(b)\neq 0$ is just Spec $A_b\cap X(k)$. Hence the set of $g\in X(k)$ such that

$$\dim G_X(f)\leqq\dim G_X(g)$$

contains an open neighborhood of x in $X(k)$. Similarly the set of $g'\in X(k)$ such that

$$m-\dim G_X(f)\leqq\dim N/\mathrm{Ker}\, g'_U\cdot N$$

contains an open neighborhood of x in $X(k)$. Since.

$$\dim M/\mathrm{Ker}\, g_U\cdot M+\dim N/\mathrm{Ker}\, g_U\cdot N=m$$

holds at any point $g\in U\cap X(k)$, we can conclude from the above facts that $\dim G_X(f)$ is locally constant in $X(k)$.

Suppose now X is an irreducible algebraic k-prescheme with k algebraically closed. Then $X(k)$ coincides with the set of closed points of X. It is a connected and dense subset of X. Hence $\dim G_X(f)$ is a constant on X. We denote it by n. Then G_X: $f\longmapsto G_X(f)$ can be viewed as a map of $X(k)$ into $G_{n,m-n}$ since there corresponds a closed point in $G_{n,m-n}$ to each n-plane in k^m naturally. Let β be an n-by-n submatrix of α with $b=\det\beta\notin R(A)$ where $R(A)$ is the radical of A. Then we have Spec $A=\bigcup\mathrm{Spec} A_b$ where the union ranges over the submatrices of the above nature; for $\bigcup\mathrm{Spec}\, A_b$ is an open subset containing all the closed points of Spec A. For brevity's sake we assume $\beta=(a_{ii'})_{i,i'=1,\cdots,n}$. We define $c_{ij}\in A_b$ $(i=1,\cdots,n;j=1,\cdots,m-n)$ by

$$\beta^{-1}\begin{pmatrix} a_{11} & \cdots\cdots & a_{1m} \\ \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots} \\ a_{n1} & \cdots\cdots & a_{nm} \end{pmatrix} = \begin{pmatrix} & c_{11} & \cdots\cdots & c_{1,m-n} \\ 1_n & \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \\ & c_{n1} & \cdots\cdots & c_{n,m-n} \end{pmatrix}$$

Recall that $G_{n,m-n}$ is covered by the affine open sets U_λ $(\lambda\in\Lambda)$ each of which is identifiable with affine space Spec $k[X_{ij}^{(\lambda)}]_{i=1,\cdots,n;j=1,\cdots,m-n}$. For $Q\in k[X_{ij}^{(\lambda)}]$ we define

$$H(Q)=Q(c_{ij})\,.$$

Then H is a homomorphism of $k[X_{ij}^{(\lambda)}]$ into A_b. H induces a morphism ${}^aH$: Spec $A_b \longrightarrow$ Spec $k[X_{ij}^{(\lambda)}]=U_\lambda$. We want to show that we can glue ${}^aH$ and get a morphism of X into $G_{n,m-n}$. For that it suffices to prove

$$G_X(f)={}^aH(f) \tag{3}$$

for any k-valued point $f\in U_b=$Spec A_b. We write f_{U_b} as f_b. Then we have

$$\begin{aligned}{}^aH(f)=H^{-1}(\mathrm{Ker}\ f_b)&=\left\{Q\in k[X_{ij}^{(\lambda)}]\,|\,Q(c_{ij})\in \mathrm{Ker}\ f_b\right\}\\ &=\left\{Q\in k[X_{ij}^{(\lambda)}]\,|\,Q\big(f_b(c_{ij})\big)=0\right\}=G_X(f)\,.\end{aligned}$$

Hence we get (3).

The morphism obtained in this way is nothing but the extension of G_X to X (by continuity). We use the same symbol G_X for it. We say that G_X is the *classifying morphism* of E (corresponding to the exact sequence (2)).

