Finsler spaces as distributions on Riemannian manifolds

Dedicated to Professor Yoshie Katusrada on her Sixtieth Birthday

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§1. **Introduction.** In the previous paper [6][1], on making use of the methods in the classical theories and the method due to M. Kurita [3], [4] we have studied a Finsler space V_n with the following fundamental function: $F=\sqrt{g_{ij}y^i y^j}+\alpha_i y^i$. Especially we have shown that the connection of E. Cartan can give rise to the affine connections on the p-manifold N of V_n in the theory of M. Kurita [4] and that the space V_n and its geometry are realizable in the N.

The principal purpose of the present paper is to show that the above two facts hold good also in a general Finsler space with the fundamental metric function of class C^4. As a consequence we have that this leads to the theory of A. Deicke [1], [2] and suggests a new method to study Finsler spaces.

§2. **Contact structure.** Let M be an n-dimensional paracompact differentiable manifold and x^i be local coordinates in a neighborhood U of any point $x \in M$. In the tangent space T_x and the dual tangent one T_x^* at x, we take a natural frame (e_i) and its dual one (e^i), and denote by y^i and p_i the components of any vectors y and p in T_x, T_x^* respectively. Further we consider the tangent bundle TM and the dual tangent one T^*M over M. We assume that M is endowed with a metric function $F(x, y)$ satisfying the following conditions;

(2.1)
(1) $F(x, y)$ is of class C^4 and is positively homogeneous of degree 1 in the y^i.
(2) $F(x, y)$ is positive if not all y^i vanish simultaneously.
(3) $g_{ij}(x, y)Z^i Z^j$ is positive definite,

where $g_{ij}(x, y)=\dfrac{1}{2}\dfrac{\partial^2 F^2}{\partial x^i \partial x^j}$.

Now we consider a mapping $\varphi: TM \rightarrow T^*M$ defined by $(x, y) \rightarrow (x, p)$ with

$$p_i=\frac{\partial F}{\partial y^i} \qquad (i=1, 2, \cdots, n). \tag{2.2}$$

1) Numbers in brackets refer to the references at the end of the paper.

Then if we put $N=\varphi(TM)$, N is a $(2n-1)$-dimensional submanifold of T^*M, which is called a p-manifold of M. Since the mapping φ is globally defined, N may be also cansidered as the figuratrix bundle over M.

If we denote the Hamiltonian function of M by $H(x, p)$ and put $g^{ij}(x, p)=\frac{1}{2}\frac{\partial^2 H^2}{\partial p_i \partial p_j}$, the following relation holds:

$$g^{ij}(x, p) g_{jk}(x, y)=\delta^i_k, \quad \text{provided} \quad p_i=g_{ij}y^j.$$

A local equation of N is given by

$$G=H(x, p)-1=0 \quad \text{or} \quad g^{ij}(x, p)p_ip_j-1=0. \tag{2.3}$$

On N we have from (2.3)

$$\frac{\partial G}{\partial p_i}=l^i\equiv\frac{y^i}{F(x, y)}, \qquad \frac{\partial G}{\partial x^i}=-\gamma^0_{0i}, \tag{2.4}$$

where $\gamma^i_{jk}=\gamma_{jhk}g^{hi}$ are the Christoffel symbols and $\gamma^0_{0i}=\gamma^h_{ji}l^jp_h$.

Next, we consider a 1-form on N

$$\omega=p_idx^i, \tag{2.5}$$

which defines a contact structure on N except for the point (x, p) corresponding to (x, y) such that $F(x, y)=0$ [3]. We can assume $\partial G/\partial p_n\neq 0$, namely $l^n\neq 0$, without loss of generality. Then we can take $2n-1$ linearly independent 1-forms ω, θ^λ and ρ_λ on N which are defined by (2.5) and

$$\theta^\lambda=dx^\lambda-\frac{\partial G/\partial p_\lambda}{\partial G/\partial p_n}dx^n, \quad \rho_\lambda=-dp_\lambda-\frac{\partial G/\partial x^\lambda}{\partial G/\partial p_n}dx^n \quad (\lambda=1, 2, \cdots, n-1).$$

By virtue of (2.4), the above forms are rewritten in

$$\theta^\lambda=dx^\lambda-\frac{l^\lambda}{l^n}dx^n, \quad \rho_\lambda=-dp_\lambda+\frac{1}{l^n}\gamma^0_{0\lambda}dx^n. \tag{2.6}$$

In this case, it is easily verified that

$$d\omega=\theta^\lambda\wedge\rho_\lambda. \tag{2.7}$$

For, in particular, dp_n, we have from (2.3) and (2.4)

$$dp_n=-\frac{1}{\partial G/\partial p_n}\left(\frac{\partial G}{\partial p_\lambda}dp_\lambda+\frac{\partial G}{\partial x^i}dx^i\right)=-\frac{1}{l^n}(-\gamma^0_{0i}dx^i+l^\lambda dp_\lambda). \tag{2.8}$$

A set $(\theta^\lambda, \omega, \rho_\lambda)$ of the above-mentioned 1-forms is called an adapted coframe on N.

§3. Adapted orthogonal coframe. We introduce the following quantities:

$$(3.1)\qquad h^i_j=\delta^i_j-l^ip_j\,,\quad \mathfrak{d}^\lambda_i=\delta^\lambda_i-\delta^n_i\frac{l^\lambda}{l^n}\,,\quad \mathfrak{d}^{\lambda i}=\mathfrak{d}^\lambda_j g^{ij}\qquad (\lambda=1,2,\cdots,n-1).$$

Then it turns out that for h^i_j, $\mathfrak{d}^\lambda_i$ and $\mathfrak{d}^{\lambda i}$

$$(3.2)\qquad h^i_jl^j=h^i_jp_i=0\,,\qquad \mathfrak{d}^\lambda_il^i=\mathfrak{d}^{\lambda i}p_i=0\,.$$

On making use of (3.1) and (3.2), we solve (2.5), (2.6) and (2.8) for dx^i and dp_i in termsof θ^λ, ω, ρ_λ and get

$$(3.3)\qquad dx^i=l^i\omega+h^i_\lambda\theta^\lambda\,,\quad dp_i=\gamma^0_{0i}\omega-\mathfrak{d}^\lambda_i\rho_\lambda-\left(\gamma^0_{0i}p_\lambda-\delta^n_i\frac{\gamma^0_{0\lambda}}{l^n}\right)\theta^\lambda\,,$$

from which it follows that

$$(3.4)\qquad dl^i=-\gamma^i_{00}\omega-\mathfrak{d}^{\lambda i}\rho_\lambda+\left\{\gamma^i_{00}p_\lambda-\gamma^i_{0\lambda}-\left(g^{ij}\gamma^0_{j\lambda}-g^{in}\frac{\gamma^0_{0\lambda}}{l^n}\right)\right\}\theta^\lambda\,,$$

where $\gamma^i_{00}=\gamma^i_{jk}l^jl^k$, $\gamma^i_{0\lambda}=\gamma^i_{j\lambda}l^j$ and $\gamma^0_{j\lambda}=\gamma^i_{j\lambda}p_i$.

Now, we consider a matrix (ζ^a_i) of rank n such that

$$(3.5)\qquad g_{ij}=\sum_{a=1}^n\zeta^a_i\zeta^a_j\,,\quad \zeta^n_i=p_i\,,\quad \zeta^\alpha_il^i=0\qquad (\alpha=1,2,\cdots,n-1).$$

Then if we denote the inverse of the matrix (ζ^a_i) by (ζ^i_a), it follows from (3.5) that

$$(3.6)\qquad g^{ij}=\sum_a\zeta^i_a\zeta^j_a\,,\quad \zeta^i_n=l^i\,,\quad \zeta^i_\alpha p_i=0\,,\quad \zeta^i_a=g^{ij}\zeta^a_j\,.$$

Hereafter Latin indices run from 1 to n, Greek indices from 1 to $n-1$. If we put

$$(3.7)\qquad h_{ij}=F\frac{\partial^2F}{\partial y^i\partial y^j}\equiv g_{ij}-p_ip_j\,,$$

$$(3.8)\qquad t^\lambda_\alpha=\zeta^j_\alpha\mathfrak{d}^\lambda_j\,,\qquad S^\alpha_\lambda=\zeta^\alpha_\lambda\,,$$

it is verified that $h_{ij}g^{jk}=h^k_i$ and the matrices (t^λ_α) and (s^α_λ) are inversive mutually, i.e. $\det(\zeta^\alpha_\lambda)\neq 0$. Since we have from (3.5) and (3.7)

$$(3.7)'\qquad \sum_\alpha\zeta^\alpha_\lambda\zeta^\alpha_\mu=g_{\lambda\mu}-p_\lambda p_\mu=h_{\lambda\mu}\,,$$

it follows from (3.8) that

$$(3.9)\qquad \det(h_{\lambda\mu})\neq 0\,,\qquad h_{\lambda\mu}t^\lambda_\alpha t^\mu_\beta=\delta_{\alpha\beta}\,.$$