3. Construction of the isomorphism in Theorem A. Let X be an irreducible algebraic prescheme over an algebraically closed field k and E a quasi-coherent O_X-Module. Suppose there is an exact sequence (2) which splits locally. Then we can construct the classifying morphism G_X: $X\longrightarrow G_{n,m-n}$ for E as shown in § 2. Let $\mathcal{E}$ be the sheaf of germs of $G_{n,m-n}$-sections of Υ^m_n. $\mathcal{E}$ actually is a Module over $G_{n,m-n}$. The inverse image of Module $\mathcal{E}$ by G_X is defined by

$$G_X^*(\mathcal{E})=O_X\times_{G_X^{-1}(O_{G_{n,m-n}})}G_X^{-1}(\mathcal{E})\,.$$

We first construct an isomorphism:

$$G_X^*(\mathcal{E})\cong E\,. \tag{4}$$

We follow the notations in the preceding sections, provided that the symbols relative to U_b are replaced by the corresponding ones relative to U with a prime. For example, we write U', A', M' for U_b, A_b, M_b and so on. In addition, a_{ij} in this section, strictly speaking, should be written as $r(a_{ij})$ with the restriction homomorphism r: $A \longrightarrow A'$. The isomorphism (4) is a collection of isomorphisms: $\Gamma(U', G_X^*(\mathcal{E})) \longrightarrow M'$. We construct the isomorphism (4) on U' in the following, assuming $\beta=(a_{ii'})_{i,i'=1,\cdots,n}$ for the convenience of notations. The rest are treated in exactly the same manner. Let $\sigma' \in M'$. Then σ' is a linear combination of the line vectors α_i $(i=1,\cdots,n)$ with coefficients in A', where $\alpha_i=(a_{il},\cdots,a_{im})$. Let $\beta_i=(0,\cdots,0,1,0,\cdots,0, c_{il},\cdots,c_{i,m-n})$ where $i=1,\cdots,n$. Then α_i with $1 \leqq i \leqq n$ can be written as linear combinations of $\beta_{i'}$. We further have

LEMMA 2. *For $k=n+1,\cdots,m$ also, α_k are linear combinations of β_i.*

PROOF. Let M_0 be the submodule of M' generated by $\alpha_1,\cdots,\alpha_n$. For any k-valued point f we have

$$f^m(\alpha_k)=f(a_{kl})f^m(\beta_1)+\cdots+f(a_{kn})f^m(\beta_n)$$

where f^m is defined as r^m in §2. Hence

$$\alpha_k-a_{kl}\beta_l-\cdots-a_{kn}\beta_n \in R(A')^m .$$

Since M' is a direct summand of A'^m, $R(A')^m \cap M'$ equals $R(A')M'$. Hence we have

$$M_0 \oplus R(A')M' = M' .$$

We therefore obtain $M'=M_0$ from the lemma of Nakayama. This completes the proof.

Let us now define R_λ-homomorphisms $e_h^{(\lambda)}$: $\tilde{R}_\lambda \longrightarrow R_\lambda$ by

$$e_h^{(\lambda)}(X_k^{(\lambda)})=\delta_{hk} .$$

For each $h=1,\cdots,n$ $e_h^{(\lambda)}$ corresponds to an element of $\Gamma(U_\lambda, \Upsilon_n^m)$, denoted by $e_h^{(\lambda)}$ again, by means of the isomorphism (1). Then $e_1^{(\lambda)},\cdots,e_n^{(\lambda)}$ constitute an R_λ-base for $\Gamma(U, \Upsilon_n^m)$. It may be called the "*canonical*" base. We take $\lambda=\{1,\cdots,n\}$, which is actually decided by the way of choosing β. Then $G_X(U') \subset U_\lambda$. We write $\tilde{e}_h^{(\lambda)}$ for $e_h^{(\lambda)} \circ G_X|U'$ where $G_X|U'$ is the restriction of G_X on U'. Then $\tilde{e}_h^{(\lambda)} \in \Gamma(U', G_X^{-1}(\mathcal{E}))$. Using Lemma 2, we can find $d_1,\cdots,d_n \in A'$ such that

$$\sigma' = d_1\beta_1+\cdots+d_n\beta_n .$$

We define

$$j_{U'}(\sigma')=d_1 \otimes \tilde{e}_1^{(\lambda)}+\cdots+d_n \otimes \tilde{e}_n^{(\lambda)} .$$

Then we have $j_{U'}(\sigma') \in \Gamma(U', G_X^*(\mathcal{E}))$.

Let us go back to U and define $j_U(\sigma)$ for $\sigma \in M$ by gluing $j_{U'}(r'(\sigma))$ where

r' is the restriction homomorphism $M \longrightarrow M'$. To do so, take $\hat{\beta}=(a_{ii'})_{i=1,\cdots,n;i'=1,\cdots,n-1,n+1}$, since the rest are treated in the same way. Suppose $\hat{b}=\det\hat{\beta}\notin R(A)$. We write $\mu=\{1,\cdots,n-1,n+1\}$ as before. Let r_λ(resp. r_μ) be the restriction homomorphism: $\Gamma(U_\lambda,\gamma_n^m)$ (resp. $\Gamma(U_\mu,\gamma_n^m)) \longrightarrow \Gamma(U_\lambda \cap U_\mu, \gamma_n^m)$. Let $\varepsilon_h^{(\lambda)}$ (resp, $\varepsilon_k^{(\mu)}$) be the image of $e_h^{(\lambda)}$ (resp. $e_k^{(\mu)}$) by r_λ (resp. r_μ). Then we have

(5) $$(\varepsilon_1^{(\lambda)},\cdots,\varepsilon_n^{(\lambda)})=(\varepsilon_1^{(\mu)},\cdots,\varepsilon_n^{(\mu)})^t M .$$

We write

$$N=\begin{pmatrix} & & c_{1,n+1} \\ & 1_{n-1} & \vdots \\ 0\cdots & & c_{n,n+1}\end{pmatrix}.$$

Let $U''=U'\cap\operatorname{Spec}A_{\hat{b}}$ and $\tilde{\varepsilon}_h^{(\lambda)}=\varepsilon_h^{(\lambda)}\circ G_X|U''$, $\tilde{\varepsilon}_k^{(\mu)}=\varepsilon_k^{(\mu)}\circ G_X|U''$. Then it follows from (5) that

(6) $$(\tilde{\varepsilon}_1^{(\lambda)},\cdots,\tilde{\varepsilon}_n^{(\lambda)})=(\tilde{\varepsilon}_1^{(\mu)},\cdots,\tilde{\varepsilon}_n^{(\mu)})^t N .$$

Let $\sigma\in M$. Let σ' be the restriction of σ on U' and σ'' that on Spec $A_{\hat{b}}$. We denote by $\hat{\beta}_i$ the line vectors of the matrix $\hat{\beta}^{-1}(a_{ij})_{i=1,\cdots,n;j=1,\cdots,m}$. Define $\hat{d}_i$ by

$$\sigma''=\hat{d}_1\hat{\beta}_1+\cdots+\hat{d}_n\hat{\beta}_n .$$

Then up to the restriction homomorphism, we have

(7) $$(\hat{d}_1,\cdots,\hat{d}_n)=(d_1,\cdots,d_n)N .$$

From (6), (7) we obtain

(8) $$d_1\otimes\tilde{\varepsilon}_1^{(\lambda)}+\cdots+d_n\otimes\tilde{\varepsilon}_n^{(\lambda)}=\hat{d}_1\otimes\tilde{\varepsilon}_1^{(\mu)}+\cdots+\hat{d}_n\otimes\tilde{\varepsilon}_n^{(\mu)} .$$

Note that $\tilde{\varepsilon}_h^{(\iota)}=\bar{\bar{e}}_h^{(\iota)}|U''$ for $\iota=\lambda,\mu$. Then it follows from (8) that we can get an element of $\Gamma(U, G_X^*(\mathcal{E}))$ by gluing the pieces together. We write it as $j_U(\sigma)$. Then

(9) $$j_U:\ M \longrightarrow \Gamma(U, G_X^*(\mathcal{E}))$$

is an A-module isomorphism.

By the same reasoning as above we have the following lemma.

LEMMA 3. *j_U does not depend on the choice of a splitting.*

LEMMA 4. *These isomorphisms j_U satisfy the condition of compatibility with the restriction homomorphisms.*

PROOF. Let U' be any open subset of U. We write A', M' for $\Gamma(U', O_X)$, $\Gamma(U' E)$ respectively. A local splitting of (2) over U gives rise to isomorphisms

$$g_U: \quad M \oplus N \cong A^m,$$
$$g_{U'}: \quad M' \oplus N' \cong A'^m.$$

We can define b', β_i' for $g_{U'}$ in the same way as b, β_i for g_U respectively. Let r be the restriction homomorphism: $\Gamma(U_b, O_X) \longrightarrow \Gamma(U'_{b'}, O_X)$. Then $\beta_i' = r^m(\beta_i)$. Hence $d_h' = r(d_h)$ where d_h' are defined for $g_{U'}$ as d_h for g_U. We can therefore conclude that $j_{U'}(\sigma')$ is the image by the restriction homomorphism of $j_U(\sigma)$ where σ' is that of σ.