Further we put

$$(3.10)\qquad r_{\alpha\beta}=\left(\zeta^j_\alpha\gamma^0_{j\lambda}-\zeta^n_\alpha\frac{\gamma^0_{0\lambda}}{l^n}\right)t^\lambda_\beta+A_{ijk}\gamma^k_{00}\zeta^i_\alpha\zeta^j_\beta\,,$$

where $$A_{ijk}=FC_{ijk}=\frac{1}{2}F\frac{\partial g_{ij}}{\partial y^k}.$$

By means of (3.8) and (3.10) we transform the coframe $(\theta^\lambda, \omega, \rho_\lambda)$ to a new adapted coframe $(\omega^\alpha, \omega^n, \omega_\alpha)$:

$$\omega^\alpha = s^\alpha_\lambda\theta^\lambda, \quad \omega^n=\omega, \quad \omega_\alpha = t^\lambda_\alpha\rho_\lambda + r_{\alpha\beta}s^\beta_\lambda\theta^\lambda. \tag{3.11}$$

As is shown later, this coframe is in fact a so-called "adapted orthogonal coframe" and further the following holds good:

$$Dl^i=-\sum_\alpha \zeta^i_\alpha\omega_\alpha \quad \text{or} \quad \omega_\alpha=-g_{ij}\zeta^j_\alpha Dl^i=-\zeta^\alpha_i Dl^i, \tag{3.12}$$

where Dl^i is the covariant differential of the unit vector l^i with respect to the connection of E. Cartan.

First, we deduce (3.12). Since we have

$$Dl^i = dl^i + \Gamma^{*i}_{jk}l^j dx^k = dl^i + (\gamma^i_{0k} - A^i_{jk}\gamma^j_{00})dx^k, \tag{3.12'}$$

on use of (3.1)$\sim$(3.5), (3.8) and (3.10) we get

$$\begin{aligned}-g_{ij}\zeta^j_\alpha Dl^i &= t^\lambda_\alpha\rho_\lambda + \left(A_{\lambda jk}\gamma^k_{00}+\gamma^0_{j\lambda}-\delta^n_j\frac{\gamma^0_{0\lambda}}{l^n}\right)\zeta^j_\alpha\theta^\lambda \\ &= t^\lambda_\alpha\rho_\lambda + r_{\alpha\beta}s^\beta_\lambda\theta^\lambda = \omega_\alpha\end{aligned}$$

Next, we shall show that the coframe $(\omega^\alpha, \omega^n, \omega_\alpha)$ is an adapted orthogonal coframe. Since we have by virtue of (3.8) and (3.10)

$$r_{\alpha\beta}s^\alpha_\lambda s^\beta_\mu = \gamma^0_{\lambda\mu} + A_{\lambda\mu i}\gamma^i_{00}$$

which is symmetric in λ and μ, it follows from (2.7) and (3.11) that

$$\omega^\alpha\wedge\omega_\alpha = \theta^\lambda\wedge\rho_\lambda = d\omega = d\omega^n. \tag{3.13}$$

We can put

$$d\omega^\alpha = \frac{1}{2}k^\alpha_{\beta\gamma}\omega^\beta\wedge\omega^\gamma + l^\alpha_\beta\omega^n\wedge\omega^\beta + m^{\alpha\beta}_\gamma\omega_\beta\wedge\omega^\gamma + h^{\alpha\beta}\omega_\beta\wedge\omega^n, \tag{3.14}$$

$$d\omega_\alpha = \frac{1}{2}u^{\beta\gamma}_\alpha\omega_\beta\wedge\omega_\gamma + v^\beta_\alpha\omega^n\wedge\omega_\beta + w_{\alpha\beta}\omega^n\wedge\omega^\beta + y^\beta_{\alpha\gamma}\omega^\gamma\wedge\omega_\beta + \frac{1}{2}z_{\alpha\beta\gamma}\omega^\beta\wedge\omega^\gamma, \tag{3.15}$$

where $k^\alpha_{\beta\gamma}=-k^\alpha_{\gamma\beta}$, $u^{\beta\gamma}_\alpha=-u^{\gamma\beta}_\alpha$ and $z_{\alpha\beta\gamma}=-z_{\alpha\gamma\beta}$. In this case, the following relations hold:

$$\begin{aligned} &k^\alpha_{\beta\gamma}=y^\alpha_{\beta\gamma}-y^\alpha_{\gamma\beta}, \quad l^\alpha_\beta=-v^\alpha_\beta, \quad m^{\alpha\beta}_\gamma-m^{\beta\alpha}_\gamma=u^{\alpha\beta}_\gamma, \\ &h^{\alpha\beta}=h^{\beta\alpha}, \quad w_{\alpha\beta}=w_{\beta\alpha}, \quad z_{\alpha\beta\gamma}+z_{\beta\gamma\alpha}+z_{\gamma\alpha\beta}=0.\end{aligned} \tag{3.16}$$

In view of (2.5), (2.6), (3.5) and (3.11), we have

(3.17) $$\omega^\alpha = \zeta^\alpha_i dx^i, \qquad \omega^a = \zeta^a_i dx^i \qquad (a=1, 2, \cdots, n),$$

from which it follows that

(3.18) $$dx^i = \zeta^i_a \omega^a = \zeta^i_\alpha \omega^\alpha + l^i \omega^n ..$$

From (3.12), (3.12)′ and (3.18) we have

(3.19) $$dl^i = -\gamma^i_{00}\omega^n - (\gamma^i_{0k} - A^i_{jk}\gamma^j_{00})\zeta^k_\alpha \omega^\alpha - \sum_\alpha \zeta^i_\alpha \omega_\alpha .$$

Since $\zeta^a_i(x, y)$ are homogeneous of degree 0 in y^i, the differentials $d\zeta^a_i$ are expressible in

$$d\zeta^a_i = \zeta^a_{i,j}dx^j + \zeta^a_{i||j}dl^j,$$

where $\zeta^a_{i,j} = \partial\zeta^a_i/\partial x^j$ and $\zeta^a_{i||j} = F\partial\zeta^a_i/\partial y^j$. In the sequel the symbols "$_{,j}$" and "$_{||j}$" are used in such ways.

On use of (3.18) and (3.19) we obtain

(3.20) $$d\zeta^\alpha_i = (\zeta^\alpha_{i,j}l^j - \zeta^\alpha_{i||j}\gamma^j_{00})\omega^n + \left\{\zeta^\alpha_{i,k} - \zeta^\alpha_{i||j}(\gamma^j_{0k} - A^i_{rk}\gamma^r_{00})\right\}\zeta^k_\beta \omega^\beta - \sum_\beta \zeta^\alpha_{i||j}\zeta^j_\beta \omega_\beta .$$

Because of the homogeneity of ζ^α_i, from (3.5) we have

(3.21) $$\zeta^\alpha_{i,j}l^i = 0, \qquad l^i\zeta^\alpha_{i||j} = -\zeta^\alpha_j .$$

If we benote by $\zeta^\alpha_{i|j}$ the first covariant derivatives of ζ^α_i with respect to the connection of E. Cartan, it follows that

(3.22) $$\zeta^\alpha_{i|j}l^j = \zeta^\alpha_{i,j}l^j - \zeta^\alpha_{i||j}\gamma^j_{00} - \zeta^\alpha_j(\gamma^j_{0i} - A^j_{ki}\gamma^k_{00}) .$$

Calculating $d\omega^\alpha$ on use of (3.18), (3.20) and applying (3.21), (3.22) to the resulting expression, by the comparison with the corresponding coefficients in (3.14) we obtain

(3.23) $$h^{\alpha\beta} = \delta^{\alpha\beta},$$

(3.24) $$l^\alpha_\beta = \zeta^\alpha_{i|j}l^j\zeta^i_\beta, \quad m^{\alpha\beta}_\gamma = -\zeta^\alpha_{i||j}\zeta^j_\beta\zeta^i_\gamma, \quad k^\alpha_{\beta\gamma} = (\zeta^\alpha_{i|k} - \zeta^\alpha_{k|i})\zeta^k_\beta\zeta^i_\gamma .$$

In particular,

(3.25) $$l^\alpha_\beta = g_{ik}\zeta^k_{\alpha|j}l^j g^{ih}\zeta^\beta_h = \zeta^i_{\alpha|j}l^j\zeta^\beta_i = -\zeta^\beta_{i|j}l^j\zeta^i_\alpha = -l^\beta_\alpha .$$

Thus the coframe $(\omega^\alpha, \omega^n, \omega_\alpha)$ satisfies the conditions (3.13), (3.23) and (3.25), which characterize an adapted orthogonal coframe.