Thus $j: U \longrightarrow j_U$ is the required sheaf isomorphism.

In conclusion we can state the

THEOREM. *Let X be an irreducible algebraic prescheme over an algebraically closed field k. Let E be a quasi-coherent O_X-Module having an exact sequence* (2) *which splits locally. Then there are a morphism G_X: $X \longrightarrow G_{n,m-n}$ and an isomorphism: $G_X^*(\mathcal{E}) \cong E$ for some positive integer n where $\mathcal{E}$ is the sheaf of germs of $G_{n,m-n}$-sections* of γ_n^m.

(Hence E turns out to be locally free.)

Now let us prove Theorem A. It is the same in essence as the theorem stated just above. There is only need of giving attention to some facts. First we note that

$$\widecheck{G_X^*(\mathcal{E})} = G_X^*(\check{\mathcal{E}}),$$

since $\mathcal{E}$ is locally free and of finite rank. The isomorphism: $G_X^*(\mathcal{E}) \cong E$ induces the one: $V(\check{\mathcal{E}}) \cong V(\widecheck{G_X^*(\mathcal{E})})$. Secondly we have

$$V(G_X^*(\check{\mathcal{E}})) = V(\check{\mathcal{E}}) \times_{G_{n,m}} X.$$

Hence we can obtain Theorem A.

4. Rational homotopy. We make the definition of rational homotopy in the first half of this section and construct the rational homotopy in Theorem B in the second one.

Let X, Y be k-preschemes where k is an arbitrary field. Let $k[T]$ be the polynomial algebra over k in one variable T and t a k-valued point of the k-scheme Spec $k[T]$. Then t induces an algebra homomorphism t^*: $k[T] \longrightarrow k$. On the other hand k is included in $\Gamma(X, O_X)$ in the natural way. The product of t^* with this inclusion is a homomorphism: $k[T] \longrightarrow \Gamma(X, O_X)$. This homomorphism induces a morphism $\tilde{t}$: $X \longrightarrow$ Spec $k[T]$ in the natural way. Now we write

$$Z = X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\ k[T].$$

Then there is a unique morphism t': $X \rightarrow Z$ such that the diagram

$$\begin{tikzcd}
 & \operatorname{Spec} k[T] \arrow[dr] & \\
X \arrow[ur, "\tilde{t}"] \arrow[r, "t'"] \arrow[dr, "id"'] & Z \arrow[u] \arrow[r] \arrow[d] & \operatorname{Spec} k \\
 & X \arrow[ur] &
\end{tikzcd}$$

is commutative. Let t_1 (resp. t_2 be the k-valued point of Spec $k[T]$ which corresponds to the natural projection:

$$k[T] \longrightarrow k[T]/(T) \cong k$$
$$(\text{resp. } k[T] \longrightarrow k[T]/(1-T) \cong k).$$

As stated above, these k-valued points give rise to morphisms t_1, t: $X \longrightarrow Z$ respectively. We can now define the rational homotopy as follows. Let us consider morphisms f_1, f_2: $X \longrightarrow Y$. Then *a rational homotopy from* f_1 *to* f_2 is by definition a morphism h: $Z \longrightarrow$Y such that $f_i = h \circ t'_i$ for $i=1,2$. We also say that f_1 *is rationally homotopic to* f_2.

Let us turn to the problem of constructing the rational homotopy in Theorem B. Let E be a quasi-coherent O_X-Module. E is supposed to be a direct summand of a free O_X-Module of finite rank. Hence for some positive integer m there are a quasi-coherent O_X-Module E_1 and an isomorphism