Noting (3.12) and

$$\Gamma^{*i}{}_{jk||h}l^j = A^i_{kh|j}l^j, \qquad \Gamma^{*i}{}_{jk||h}l^jl^k = 0,$$

on use of (3.12), (3.18) and (3.20) we calculate $d\omega_\alpha$ and in view of (3.15) get

$$
(3.26)\quad \begin{cases} u_\alpha^{\beta\gamma} = \zeta_{i||j}^\alpha(\zeta_\beta^i\zeta_\gamma^j - \zeta_\gamma^i\zeta_\beta^j), \quad v_\alpha^\beta = l_\beta^\alpha = -l_\alpha^\beta, \\ y_{\alpha\gamma}^\beta = -(\zeta_{j|k}^\beta - A_{jk|r}^i l^r\zeta_i^\beta)\zeta_\alpha^j\zeta_\gamma^k, \\ w_{\alpha\beta} = -R_{jhk}^{\ i} l^j l^k\zeta_i^\alpha\zeta_\beta^h = R_{ijkh} l^i l^h \zeta_\alpha^j\zeta_\beta^k, \\ z_{\alpha\beta\gamma} = R_{jkh}^{\ i} l^j\zeta_i^\alpha\zeta_\beta^k\zeta_\gamma^h = R_{jikh} l^j\zeta_\alpha^i\zeta_\beta^k\zeta_\gamma^h, \end{cases}
$$

where $R_{jhk}^{\ i} = g^{im}R_{jmhk}$, being components of the curvature tensor of E. Cartan.

In this case, it is verified that (3.24) and (3.26) satisfy (3.16).

§4. Equations of structure and connections. We know that with respect to an adapted orthogonal coframe $\pi = (\omega^\alpha, \omega^n, \omega_\alpha)$ the equations of structure can be uniquely represent as follows [4]:

$$
(4.1)\quad \begin{cases} d\omega^n = \omega^\alpha \wedge \omega_\alpha, \quad d\omega^\alpha = \omega^\beta\wedge\omega_\beta^\alpha + \omega_\beta\wedge\mu^{\beta\alpha} + \omega_\alpha\wedge\omega^n, \\ d\omega_\alpha = \omega^\beta\wedge\nu_{\beta\alpha} - \omega_\beta\wedge\omega_\alpha^\beta - w_{\alpha\beta}\omega^\beta\wedge\omega^n + \Phi_\alpha, \end{cases}
$$

where

$$
(4.2)\quad \omega_\beta^\alpha = \frac{1}{2}\sum_\gamma (k_{\beta\gamma}^\alpha + k_{\beta\alpha}^\gamma + k_{\gamma\alpha}^\beta)\omega^\gamma - l_\beta^\alpha\omega^n + \frac{1}{2}\sum_\gamma (u_\alpha^{\beta\gamma} + u_\gamma^{\beta\alpha} + u_\beta^{\gamma\alpha})\omega_\gamma,
$$

$$
(4.3)\quad \begin{cases} \mu^{\alpha\beta} = \dfrac{1}{2}\sum_\gamma (m_\gamma^{\alpha\beta} + m_\gamma^{\beta\alpha} + u_\alpha^{\gamma\beta} + u_\beta^{\gamma\alpha})\omega^\gamma, \\ \nu_{\alpha\beta} = \dfrac{1}{2}\sum_\gamma (k_{\gamma\beta}^\alpha + k_{\gamma\alpha}^\beta + y_{\alpha\beta}^\gamma + y_{\beta\alpha}^\gamma)\omega_\gamma, \quad \Phi_\alpha = \dfrac{1}{2} z_{\alpha\beta\gamma}\omega^\beta\wedge\omega^\gamma. \end{cases}
$$

From (3.24) we have $1/2\ (k_{\beta\gamma}^\alpha + k_{\beta\alpha}^\gamma + k_{\gamma\alpha}^\beta) = -\zeta_{i|j}^\alpha\zeta_\beta^i\zeta_\gamma^j$. On the other hand, if we denote by $\zeta_i^\alpha|_j$ the second covariant derivativesof ζ_i^α with respect to the connection of E. Cartan, since A_{jk}^i are symmetric in j and k, by (3.26) we have

$$
(4.4)\quad u_\alpha^{\beta\gamma} = \zeta_i^\alpha|_j(\zeta_\beta^i\zeta_\gamma^j - \zeta_\gamma^i\zeta_\beta^j),
$$

from which it follows that $1/2\ (u_\alpha^{\beta\gamma} + u_\gamma^{\beta\alpha} + u_\beta^{\gamma\alpha}) = \zeta_i^\alpha|_j\zeta_\beta^i\zeta_\gamma^j$. Hence (4.2) is expressible in

$$
(4.5)\quad \omega_\beta^\alpha = \Gamma_{\beta\gamma}^\alpha\omega^\gamma + \Gamma_{\beta n}^\alpha\omega^n + \Gamma_{\ \beta}^{\alpha\gamma}\omega_\gamma,
$$

where

$$
(4.6)\quad \Gamma_{\beta\gamma}^\alpha = -\zeta_{i|j}^\alpha\zeta_\beta^i\zeta_\gamma^j, \quad \Gamma_{\beta n}^\alpha = -l_\beta^\alpha = -\zeta_{i|j}^\alpha\zeta_\beta^i l^j, \quad \Gamma_{\ \beta}^{\alpha\gamma} = \zeta_i^\alpha|_j\zeta_\beta^i\zeta_\gamma^j.
$$

Similarly by virtue of (3.24), (3.26) and (4.4), (4.3) is reducible to

$$
(4.7)\quad \begin{cases} \mu^{\alpha\beta} = -A_{ijk}\zeta_\alpha^i\zeta_\beta^j\zeta_\gamma^k\omega^\gamma, \quad \nu_{\alpha\beta} = -A_{jk|r}^i l^r\zeta_\alpha^j\zeta_\beta^k\zeta_i^\gamma\omega_\gamma, \\ \Phi_\alpha = \dfrac{1}{2}R_{ijhk} l^i\zeta_\alpha^j\zeta_\beta^h\zeta_\gamma^k\omega^\beta\wedge\omega^\gamma. \end{cases}
$$

A coframe transformation between any two adapted orthogonal coframes π and π' is given as follows:

$$(4.8)\qquad \pi=\begin{pmatrix}\omega^1\\ \vdots\\ \omega^n\\ \omega_1\\ \vdots\\ \omega_{n-1}\end{pmatrix},\quad \pi'=\begin{pmatrix}\omega'^1\\ \vdots\\ \omega'^n\\ \omega'_1\\ \vdots\\ \omega'_{n-1}\end{pmatrix},\quad \pi'=\phi\pi,\quad \phi=(\phi^A_B)=\begin{pmatrix}\phi^\alpha_\beta & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \phi^\alpha_\beta\end{pmatrix},\quad (A,B=1,2,\cdots,2n-1)$$

where the matrix (ϕ^α_β) is orthogonal. Under the transformation ϕ, the following relations are satisfied [4]:

$$(4.9)\qquad \omega'^\alpha=\phi^\alpha_\beta\omega^\beta,\quad \omega'_\alpha=\sum_\beta\phi^\alpha_\beta\omega_\beta,\quad \omega'^n=\omega^n,$$

$$(4.10)\qquad \omega'^\alpha_\beta=\sum_\gamma\phi^\beta_\gamma\omega^\delta_\gamma\phi^\alpha_\delta-\sum_\gamma d\phi^\alpha_\gamma\phi^\beta_\gamma=(\phi^{-1})^\gamma_\beta\omega^\delta_\gamma\phi^\alpha_\delta+d(\phi^{-1})^A_\beta\phi^\alpha_A,$$

$$(4.11)\qquad \begin{cases}\mu'^{\alpha\beta}=(\phi^{-1})^\alpha_\gamma(\phi^{-1})^\beta_\delta\mu^{\gamma\delta}, & w'_{\alpha\beta}=\phi^\gamma_\alpha\phi^\delta_\beta w_{\gamma\delta},\\ \nu'_{\alpha\beta}=\phi^\gamma_\alpha\phi^\delta_\beta\nu_{\gamma\delta}, & \Phi'_\alpha=\phi^\beta_\alpha\Phi_\beta,\end{cases}$$

where ω'^α_β, $\mu'^{\alpha\beta},\cdots$ are the forms with respect to π' corresponding to ω^α_β, $\mu^{\alpha\beta},\cdots$ with respect to π. From (4.10) and (4.11) it is seen that $u^{\alpha\beta}$, $w_{\alpha\beta}$ $\nu_{\alpha\beta}$ and Φ_α are tensorial forms with respect to adapted orthogonal coframes, while ω^α_β are connection-like. The latter fact enables us to define some connections on N. First, put

$$(4.12)\qquad \overset{0}{\Gamma}=(\overset{0}{\omega}{}^A_B)=\begin{pmatrix}\omega^\alpha_\beta & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \omega^\alpha_\beta\end{pmatrix}.$$