$$g_1\colon \quad E \oplus E_1 \cong O_X^m . \tag{10}$$

Let us consider another decomposition

$$g_2\colon \quad E \oplus E_2 \cong O_X^m \tag{11}$$

where E_2 is an O_X-Module. Suppose X is an irreducible algebraic prescheme with k algebraically closed. From the decompositions (10), (11) we obtain the corresponding classifying morphisms G_1, G_2: $X \dashrightarrow G_{n,m-n}$ for some integer n. Let q_X be the projection of $Z = X \times \operatorname{Spec} k[T]$ on the first factor X. We set $E_Z = q_X{}^*(E)$. Let U be an affine open set in X. We write A, M, W for $\Gamma(U, O_X)$, $\Gamma(U, E)$, $q_X{}^{-1}(U)$ respectively. Then W can be identified with Spec $(A \otimes k[T])$ and, moreover, $q_X | W$ corresponds to the inclusion: $A \subset A \otimes k[T]$ given by $a \longmapsto a \otimes 1$ for $a \in A$. Hence there is a natural isomorphism:

$$q_X^*(E) \,|\, W\Big(= (q_X \,|\, W)^*(E)\Big) \cong (k[T] \otimes M)^{\sim}$$

where $(k[T]\otimes M)^{\sim}$ is the O_W-Module associated to $A\otimes k[T]$-module $k[T]\otimes M$. We write the module $k[T]\otimes M$ by M_Z below. The decompositions (10), (11) give rise to those of the A-module A^m:

$$M\oplus N_i \cong A^m \qquad (i=1,2) \tag{12}$$

respectively. We further have the $A\otimes k[T]$-module decompositions

$$M_Z\oplus k[T]\otimes N_i \cong (A\otimes k[T])^m \qquad (i=1,2) \tag{13}$$

from (12). (13) gives the inclusions: $M_Z \subsetneq (A\otimes k[T])^m$. We denote them by $g_1(W)$, $g_2(W)$ respectively. We define $(1-T)g_1(W)$, $(T)g_2(W)$ by $(1-T)g_1(W)\sigma=(1\otimes(1-T))g_1(W)(\sigma)$, $(T)g_2(W)(\sigma)=(1\otimes(T))g_2(W)(\sigma)$ for $\sigma\in M_Z$. We set:

$$g_W=(1-T)g_1(W)\oplus(T)g_2(W). \tag{14}$$

g_W induces a morphism ${}^a g_W$: $E_Z|W \longrightarrow O_Z^{2m}$. Since the affine ope nsets W cover Z, we finally obtain a morphism g^*: $E_Z\longrightarrow O_Z^{2m}$. The image of g_W is a direct summand of $(A\otimes k[T])^{2m}$, as easily seen. Hence we can construct a classifying morphism G_Z: $Z\longrightarrow G_{n,2m-n}$ by means of g^*.

Let U_λ be an affine open set defined in § 1. Hence U_λ is.

We denote by R, R' polynomial rings $k[X_{ij}]_{i=1,\cdots,n;j=1,\cdots,m-n}$, $k[Y_{ik}]_{i=1,\cdots,n;k=1,\cdots,2m-n}$ respectively. Consider the epimorphisms s_1, s_2: $R'\longrightarrow R$ that are defined by

$$s_1(Y_{ik})=X_{ik} \quad \text{if} \quad 1\leqq k\leqq m-n, \quad \text{otherwise} \quad s_1(Y_{ik})=0$$

$$s_2(Y_{ik})=X_{i,k-m} \quad \text{if} \quad m+1\leqq k\leqq 2m-n, \quad \text{otherwise} \quad s_2(Y_{ik})=0.$$

Let λ be a subset with card. $\lambda=n$ of $\{1,\cdots,m\}$. Add m to each element of λ. Then we have a subset of $\{1,\cdots,2m\}$. We write it as $\lambda+m$. The meaning of $s_1^{(\lambda)}$, $s_2^{(\lambda)}$ is evident. These epimorphisms induce morphisms: $U_\lambda\longrightarrow U'_\lambda$, $U_\lambda\longrightarrow U'_{\lambda+m}$ respectively where U_λ, U'_λ are the affine open sets in $G_{n,m-n}$, $G_{n,2m-n}$ defined in § 1 respectively. Gluing these morphisms, we obtain two closed immersions $G_{n,m-n}\subsetneq G_{n,2m-n}$. We denote them by s_1, s_2 again.

LEMMA 5. *s_1, s_2 are rationally homotopic to each other.*

PROOF. Beginning with the epimorphism s: $R'\longrightarrow R\otimes k[T]$ that is defined by $s(Y_{ik})=X_{ik}\otimes T$ if $1\leqq k\leqq m-n$, $s(Y_{ik})=X_{i,k-m}\otimes(1-T)$ if $m+1\leqq k\leqq 2m-n$, we can construct a morphism: $X\times_{\mathrm{Spec}\,k}\mathrm{Spec}\ k[T]\longrightarrow Y$ exactly as above. This morphism is the required rational homotopy.