Then under the transformation ϕ,

$$\overset{0}{\omega}{}'^A_B=(\phi^{-1})^C_B\overset{0}{\omega}{}^D_C\phi^A_D+d(\phi^{-1})^C_B\phi^A_C\qquad (A,B,C,D=1,2,\cdots,2n-1)$$

and hence $\overset{0}{\Gamma}$ defines a connection on N evidently. We shall call this connection the K_0-connection in the sequel. The torsion and curvature forms for the K_0-connection are given by

$$(4.13)\qquad \begin{cases}\overset{0}{\tau}{}^\alpha=d\omega^\alpha-\omega^\beta{}_\wedge\omega^\alpha_\beta=\omega_\beta{}_\wedge\mu^{\beta\alpha}+\omega_\alpha{}_\wedge\omega^n, & \overset{0}{\tau}{}^n=d\omega^n=\omega^\alpha{}_\wedge\omega_\alpha,\\ \overset{0}{\tau}{}^{\alpha+n}=d\omega_\alpha-\sum_\beta\omega_\beta{}_\wedge\omega^\alpha_\beta=\omega^\beta{}_\wedge\nu_{\beta\alpha}-w_{\alpha\beta}\omega^\beta{}_\wedge\omega^n+\Phi_\alpha, & \end{cases}$$

$$(4.14)\qquad \overset{0}{\Omega}=\begin{pmatrix}\overset{0}{\Omega}{}^\alpha_\beta & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \overset{0}{\Omega}{}^\alpha_\beta\end{pmatrix},\qquad \overset{0}{\Omega}{}^\alpha_\beta=d\omega^\alpha_\beta-\omega^\gamma_\beta{}_\wedge\omega^\alpha_\gamma.$$

On making use of (3.26) and (4.7), we can express (4.13) as follows:

$$(4.15)\quad \begin{cases} \overset{0}{\tau}{}^{\alpha} = -\sum\limits_{\beta} A^{i}_{jk}\zeta^{\alpha}_{i}\zeta^{j}_{\beta}\zeta^{k}_{\gamma}\omega_{\beta\wedge}\omega^{\gamma} + \omega_{\alpha\wedge}\omega^{n}, \qquad \overset{0}{\tau}{}^{n} = -\sum\limits_{\delta}\delta_{\beta\gamma}\omega_{\beta\wedge}\omega^{\gamma}, \\ \overset{0}{\tau}{}^{n+\alpha} = \dfrac{1}{2}R^{j}_{ihk}l^{i}\zeta^{\alpha}_{j}\zeta^{h}_{\beta}\zeta^{k}_{\gamma}\omega^{\beta}{}_{\wedge}\omega^{\gamma} - R^{j}_{ihk}l^{i}l^{h}\zeta^{\alpha}_{j}\zeta^{k}_{\beta}\omega^{\beta}{}_{\wedge}\omega^{n} \\ \qquad\qquad + \sum\limits_{\beta} A^{i}_{jk|r}l^{r}\zeta^{\alpha}_{i}\zeta^{j}_{\beta}\zeta^{k}_{\gamma}\omega_{\beta\wedge}\omega^{\gamma} \\ \quad = \dfrac{1}{2}R^{j}_{ihk}l^{i}\zeta^{\alpha}_{j}\zeta^{h}_{b}\zeta^{k}_{c}\omega^{b}{}_{\wedge}\omega^{c} + \sum\limits_{\beta} A^{i}_{jk|r}l^{r}\zeta^{\alpha}_{i}\zeta^{j}_{\beta}\zeta^{k}_{a}\omega_{\beta\wedge}\omega^{a}. \end{cases}$$

Next, if we put

$$(4.16)\qquad \Gamma = (\omega^{A}_{B}) = \begin{pmatrix} \omega^{\alpha}_{\beta} & \omega^{\alpha}_{n} & 0 \\ \omega^{n}_{\beta} & 0 & 0 \\ 0 & 0 & \omega^{\alpha}_{\beta} \end{pmatrix}, \qquad \omega^{n}_{\alpha} = -\omega^{\alpha}_{n} = \omega_{\alpha},$$

we can verify by (4.9) and (4.10) that Γ defines another connection on N. In the sequel, this connection will be called the K-connection. The torsion and curvature forms for the K-connection are given by

$$(4.17)\qquad \tau^{\alpha} = \omega_{\beta\wedge}\mu^{\beta\alpha}, \quad \tau^{n} = 0, \quad \tau^{n+\alpha} = \omega^{\beta}{}_{\wedge}\nu_{\beta\alpha} - w_{\alpha\beta}\omega^{\beta}{}_{\wedge}\omega^{n} + \Phi_{\alpha},$$

$$(4.18)\qquad \Omega = \begin{pmatrix} \Omega^{\alpha}_{\beta} & \Omega^{\alpha}_{n} & 0 \\ \Omega^{n}_{\beta} & 0 & 0 \\ 0 & 0 & \overset{0}{\Omega}{}^{\alpha}_{\beta} \end{pmatrix}, \qquad \begin{aligned} &\Omega^{\alpha}_{\beta} = \overset{0}{\Omega}{}^{\alpha}_{\beta} + \omega_{\beta\wedge}\omega_{\alpha}, \quad \Omega^{n}_{\alpha} = d\omega_{\alpha} - \omega^{\beta}_{\alpha\wedge}\omega_{\beta}, \\ &\Omega^{\alpha}_{n} = -d\omega_{\alpha} + \sum_{\beta}\omega_{\beta\wedge}\omega^{\alpha}_{\beta}. \end{aligned}$$

From (4.13), (4.15) and (4.17) we have

$$(4.19)\qquad \tau^{\alpha} = -\sum_{\beta} A^{i}_{jk}\zeta^{\alpha}_{i}\zeta^{j}_{\beta}\zeta^{k}_{\gamma}\omega_{\beta\wedge}\omega^{\gamma}, \quad \tau^{n} = 0, \quad \tau^{n+\alpha} = \overset{0}{\tau}{}^{n+\alpha}.$$

Now we shall calculate the curvature forms exactly and express in more concrete forms. For this purpose it needs to regulate to some extent the results obtained hitherto. First, since $dp_{i} = \Gamma^{*h}_{ij}p_{h}dx^{j} + g_{ij}Dl^{j}$, it follows from (3.12) and (3.18) that

$$(4.20)\qquad dp_{i} = (\gamma^{0}_{ij} + A_{ijk}\gamma^{k}_{00})\zeta^{j}_{a}\omega^{a} - \zeta^{\beta}_{i}\omega_{\beta},$$

which is obtained directly also from (3.3). Then we can express (3.20) and (4.20) in a single form

$$(4.21)\qquad d\zeta^{a}_{i} = (\zeta^{a}_{i|j} + \Gamma^{*h}_{ij}\zeta^{a}_{h})\zeta^{j}_{b}\omega^{b} - \sum_{\beta}\zeta^{a}_{i||j}\zeta^{j}_{\beta}\omega_{\beta}.$$

Since $(d\zeta^{a}_{i})\zeta^{j}_{a} + \zeta^{a}_{i}(d\zeta^{j}_{a}) = 0$, from (4.21) we obtain

$$(4.22)\qquad d\zeta^{i}_{a} = -(\zeta^{b}_{j|k}\zeta^{i}_{b} + \Gamma^{*i}_{jk})\zeta^{j}_{a}\zeta^{k}_{c}\omega^{c} + \sum_{\beta}\zeta^{b}_{j||k}\zeta^{i}_{b}\zeta^{j}_{a}\zeta^{k}_{\beta}\omega_{\beta},$$

which is valid for every value of a. In fact, if we put $a=n$, then (4.22) implies (3.19). Next, if we let Greek indices in (3.24), (3.26) and (4.7) be suitably equal to n, we have

$$\text{(4.23)} \quad \begin{cases} l^n_\beta = k^n_{\beta\gamma} = k^\alpha_{nn} = \mu^{\beta n} = \nu_{n\alpha} = z_{\alpha nn} = 0\,, & m^{n\beta}_\gamma = -\delta^\beta_\gamma\,, \\ m^{\alpha\beta}_n = \delta^{\alpha\beta}\,, \quad k^\alpha_{n\beta} = -k^\alpha_{\beta n} = l^\alpha_\beta\,, & y^\beta_{\alpha n} = \nu^\beta_\alpha\,, \quad z_{\alpha n\gamma} = -z_{\alpha\gamma n} = w_{\alpha\gamma}\,. \end{cases}$$

In particular, for ω^α_β in (4.5) with (4.6)

$$\text{(4.24)} \quad \begin{cases} \Gamma^n_{bc} = \Gamma^a_{nc} = \Gamma^{n\gamma}_n = 0\,, \quad \Gamma^{n\gamma}_\beta = -\Gamma^{\beta\gamma}_n = \delta_{\beta\gamma}\,, \\ \omega^n_n = 0\,, \quad \omega^n_\alpha = -\omega^\alpha_n = \omega_\alpha\,, \quad \omega^a_b = -\omega^b_a\,. \end{cases}$$

In consequece of (4.23) and (4.24), the equations of structure and connection forms are expressible in the following simpler forms:

$$\text{(4.25)} \quad d\omega^a = \frac{1}{2} k^a_{bc}\omega^b{}_\wedge\omega^c + m^{a\beta}_c\omega_{\beta\wedge}\omega^c = \omega^b{}_\wedge\omega^a_b + \omega_{\beta\wedge}\mu^{\beta a}\,,$$

$$\text{(4.26)} \quad \begin{aligned} d\omega_\alpha &= \frac{1}{2} u^{\beta\gamma}_\alpha\omega_{\beta\wedge}\omega_\gamma + y^\beta_{a\alpha}\omega^a{}_\wedge\omega_\beta + \frac{1}{2} z_{\alpha bc}\omega^b{}_\wedge\omega^c \\ &= \omega^a{}_\wedge\nu_{a\alpha} + \sum_\beta \omega_{\beta\wedge}\omega^\alpha_\beta + \frac{1}{2} z_{\alpha bc}\omega^b{}_\wedge\omega^c\,, \end{aligned}$$

$$\text{(4.27)} \quad \begin{cases} \omega^a_b = \Gamma^a_{bc}\omega^c + \Gamma^{a\gamma}_b\omega_\gamma\,, \\ \Gamma^a_{bc} = -\zeta^a_{i|j}\zeta^i_b\zeta^j_c\,, \quad \Gamma^{a\gamma}_b = \zeta^a_i|_j\zeta^i_b\zeta^j_\gamma\,. \end{cases}$$

Now, put

$$\text{(4.28)} \quad d\Gamma^a_{bc} = \Gamma^a_{bcd}\omega^d + \Gamma^{a\ \gamma}_{bc}\omega_\gamma\,, \qquad d\Gamma^{a\alpha}_b = \Gamma^{a\alpha}_{b\ c}\omega^c + \Gamma^{a\alpha\beta}_b\omega_\beta$$

and calculate $d\Gamma^a_{bc}$, $d\Gamma^{a\alpha}_b$ on use of (3.18), (3.19), (4.22) and (4.27). Then by the comparison with the corresponding coefficients in (4.28) we have

$$\text{(4.29)} \quad \begin{cases} \Gamma^a_{bcd} = (-\zeta^a_{i|jk} + \zeta^a_{h|j}\zeta^e_{i|k}\zeta^h_e + \zeta^a_{i|h}\zeta^e_{j|k}\zeta^h_e)\zeta^i_b\zeta^j_c\zeta^k_d\,, \\ \Gamma^{a\ \gamma}_{bc} = (\zeta^a_{i|j||k} - \zeta^a_{h|j}\zeta^e_{i||k}\zeta^h_e - \zeta^a_{i|h}\zeta^e_{j||k}\zeta^h_e)\zeta^i_b\zeta^j_c\zeta^k_\gamma\,, \end{cases}$$

$$\text{(4.30)} \quad \begin{cases} \Gamma^{a\alpha}_{b\ c} = (\zeta^a_i|_{j|k} - \zeta^a_h|_j\zeta^e_{i|k}\zeta^h_e - \zeta^a_i|_h\zeta^e_{j|k}\zeta^h_e)\zeta^i_b\zeta^j_\alpha\zeta^k_c\,, \\ \Gamma^{a\alpha\beta}_b = (-\zeta^a_i|_{j||k} + \zeta^a_h|_j\zeta^e_{i||k}\zeta^h_e + \zeta^a_i|_h\zeta^e_{j||k}\zeta^h_e)\zeta^i_b\zeta^j_\alpha\zeta^k_\beta\,. \end{cases}$$

Further we put

$$\text{(4.31)} \quad \begin{aligned} &\Omega^a_b(=d\omega^a_b - \omega^c_{b\wedge}\omega^a_c) \\ &= -\frac{1}{2} R^a_{b\ cd}\omega^c{}_\wedge\omega^d - \sum_d P^a_{b\ c\alpha}\omega_{\alpha\wedge}\omega^c - \frac{1}{2}\sum_{\alpha,\beta} S^a_{b\ \alpha\beta}\omega_{\alpha\wedge}\omega_\beta\,, \end{aligned}$$

where $R^a_{b\ cd} = -R^a_{b\ dc}$, $S^a_{b\ \alpha\beta} = -S^a_{b\ \beta\alpha}$.

Calculating Ω^a_b by means of (4.25)~(4.28) and comparing with coefficients in the right hand in (4.31), we get

$$\text{(4.32)}\quad \begin{cases} R^{a}_{b\,cd} = -(\Gamma^{a}_{be}K^{e}_{cd} + \Gamma^{a\beta}_{b}z_{\beta cd} + \Gamma^{a}_{ec}\Gamma^{e}_{bd} - \Gamma^{a}_{ed}\Gamma^{e}_{bc} + \Gamma^{a}_{bdc} - \Gamma^{a}_{bcd}), \\ P^{a}_{b\,c\alpha} = -(\Gamma^{a}_{be}m^{e\alpha}_{c} - \Gamma^{a\beta}_{b}y^{\alpha}_{\beta c} + \Gamma^{a\alpha}_{e}\Gamma^{e}_{bc} - \Gamma^{e\alpha}_{b}\Gamma^{a}_{ec} + \Gamma^{a\ \alpha}_{bc} - \Gamma^{a\alpha}_{b\ c}), \\ S^{a}_{b\,\alpha\beta} = \Gamma^{a\gamma}_{b}u^{\alpha\beta}_{\gamma} - \Gamma^{e\alpha}_{b}\Gamma^{a\beta}_{e} + \Gamma^{e\beta}_{b}\Gamma^{a\alpha}_{e} + \Gamma^{a\beta\alpha}_{b} - \Gamma^{a\alpha\beta}_{b}. \end{cases}$$

Finally, if we substitute (3.24), (3.26), (3.27), (3.29) and (4.30) in the right hands in (4.32) and apply the Ricci's formulas [5] to the results, we have

$$\text{(4.33)}\quad \begin{cases} R^{a}_{bcd} = R^{\ i}_{jhk}\zeta^{a}_{i}\zeta^{j}_{b}\zeta^{h}_{c}\zeta^{k}_{d}, \qquad P^{a}_{b\,c\alpha} = P^{\ i}_{jhk}\zeta^{a}_{i}\zeta^{j}_{b}\zeta^{h}_{c}\zeta^{k}_{\alpha}, \\ S^{a}_{bcd} = S^{\ i}_{jhk}\zeta^{a}_{i}\zeta^{j}_{b}\zeta^{h}_{c}\zeta^{k}_{d}, \end{cases}$$

where $R^{\ i}_{jhk}$, $P^{\ i}_{jhk}$ and $S^{\ i}_{jhk}$ are the components of the curvature tensors of E. Cartan.

Thus we have seen that the forms $\overset{0}{\Omega}{}^{\alpha}_{\beta}$ for the K_0-connection are given by

$$\text{(4.34)}\quad \overset{0}{\Omega}{}^{\alpha}_{\beta} = -\frac{1}{2}R^{\alpha}_{\beta cd}\omega^{c}{}_{\wedge}\omega^{d} - \sum_{\gamma}P^{\alpha}_{\beta c\gamma}\omega_{\gamma\wedge}\omega^{c} + \frac{1}{2}\sum_{\gamma,\varepsilon}(-S^{\alpha}_{\beta\gamma\varepsilon} + \delta^{\alpha}_{\gamma}\delta_{\beta\varepsilon} - \delta^{\alpha}_{\varepsilon}\delta_{\beta\gamma})\omega_{\gamma\wedge}\omega_{\varepsilon}$$

and for the K-connection

$$\text{(4.35)}\quad \Omega = \begin{pmatrix} \Omega^{a}_{b} & 0 \\ 0 & \overset{0}{\Omega}{}^{\alpha}_{\beta} \end{pmatrix}, \qquad \Omega^{a}_{b};\ (4.31)\ \text{with}\ (4.33).$$

§5. Finsler space M as a distribution on N. By means of the matrices (ζ^{i}_{a}) and (ζ^{a}_{i}) we transform the frame (e_i) and the coframe (e^i) in §2 to (e_a) and (e^a):

$$\text{(5.1)}\quad e_a = \zeta^{i}_{a}e_i, \qquad e^a = \zeta^{a}_{i}e^i = \omega^a.$$

If we denote by $\overset{c}{\omega}{}^{i}_{j}$ the connection forms of E. Cartan with respect to (e^i), they are given by

$$\text{(5.2)}\quad \overset{c}{\omega}{}^{i}_{j} = A^{i}_{jk}Dl^{k} + \Gamma^{*i}_{jk}dx^{k},$$

which are, under (5.1), transformable to

$$\text{(5.3)}\quad \overset{c}{\omega}{}^{a}_{b} = \zeta^{j}_{b}\overset{c}{\omega}{}^{i}_{j}\zeta^{a}_{i} + d\zeta^{j}_{b}\zeta^{a}_{j}.$$