LEMMA 6. *The diagram*:

$$
\begin{tikzcd}
& & G_{n,m-n} \arrow[dr, "s_1"] & \\
X \arrow[urr, "G_1"] \arrow[drr, "G_2"'] \arrow[r, shift left, "t'_1"] \arrow[r, shift right, "t'_2"'] & X\times \operatorname{Spec} k[T] \arrow[rr, "G_Z"] & & G_{n,2m-n}\,. \\
& & G_{n,m-n} \arrow[ur, "s_2"'] &
\end{tikzcd}
$$

has the commutative upper and lower triangles.

PROOF. Let f be a k-valued point of X. Then $t'_i \circ f$ are k-valued points of Z where $i=1, 2$. We write x, z_i for the closed points corresponding to f, $t'_i \circ f$ respectively. We follow the notations in the earlier part of this section. Suppose $x \in U$. Then $z_i \in W$. To $t'_i \circ f$ there correspond k-homomorphisms: $A \otimes k[T] \longrightarrow k$, which are denoted by $\tilde{f}_i$ respectively. Take arbitrary $\sigma \in M$ and $P \in k[T]$. Then from (14) we obtain

$$\tilde{f}_1^{2m}\Big(g_W(\sigma \otimes P(T))\Big) = \Big(f_U^m(g_1(W)(\sigma))(P(0), \cdots, 0\Big)$$

$$\tilde{f}_2^{2m}\Big(g_W(\sigma \otimes P(T))\Big) = \Big(0, \cdots, f_U^m(g_2(W)(\sigma))P(1)\Big)$$

where f_U is $f \circ r_U$ in § 2. We therefore have

$$G_Z \circ t'_i(x) = s_i \circ G_i(x) \tag{15}$$

with x ranging over the closed points of X. Since the set of closed points is dense, (15) holds for any point x of X. This completes the proof.

It is seen from the above two lemmas that $s_1 \circ G_1$ and $s_2 \circ G_2$ are rationally homotopic. Hence we get Theorem B.

5. $\boldsymbol{B}_k$ and $\boldsymbol{B}_k^s$. In this section we construct the direct limit of Grassmannian k-schemes $G_{n,n}$ $(n=1, 2, \cdots)$ in the category of k-schemes and then define the classifying k-space B_k. We shall further prove a proposition.

Consider the polynomial ring $k[X_1, \cdots, X_n]$ in n variables. We use the notation A_n for it. Substituting the zero for X_{n+1}, we get a homomorphism $i_{n,n+1}\colon A_{n+1} \longrightarrow A_n$. It induces a closed immersion $j_{n,n+1}\colon \operatorname{Spec} A_n \longrightarrow \operatorname{Spec} A_{n+1}$. We further put $i_{n,m} = i_{m-1,m} \cdot \cdots \cdot i_{n,n-1}$, $j_{n,m} = j_{m-1,m} \cdot \cdots \cdot j_{n,n-1}$ for integers m with $n<m$. Thus we get an inverse system $(A_n, i_{n,m})$ of rings and a direct system $(\operatorname{Spec} A_n, j_{n,m})$ of affine schemes. The direct limit of the latter in the category of schemes is equal to Spec lim inv. A_n.

Consider the Grassmannian k-scheme $G_{n,n}$. Take arbitrary $\lambda \in \Lambda$ and set $U_1 = U_\lambda$. We write Λ_n for Λ from now on. We define an element $\mu \in \Lambda_m$ by $\mu = \lambda \cup \{2n+1, \cdots, 2m-1\}$. Instead of U_μ we write U_m. Then $\Gamma(U_m, G_{m,m})$ can be viewed as $A_m{}^2$. Hence the system U_m with the natural immersions can be identified with a cofinal subsystem of (Spec $A_n, j_{n,m}$). We denote the direct limit by $V_{n,\lambda}$. Then we have the natural closed immersions j_m: $U_m \longrightarrow V_{n,\lambda}$. We want to glue $V_{n,\lambda}$ where n ranges over the positive integers and λ over Λ_n. Let $\lambda, \mu \in \Lambda_n$. We put $d = \det M$ and $d' = \det M'$ (see §1 for M and M'). If we begin with U_μ, then we have another direct system: $U_1' \longrightarrow U_2' \longrightarrow U_m' \longrightarrow \cdots$. It is readily checked that $U_m \cap U_m' = (U_m)_d = U(_m')_{d'}$ and that lim dir. $(U_m)_d = (V_{n,\lambda})_d$, lim dir. $(U_m')_{d'} = (V_{n,\mu})_{d'}$. The morphism of the direct systems:

$$\begin{array}{ccccc} (U_1)_d & \longrightarrow & (U_2)_d & \longrightarrow & \cdots \\ \Big\downarrow \| & & \Big\downarrow \| & & \\ (U_1')_{d'} & \longrightarrow & (U_2')_{d'} & \longrightarrow & \cdots \end{array}$$

gives rise to an isomorphism: $(V_{n,\lambda})_d \longrightarrow (V_{n,\mu})_{d'}$. These isomorphisms satisfy the condition of compatibility. Thus we can obtain a k-prescheme V_n. In addition V_n is contained in V_m as an open sub-prescheme for $m > n$. We define $B^s{}_k = \bigcup_{n=1}^{\infty} V_n$. Then $B^s{}_k$ can be viewed as a k-scheme. We can further consider $G_{n,n}$ as a sub-scheme of $B^s{}_k$ in the natural way, so that we have a sequence of sub-schemes: $\cdots \subset G_{n,n} \subset G_{n-1,n-1} \subset \cdots \subset B^s{}_k$. We define B_k to be the union of $G_{n,n}$ $(n=1, 2, \cdots)$. Then there is a natural injection π: $B_k \longrightarrow B^s{}_k$. Using π, we introduce the structure of a geometrical k-space into B_k. In other words the structure sheaf of B_k is defined to be the inverse image by π of that of $B^s{}_k$. π turns out to be a morphism.

PROPOSITION 3. *B_k is isomorphic to the direct limit of $G_{n,n}$ in the category of geometrical k-spaces.*

PROOF. We denote by B the direct limit of $G_{n,n}$. Then there is a morphism $\bar{j}$: $B \longrightarrow B_k$. Let $x \in B_k$. Then $x \in G_{n,n}$ for some n. To x there corresponds a prime ideal I_x in A_{n^2}. We write I for the inverse image of I_x by the natural morphism: lim inv. $A_n \longrightarrow A_{n^2}$. Then the proposition follows from the fact: $O_{B_k,x}$ is isomorphic to (lim inv. $A_n)_I$ = lim inv. $O_{G_{m,m},x}$.

PROPOSITION 4. *Let X be a quasi-compact reduced k-prescheme and G_X a k-morphism: $X \longrightarrow B_k$. Then G_X decomposes into $X \longrightarrow G_{n,n} \subsetneq B_k$ for some n.*

PROOF. It suffices to prove $G_X(X(k)) \subset G_{n,n}$ for some n. (See *I*, 5.2.2, [1]). Suppose the contrary. Then there are closed points x_n $(n=1,2,\cdots)$ of X such that $x'_n \in G_{n+1,n+1} - G_{n,n}$, where $x'_n = G_X(x_n)$. We set $S = \{x'_n | n = 1,2,\cdots\}$. Since x'_n are closed in $B^s{}_k$, they are so in B_k too. Let S' be any subset of S. Then $S' \cap G_{m,m}$ are closed in $G_{m,m}$ for any m. Hence S is a closed discrete subset in a quasi-compact set $G_X(X)$. Therefore it is finite. This contradiction proves the proposition.

6. Proof of the main theorem. Let X be an irreducible noetherian scheme over an algebraically closed field k. A coherent O_X-Module will be called projective if it is a direct summand of a free O_X-Module of finite rank. Hence a projective O_X-Module is locally free (see §3). Let $KP(X)$ be the Grothendieck group of classes of projective O_X-Modules. Then each $\xi \in KP(X)$ can be written in the form: $[E] - l$ where $[E]$ is the class of a projective O_X-Module E and l a positive integer. For E there is a coherent O_X-Module F such that $E \oplus F \cong O_X{}^m$ for some positive integer m. Hence we can construct a classifying morphism G_X: $X \longrightarrow G_{n,m-n}$ by the use of this direct sum decomposition, where n is the rank of E. We restrict ourselves to the case where $2n = m$ from now on. We view G_X as a morphism: $X \longrightarrow G_{n,n} \times (l-n)$, and further as one: $X \longrightarrow B_k \times (l-\mathrm{n})$. We define $\varphi(\xi)$ to be the rational homotopy class $\in [X, B_k \times Z]_{\mathrm{rat}}$ containing G_X.