If we substitute (5.2) in the right hand of (5.3) and calculate on use of (3.12), (3.18) and (4.22), it follows from (4.27) that

$$\overset{c}{\omega}{}^{a}_{b} = -\zeta^{a}_{i|j}\zeta^{i}_{b}\zeta^{j}_{c}\omega^{c} + \sum_{\beta}\zeta^{a}_{i}|_{j}\zeta^{i}_{b}\zeta^{j}_{\beta}\omega_{\beta} = \omega^{a}_{b}.$$

Hence we can state

THEOREM 1. *The forms ω^a_b are the connection forms of E. Cartan, which give rise to the K-connection and K_0-connection on N.*

Now we consider quadratic differential forms on N

$$A=\sum_a \omega^a\omega^a\,, \qquad \bar{A}=\sum_\alpha \omega^\alpha\omega^\alpha\,,$$

which are independent of the choice of adapted orthogonal coframes and are also quadratic differential forms on M. In this case, it follows from (3.5), (3.7)′, (3.9) and (3.17) that

$$\begin{aligned} A &= \delta_{ab}\omega^a\omega^b=\delta_{ab}\zeta^a_i\zeta^b_j dx^i dx^j=g_{ij}dx^i dx^j\,, \\ \bar{A} &= \delta_{\alpha\beta}\omega^\alpha\omega^\beta=h_{\lambda\mu}t^\lambda_\alpha t^\mu_\beta\omega^\alpha\omega^\beta=h_{\lambda\mu}\theta^\lambda\theta^\mu=h_{ij}dx^i dx^j\,. \end{aligned} \tag{5.4}$$

In addition to A and $\bar{A}$, we can take the invariant forms on N

$$B=\sum_\alpha \omega_\alpha\omega_\alpha\,, \qquad d\omega^n=\omega^\alpha{}_\wedge\omega_\alpha\,,$$

which are expressible in

$$\begin{aligned} B &= \sum_\alpha \zeta^\alpha_i\zeta^\alpha_j Dl^i Dl^j=h_{ij}Dl^i Dl^j=g_{ij}Dl^i Dl^j\,, \\ d\omega^n &= -\sum_\alpha \zeta^\alpha_i\zeta^\alpha_j dx^i{}_\wedge Dl^j=-h_{ij}dx^i{}_\wedge Dl^j=-g_{ij}dx^i{}_\wedge Dl^j\,. \end{aligned} \tag{5.5}$$

Then it is easily verfied that the tensors on N corresponding to the above four forms are all parallel with respect to the K-connection.

In particular, the tensor corresponding to the A has the following components:

$$\left\{\begin{array}{ll} \left.\begin{array}{l} g_{ij} \text{ with respect to } (e_i) \\ \delta_{ab} \text{ with respect to } (e_a) \end{array}\right\} & \text{on } M, \\ \begin{pmatrix} \delta_{ab} & 0 \\ 0 & 0 \end{pmatrix} \text{ with respect to } \pi=(\omega^a,\, \omega_\alpha) & \text{on } N. \end{array}\right. \tag{5.6}$$

Since $D\delta_{AB}=-\omega^B_A-\omega^A_B=0$, a tensor δ_{AB} on N is parallel with respect to the K-connection. Accordingly we can adopt this tensor as the fundamental metric tensor and consider the N as a $(2n-1)$-dimensional Riemannian manifold, which has the torsion as is shown later.

As the coframe $\pi=(\omega^a,\, \omega_\alpha)$ is a base of the dual tangent space $T^*_{(x,p)}$ at a point $(x,\, p)\in N$, there will exist the base dual to π in the tangent space $T_{(x,p)}$. If we denote this base by $(e_a,\, e^\alpha)$, which will be called an adapted orthogonal frame on N, the following relations hold:

$$\langle e_a,\, \omega^b\rangle=\delta^b_a\,, \quad \langle e_a,\, \omega_\alpha\rangle=\langle e^\alpha,\, \omega^a\rangle=0\,, \quad \langle e^\alpha,\, \omega_\beta\rangle=\delta^\alpha_\beta\,. \tag{5.7}$$

$$(5.8)\qquad \langle e_a, e_b\rangle=\delta_{ab}\,,\quad \langle e_a, e^\alpha\rangle=0\,,\quad \langle e^\alpha, e^\beta\rangle=\delta^{\alpha\beta}\,,\quad \langle\omega^a, \omega^b\rangle=\delta^{ab}\,,$$
$$\langle\omega^a, \omega_\alpha\rangle=0\,,\quad \langle\omega_\alpha, \omega_\beta\rangle=\delta_{\alpha\beta}\,,\qquad (\langle\,,\rangle;\ \text{inner product}).$$

A. Deicke showed that it was, in general, impossible to imbed an n-dimensional Finsler space in a $(2n-1)$-dimensional Riemannian madifold R_{2n-1} without torsion, while it was possible to do so in R_{2n-1} with the metric and torsion chosen suitably [1], [2].

Now we shall show that it is possible to realize the M as a distribution on N in the similar way as A. Deicke.

We consider a system of differential equations

$$(5.9)\qquad \omega_\alpha=-\zeta^\alpha_i Dl^i=0\,,$$

which is equivalent to

$$(5.9)'\qquad Dl^i=dl^i+\Gamma^{*i}_{jk}l^j dx^k=0\,.$$

Then we obtain as the complete integrability condition for (5.9)$'$

$$(5.10)\qquad R^i_{0hk}=0\qquad (\text{or}\ \ R^a_{ncd}=0)\,,$$

which implies that M is a space with the absolute parallelism of E. Cartan. Therefore the system (5.9) is, in general, not completely integrable, that is, it does not define a family of submanifolds of N, but defines an n-dimensional distribution $\mathfrak{M}$ on N. In this case, a local base for the distribution $\mathfrak{M}$ is given by (e_a) $(a=1,2,\cdots,n)$, which is also a local base for M. And further it follows from (4.16), (5.6) and the Theorem 1 that the metric and connection on $\mathfrak{M}$ induced from those on N identify with the metric and connection on M.

The metric $d\sigma^2$ on N depends on only the local length and angular metric on M. In fact, if ds is the distance between the centres of two neighboring line-elements in M and $d\phi$ is the angle between the directions, it follows from (5.4), (5.5) and (5.8) that

$$d\sigma^2=\langle\omega^a e_a+\omega_\alpha e^\alpha\,,\ \omega^b e_b+\omega_\beta e^\beta\rangle=\sum_a\omega^a\omega^a+\sum_\alpha\omega_\alpha\omega_\alpha$$
$$=g_{ij}dx^i dx^j+g_{ij}Dl^iDl^j=ds^2+d\phi^2\,.$$

The autoparallel curves in N with respect to the K-connection do not coincide with the geodesics. In fact, if we put

$$(5.11)\qquad \tau^A=\frac{1}{2}T^A_{BC}\omega^B{}_\wedge\omega^C\,,\quad T^A_{BC}+T^A_{CB}=0\quad (\omega^{n+\alpha}=\omega_\alpha\,;\ A,B,C=1,2,\cdots,2n-1),$$

from (4.15) and (4.19) we have

$$(5.12)\qquad \begin{aligned} &T^a_{bc}=T^A_{n+\beta,n+\gamma}=0\,, \qquad T^a_{c,n+\beta}=-T^a_{n+\beta,c}=A^i_{jk}\zeta^a_i\zeta^j_\beta\zeta^k_c\,,\\ &T^{n+a}_{bc}=R^j_{ihk}l^i\zeta^a_j\zeta^h_b\zeta^k_c\,, \qquad T^{n+a}_{n+\beta,c}=-T^{n+a}_{c,n+\beta}=A^i_{jk|r}l^r\zeta^a_i\zeta^j_\beta\zeta^k_c\,, \end{aligned}$$

from which it follows that all of T^A_{BC} are not skew-symmetric in all indices A, B and C, that is, the above assertion is true.

Now we shall introduce a new connection on N, which is in fact due to A. Deicke [2].

If we define a matrix (ψ^a_b) by $\begin{pmatrix}\psi^a_\beta & 0\\ 0 & 1\end{pmatrix}$, (ψ^a_β) being orthogonal, the coframe transformation in (4.8) is expressible in $\psi=(\psi^A_B)=\begin{pmatrix}\psi^a_b & 0\\ 0 & \psi^a_\beta\end{pmatrix}$. Then we have

$$(5.13)\qquad \begin{aligned} &\omega'^A=\psi^A_B\omega^B\,, \quad e'_A=(\psi^{-1})^B_A e_B \qquad (e_{n+a}=e^a)\,,\\ &S'^a_{bc}\omega'^c=\psi^a_d(\psi^{-1})^e_b(S^d_{ec}\omega^c)\,, \quad \sum_\gamma W'^a_{b\gamma}\omega'_\gamma=\psi^a_d(\psi^{-1})^e_b(\sum_\gamma W^d_{e\gamma}\omega_\gamma)\,, \end{aligned}$$

where S'^a_{bc}, W'^a_{bc} and S^a_{bc}, W^a_{bc} are the components of any two tensors on M with respect to (e'_a) and (e_a) respectively.