LEMMA 7. *$\varphi(\xi)$ is uniquely determined by ξ.*

PROOF. First we replace E, F, m by $E \oplus O_X{}^k$, $F \oplus O_X{}^k$, $m+2k$ respectively. Hence l must be replaced by $l+k$. In this case we easily see that G_X does not change as a morphism: $X \longrightarrow B_k$. Consequently $\varphi(\xi)$ also does so.

Secondly suppose we have $E \oplus F' \cong O_X{}^m$ also for some coherent F'. Using this decomposition, we construct a classifying morphism G_X'. Then G_X' is rationally homotopic to G_X by means of Theorem *B*. Hence $\varphi(\xi)$ does not change.

Finally let $[E'] - l'$ be any other form of expressing ξ. Then $E \oplus O_X{}^k = E' \oplus O_X{}^{k'}$ for some positive integers k, k'. Suppose $E' \oplus F' = O_X{}^{m'}$ for some coherent F' and some positive integer m'. Let G_X' be the classifying morphism obtained from this decomposition. By the above first and second steps we see that G_X' is rationally homotopic to G_X. Hence $\varphi(\xi)$ does not change, even though we start by $\xi = [E'] - l'$. This completes the proof of Lemma 7.

LEMMA 8. *φ: $KP(X) \longrightarrow [X, B_k \times Z]_{\mathrm{rat}}$ is surjective.*

PROOF. Let $[f]$ be the rational homotopy class $\in [X, B_k \times Z]_{\mathrm{rat}}$ containing a morphism f: $X \longrightarrow B_k \times l$ where $l \in Z$. Then $f(X) \subset G_{n,n}$ for some positive integer n.

φ sends $f^*(E)-(l-n)$ to $[f]$. This completes the proof of the subjectivity of φ.

From now on suppose further X is non-singular quasi-projective. Then we have the following lemma.

LEMMA 9. *Let* Y *be a k-scheme of the same kind as X. Let f, g be morphisms*: $X \longrightarrow Y$ *which are rationally homotopic. Then* f', g': $K(Y) \longrightarrow K(X)$ *coincide.*

PROOF. Let h: $X \times \operatorname{Spec} k[T] \longrightarrow Y$ be a rational homotopy from f to g. Then we have $f=h\circ t_1'$ and $g=h\circ t_2'$ (for t_1', t_2' see §4). Let p be the projection: X Spec $k[T] \longrightarrow X$. Then $p\circ t_1'$, $p\circ t_2'$ are the identity. On the other hand p': $K(X) \longrightarrow K(X \times \operatorname{Spec} k[T])$ is also an isomorphism. For this fact see [2]. Hence $(t_1')=(t_2')$. We therefore have

$$f' = (t_1')' \circ h' = (t_2')' \circ h' = g' .$$

This completes the proof.

Let $[f] \in [X, B_k \times Z]_{\mathrm{rat}}$ be an arbitrary class with $f(X) \subset G_{n,n} \times l$ for some n, l. Let E be the universal bundle over $G_{n,n}$. Then it is easily seen from the above lemma that

$$f'\left(\gamma_X(E)\right) - (l-n)$$

is uniquely determined by the class $[f]$. We write $\psi([f])$ for it. Then ψ can be viewed as a map of $[X, B_k \times Z]$ into $K(X)$.

Let ι be the natural homomorphism: $KP(X) \longrightarrow K(X)$, i.e. the one sending $[E]$ to $\gamma_X(E)$ for a projective O_X-Module E. Then we have the commutative triangle:

$$\begin{tikzcd} KP(X) \arrow[r, "\varphi"] \arrow[d, "\iota"] & {[X, B_k \times Z]_{\mathrm{rat}}} \arrow[dl, "\psi"] \\ K(X) & \end{tikzcd}$$

as will be easily checked. Hence we have obtained the main theorem.

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