If we put

$$(5.14)\qquad \tilde{\Gamma}=(\tilde{\omega}^A_B)=\begin{pmatrix}\omega^a_b & \tilde{\omega}^a_{n+\beta}\\ \tilde{\omega}^{n+a}_b & \omega^a_\beta\end{pmatrix}, \qquad \begin{aligned} \tilde{\omega}^{n+a}_b&=-\tilde{\omega}^b_{n+a}\\ &=(A^i_{hk}+R^{\ i}_{j\ hk}l^j)\zeta^a_i\zeta^h_b\zeta^k_c\omega^c-\sum_\gamma A^i_{jk|r}l^r\zeta^j_b\zeta^k_\gamma\zeta^a_i\omega_\gamma\,, \end{aligned}$$

as it is seen because of (5.13) that the forms $\tilde{\omega}^{n+a}_b$ are tensorial, the $\tilde{\Gamma}$ defines a connection on N surely. In the sequel, this connection will be called the D-connection.

For the D-connection, we obtain the components of the torsion tensor in the similar way as in (5.11):

$$(5.15)\qquad \begin{aligned} &\tilde{T}^a_{bc}=\tilde{T}^A_{n+\beta\, n+\gamma}=\tilde{T}^{n+a}_{n+\beta,c}=0\,, \quad \tilde{T}^a_{n+\beta,c}=-\tilde{T}^a_{c,n+\beta}=R^j_{ihk}l^i\zeta^\beta_j\zeta^h_a\zeta^k_c\,,\\ &\tilde{T}^{n+a}_{bc}=-\tilde{T}^b_{n+a,c}=-R_{jihk}l^j\zeta^i_a\zeta^h_b\zeta^k_c\,. \end{aligned}$$

Then it follows from (5.15) that the autoparallel curves in N with respect to the D-connection coincide with the geodesics.

If we put

$$(5.16)\qquad \tilde{\Gamma}_0=(\tilde{\omega}^A_B)=\begin{pmatrix}\omega^a_\beta\ 0\ \tilde{\omega}^a_{n+\beta}\\ 0\ 0 \qquad ,\\ \tilde{\omega}^{n+a}_b \quad \omega^a_\beta\end{pmatrix} \qquad \begin{aligned} \tilde{\omega}^{n+a}_b&=-\tilde{\omega}^b_{n+a}\\ &=A^i_{jk}\zeta^a_i\zeta^j_b\zeta^k_c\omega^c-\sum_\gamma A^i_{jk|0}\zeta^a_i\zeta^j_b\zeta^k_\gamma\omega_\gamma\,, \end{aligned}$$

$\tilde{\Gamma}_0$ defines a connection on N, too, which will be called the D_0-connection.

Thus we have

COROLLARY 1.1. *The connection forms ω^a_b, the torsion tensor A^i_{jk} and the curvature tensor $R_{j\ hk}^{\ i}$ of E. Cartan give rise to the D-connection and*

D_0-connection on N.

On use of the tensors S^a_{bc} and W^a_{bc} in (5.13), we can define a general connection on N:

$$(5.17)\qquad \bar{\Gamma}=(\bar{\omega}^A_B)=\begin{pmatrix}\omega^a_b & \bar{\omega}^{\ a}_{n+\beta}\\ \bar{\omega}^{n+\alpha}_b & \omega^\alpha_\beta\end{pmatrix},\qquad \bar{\omega}^{n+\alpha}_b=-\bar{\omega}^{\ b}_{n+\alpha}=S^\alpha_{bc}\omega^c+\sum_\gamma W^\alpha_{b\gamma}\omega_\gamma\,.$$

In this case, the autoparallel curves in N with respect to the above connection $\bar{\Gamma}$ coincide with the geodesics if and only if the $\bar{\Gamma}$ becomes the D-connection. In fact, the torsion forms for $\bar{\Gamma}$ are given by

$$\bar{\tau}^a=\tau^a+\sum_\alpha\omega_\alpha{}_\wedge\bar{\omega}^{n+\alpha}_a,\qquad \bar{\tau}^{n+\alpha}=\overset{0}{\tau}{}^{n+\alpha}-\omega^b{}_\wedge\bar{\omega}^{n+\alpha}_b,$$

which are, by virtue of (4.15), (4.19) and (5.17), expressible in

$$(5.18)\qquad \begin{cases}\bar{\tau}^a=\sum_\beta(-A^a_{\beta c}+S^\beta_{ac})\omega_\beta{}_\wedge\omega^c+\sum_{\beta,\gamma}W^\beta_{a\gamma}\omega_\beta{}_\wedge\omega_\gamma\,,\\[2mm] \bar{\tau}^{n+\alpha}=\dfrac{1}{2}(R^{\ \alpha}_{n\ bc}-S^\alpha_{bc}+S^\alpha_{cb})\omega^b{}_\wedge\omega^c+\sum_\beta(A^\alpha_{\beta b|n}+W^\alpha_{b\beta})\omega_\beta{}_\wedge\omega^b,\\[2mm] A^a_{\beta c}=A^i_{jk}\zeta^a_i\zeta^j_\beta\zeta^k_c\,,\quad R^{\ \alpha}_{n\ bc}=R^{\ j}_{i\ hk}\zeta^\alpha_j l^i\zeta^h_b\zeta^k_c,\quad A^\alpha_{\beta b|n}=A^i_{jk|r}l^r\zeta^\alpha_i\zeta^j_\beta\zeta^k_b\,,\end{cases}$$

from which it follows that

$$(5.19)\qquad \begin{aligned}&\bar{T}^a_{bc}=\bar{T}^{n+\alpha}_{n+\beta,n+\gamma}=0\,,\quad \bar{T}^{\ a}_{n+\beta,c}=-\bar{T}^{\ a}_{c,n+\beta}=-A^a_{\beta c}+S^\beta_{ac}\,,\\ &\bar{T}^{\ a}_{n+\beta,n+\gamma}=W^\beta_{a\gamma}-W^\gamma_{a\beta}\,,\quad \bar{T}^{n+\alpha}_{bc}=R^{\ \alpha}_{n\ bc}-S^\alpha_{bc}+S^\alpha_{cb}\,,\\ &\bar{T}^{n+\alpha}_{n+\beta,b}=-\bar{T}^{n+\alpha}_{b,n+\beta}=A^\alpha_{\beta b|n}+W^\alpha_{b\beta}\,.\end{aligned}$$

Let $\bar{T}^A_{BC}$ be skew-symmetric in all indices A, B and C. Then, since A^a_{bc} and $A^a_{bc|n}$ are symmetric in all indices a, b and c, from (5.19) we have

$$\begin{aligned}&S^a_{bc}=A^a_{bc}+R^{\ a}_{n\ bc}=(A^i_{hk}+R^{\ i}_{j\ hk}l^j)\zeta^a_i\zeta^h_b\zeta^k_c\,,\\ &W^a_{b\gamma}=-A^a_{b\gamma|n}=-A^i_{jk|r}l^r\zeta^a_i\zeta^j_b\zeta^k_\gamma\,,\end{aligned}$$

and hence $\bar{\omega}^{n+\alpha}_b=\hat{\omega}^{n+\alpha}_b$, that is, the connection $\bar{\Gamma}$ becomes the D-connection.

For the connection $\bar{\Gamma}$, the following facts are still valid:

The fundamental tensor δ_{AB} on N is parallel and the induced connection on $\mathfrak{M}$ identifies with the connection of E. Cartan.

Thus, making summary of the results obtained, we can state.

THEOREM 2. *An n-dimensional Finsler space M endowed with the connection of E. Cartan is realizable as an n-dimensional distribution $\mathfrak{M}$ on its p-manifold N as follows:*

(1) *The N is a $(2n-1)$-dimensional Riemannian manifold with the metric whose components are given by δ_{AB} with respect to an adapted or-*

thogonal coframe.

(2) *The N is endowed with a connection* $\overline{\Gamma}$ *defined by* (5.17), *which is metric but not symmetric.*

(3) *Any transformation between adapted orthogonal coframes is confined to such as in* (4.8) (*or* 5.13). *A set of such transformations forms an orthogonal group, which is the fundamental group of N.*

(4) *The metric on N depends on only the local length and angular metric on M.*

(5) *The metric and connection on* $\mathfrak{M}$ *induced from those on N identify with the metric and connection on M.*

(6) *The autoparallel curves in N coincide with the geodesics if and only if the connection* $\overline{\Gamma}$ *becomes the D-connection.*

Corresponding to the connection $\tilde{\overline{\Gamma}}$, we have a connection $\overline{\Gamma}_0$ on N:

$$\overline{\Gamma}_0 = (\underset{0}{\overline{\omega}}) = \begin{pmatrix} \omega^{\alpha}_{\beta} & 0 \quad \overline{\omega}^{\alpha}_{n+\beta} \\ 0 & \\ \overline{\omega}^{n+\alpha}_{b} & \omega^{\alpha}_{\beta} \end{pmatrix}, \qquad \overline{\omega}^{n+\alpha}_{b} = -\overline{\omega}^{b}_{n+\alpha} = S^{\alpha}_{bc}\omega^{c} + \sum_{\gamma} W^{\alpha}_{b\gamma}\omega_{\gamma}\,. \tag{5.20}$$

Though the choice of connections on N is highly arbitrary, it is enough for the practical use to take the K-connection or the D-connection. The curvature tensor for the former is considerably simple, while the conclusion of (6) in the Theorem 2 holds only for the latter.

Let M be an n-dimensional Finsler space with the absolute parallelism of E. Cartan. Then since (5.9)′ is completely integrable, there exists a solution $l=l(x)$ satisfying the following condition: 1) $g_{ij}l^{i}l^{j}=1$ along the solution, 2) $l^{i}=l^{i}_{0}$ when $x^{i}=x^{i}_{0}$ (l^{i}_{0}, x^{i}_{0}; constants). Corresponding to such a solution, we have an n-dimensional submanifold of N through a point $(\overset{0}{x}_0, \overset{0}{p})$ $(p_i = g_{ij}l^{j}_{0})$. Consequently M is realizable as a family of such submanifolds. In other words, the distribution $\mathfrak{M}$ is involutive. In this case, it follows from (5.10), (5.15) and (5.19) that the D-connection is Riemannian. Hence we have

COROLLARY 2.1. *An n-dimensional Finsler space M with the absolute parallelism of E. Cartan is realizable as an n-dimensional involutive distribution* $\mathfrak{M}$ *on its p-manifold N* (*or as a family of n-dimensional submanifolds of N*) *as follows:*

(a) *The matters* (1), (3), (4) *and* (5) *in the Theorem* 2 *are valid.*

(b) *The N is endowed with a connection* $\overline{\Gamma}$ *defined by* (5.17). *The connection* $\overline{\Gamma}$ *is Riemannian if and only if it becomes the D-connection.*

§6. Contact tensor calculus on N. We have seen that any transformation between adapted orthogonal coframes is given by (4.8). Under such a transformation, the form $\omega^n = p_i dx^i$ is invariant, that is, the contact structure on N is invariant. We may therefore consider the transformation as a homogeneous contact transformation. In order to study a Finsler space M, it needs to have a tensor calculus on the p-manifold N which is appropriate to homogeneous contact transformations.

Let N be in general endowed with the $\bar{\Gamma}$-connection. For the sake of brevity, we consider only a proper tensor of type $(1, 1)$ on N whose components with respect to (e_A) are $T^A_B(x, p)$. A proper tensor means that the components are homogeneous of degree zero in p_i. Since $p_i = g_{ij} l^j$, $T^A_B(x, p)$ are expressible in $T^A_B(x, l)$ (or $T^A_B(x, y)$), being homogeneous of degree zero in l^i (or y^i).

The covariant differentials of T^A_B are given by

$$DT^A_B = dT^A_B + \omega^A_D T^D_B - \omega^D_B T^A_D = \nabla_D T^A_B \cdot \omega^D, \tag{6.1}$$

where $\nabla_D T^A_B$ are covariant derivatives and the components of a tensor of type $(1, 2)$ on N. If we put

$$\partial_a T^A_B = \frac{\partial T^A_B}{\partial x^i} \zeta^i_a, \quad \partial^\alpha T^A_B = T^A_{B||i} \zeta^i_\alpha, \quad \nabla^\alpha = \nabla_{n+\alpha}, \tag{6.2}$$
$$\omega^A_B = \Gamma^A_{Ba} \omega_a + \Gamma^{A\alpha}_B \omega_\alpha,$$

from (3.18), (3.19) (or (4.22)), (6.1) and (6.2) we have

$$\begin{cases} \nabla_a T^A_B = \partial_a T^A_B - T^A_{B||i} \Gamma^{*i}_{0j} \zeta^j_a + \Gamma^A_{Da} T^D_B - \Gamma^D_{Ba} T^A_D, \\ \nabla^\alpha T^A_B = -\partial^\alpha T^A_B + \Gamma^{A\alpha}_D T^D_B - \Gamma^{D\alpha}_B T^A_D. \end{cases} \tag{6.3}$$

Now we consider a mapping Φ: a tensor space on $M \to$ a linear space on N defined as follows:
By Φ a proper tensor $T^a_b(x, y)$ with an element of support y^i at any point $x \in M$ corresponds to an object T^A_B at $(x, p) \in N$ whose components are given by

$$(T^A_B) = \begin{pmatrix} T^a_b & 0 \\ 0 & 0 \end{pmatrix}, \tag{6.4}$$

where T^a_b are expressed in terms of $(x^i p_i)$ by means of $l^i = g^{ij} p_j$.

Then, under the homogeneous contact transformations, the object T^A_B becomes a tensor on N. Given, conversely, a tensor T^A_B as (6.4) at any point $(x, p) \in N$, an element of support y^i can be determined uniquely and a tensor $T^a_b(x, y)$ at $x \in M$ follows. Consequently the mapping Φ is an isomorphism of a tensor space on M into a tensor space on N.

In the sequel we shall call a tensor on N corresponding to a tensor on M by Φ an M-tensor on N and denote simply by T^a_b.

For any M-tensor T^a_b, since $T^a_b = T^i_j \zeta^a_i \zeta^j_b$ (T^i_j are considered with respect to (e_i)), it follows from (4.27), (6.2) and (6.3) that

$$\begin{aligned} &\nabla_c T^a_b = T^i_{j|k} \zeta^a_i \zeta^j_b \zeta^k_c, \qquad T^i_{j|k} = \nabla_c T^a_b \cdot \zeta^i_a \zeta^b_j \zeta^c_k, \\ &\nabla^\alpha T^a_b = -T^i_j|_k \zeta^a_i \zeta^j_b \zeta^k_\alpha, \qquad T^i_j|_k = -\sum_\alpha \nabla^\alpha T^a_b \cdot \zeta^i_a \zeta^b_j \zeta^\alpha_k \end{aligned} \tag{6.5}$$

Next, we shall get the curvature tensor. The curvature form $\bar{\Omega}^A_B$ is given by

$$\bar{\Omega}^A_B = d\omega^A_B - \omega^D_B \wedge \omega^A_D \tag{6.6}$$

Then, put

$$\bar{\Omega}^A_B = -\frac{1}{2} \bar{R}^A_{B\,cd}\, \omega^c \wedge \omega^d - \sum_\gamma \bar{P}^A_{B\,c\gamma}\, \omega_\gamma \wedge \omega^c + \frac{1}{2} \sum_{\beta,\gamma} \bar{S}^A_{B\,B\gamma}\, \omega_B \wedge \omega_\gamma \tag{6.7}$$

where $\bar{R}^A_{B\,cd} = -\bar{R}^A_{B\,dc}$, $\bar{S}^A_{B\,B\gamma} = -\bar{S}^A_{B\,\gamma\beta}$.

On making use of (4.4), (4.7), (4.25), 4.26), (4.27) and (6.2) we calculate the right side in (6.6) and by the comparison with the coefficients in the right hand in (6.7) we have

$$\begin{aligned} \bar{R}^A_{B\,cd} &= (\partial_d \Gamma^A_{B\,c} - \Gamma^A_{B\,c||i} \Gamma^{*i}_{0j} \zeta^j_d - \Gamma^A_{B\,a} \Gamma^a_{cd} + \Gamma^D_{B\,c} \Gamma^A_{D\,d}) - c|d - \Gamma^{A\alpha}_B R_{n\alpha cd}, \\ \bar{P}^A_{B\,c\gamma} &= \partial^\gamma \Gamma^A_{B\,c} - \partial_c \Gamma^{A\gamma}_B + \Gamma^{A\gamma}_{B\ ||i} \Gamma^{*i}_{0j} \zeta^j_c + \Gamma^A_{B\,a} (\Gamma^{a\gamma}_c + A^a_{c\gamma}) \\ &\qquad - \Gamma^{A\alpha}_B (\Gamma^\gamma_{\alpha c} + A^\gamma_{\alpha c|n}) - \Gamma^D_{B\,c} \Gamma^{A\gamma}_D + \Gamma^{D\gamma}_B \Gamma^A_{D\,c}, \\ \bar{S}^A_{B\,\gamma\varepsilon} &= \partial^\varepsilon \Gamma^{A\gamma}_B - \Gamma^{A\gamma}_B \Gamma^{\gamma\varepsilon}_\alpha + \Gamma^{D\varepsilon}_B \Gamma^{A\gamma}_D - \gamma|\varepsilon, \end{aligned} \tag{6.8}$$

where $c|d$ and $\gamma|\varepsilon$ indicate the terms written down with c, d and with γ, ε interchanged respectively. For the practical use of (6.8), we choose the K-connection or the D-connection and have to calculate further.

In this paper we have combined the theory of M. Kurita with that of A. Deicke and suggested only a new method to study Finsler spaces. We shall leave the practical applications of this method in later papers.

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