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Elliptic surfaces and contact conics for a 3-nodal quartic

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Abstract. Let $\mathcal{Q}$ be an irreducible 3-nodal quartic and let $\mathcal{C}$ be a smooth conic such that $\mathcal{C} \cap \mathcal{Q}$ does not contain any node of $\mathcal{Q}$ and the intersection multiplicity at $z \in \mathcal{C} \cap \mathcal{Q}$ is even for each z. In this paper, we study geometry of $\mathcal{C} + \mathcal{Q}$ through that of integral sections of a rational elliptic surface which canonically arises from $\mathcal{Q}$ and $z \in \mathcal{C} \cap \mathcal{Q}$. As an application, we construct Zariski pairs $(\mathcal{C}_1 + \mathcal{Q}, \mathcal{C}_2 + \mathcal{Q})$, where $\mathcal{C}_i$ $(i = 1, 2)$ are smooth conics tangent to $\mathcal{Q}$ at four distinct points.

Key words: Elliptic surface, section, contact conic, Zariski pair.

Introduction

In this article, all varieties are defined over the field of complex numbers $\mathbb{C}$. Let $\varphi : S \to \mathbb{P}^1$ be a rational elliptic surface with a section O. Here a section means an irreducible curve on S intersecting a fiber at one point or a morphism $s : \mathbb{P}^1 \to S$ such that $\varphi \circ s = \mathrm{id}_{\mathbb{P}^1}$ (note that these two notions can be canonically identified). It is known that, if φ has a reducible singular fiber, S coincides with a rational elliptic surface $S_{\mathcal{Q},z_o}$ associated with a reduced plane quartic $\mathcal{Q}$, which is not 4 distinct lines meeting at one point, in $\mathbb{P}^2$ and a smooth point z_o on $\mathcal{Q}$ obtained in the following way:

(i) Let S_o be the minimal resolution of the double cover of $\mathbb{P}^2$ branched along $\mathcal{Q}$.

(ii) Choose a smooth point z_o of $\mathcal{Q}$. The pencil of lines through z_o gives rise to a pencil $\Lambda_{\mathcal{Q},z_o}$ of curves of genus 1 on S_o.

(iii) By resolving the base points of $\Lambda_{\mathcal{Q},z_o}$, we have a rational elliptic surface $\varphi : S_{\mathcal{Q},z_o} \to \mathbb{P}^1$. We denote the generically 2 to 1 morphism from $S_{\mathcal{Q},z_o}$ to $\mathbb{P}^2$ by $f_{\mathcal{Q},z_o} : S_{\mathcal{Q},z_o} \to \mathbb{P}^2$.

For details, see [3], [15], for example.

Under the above circumstance, O is mapped to z_o by $f_{\mathcal{Q},z_o}$. Let $\mathrm{MW}(S_{\mathcal{Q},z_o})$ be the set of sections of $\varphi : S_{\mathcal{Q},z_o} \to \mathbb{P}^1$. $\mathrm{MW}(S_{\mathcal{Q},z_o})$ is

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identifies with $E_{\mathcal{Q},z_o}(\mathbb{C}(t))$, the set of $\mathbb{C}(t)$-rational points of the generic fiber, $E_{\mathcal{Q},z_o}$, of $\varphi : S_{\mathcal{Q},z_o} \to \mathbb{P}^1$. $\mathrm{MW}(S_{\mathcal{Q},z_o})$ is endowed with a structure of an abelian group as O is the zero element. We denote its addition and the multiplication-by-m map ($m \in \mathbb{Z}$) by $\dot{+}$ and $[m]$, respectively. For two sections $s_1, s_2 \in \mathrm{MW}(S_{\mathcal{Q},z_o})$, $s_1 \dot{+} s_2$ and $[m]s_i (i = 1, 2)$ give rise to new curves on $S_{\mathcal{Q},z_o}$, and their images $f_{\mathcal{Q},z_o}(s_1), f_{\mathcal{Q},z_o}(s_2), f_{\mathcal{Q},z_o}(s_1 \dot{+} s_2)$ and $f_{\mathcal{Q},z_o}([m]s_i)$ in $\mathbb{P}^2$ are expected to have interesting geometric properties. In previous articles [3], [13], [14], [15], we study geometry of $f_{\mathcal{Q},z_o}(s_1), f_{\mathcal{Q},z_o}(s_2), f_{\mathcal{Q},z_o}(s_1 \dot{+} s_2), f_{\mathcal{Q},z_o}([m]s_i)$ and $\mathcal{Q}$. As an application, we gave Zariski pairs whose irreducible components are those of these curves.

In this article, we continue to study geometry of plane curves along this line. More precisely, we study irreducible 3-nodal quartics and their contact conics. Here we call a smooth conic $\mathcal{C}$ a contact conic to a reduced plane curve $\mathcal{B}$ if the following condition is satisfied:

($*$) Let $I_x(\mathcal{C}, \mathcal{B})$ denotes the intersection multiplicity at $x \in \mathcal{C} \cap \mathcal{B}$. For $\forall x \in \mathcal{C} \cap \mathcal{B}$, $I_x(\mathcal{C}, \mathcal{B})$ is even and $\mathcal{B}$ is smooth at x.

An arrangement of rational curves consisting of a 3-nodal quartic and its contact conic can be regarded as a special case of rational curve arrangements studied in [2]. In [2], E. Artal Bartolo and the second author studied the topology of reducible curves having two irreducible components $\mathcal{C}$ and $\mathcal{D}$ such that

(i) $\mathcal{C}$ is a smooth conic,
(ii) $\mathcal{D}$ is a nodal rational curve of degree n, i.e., an irreducible curve with $(n-1)(n-2)/2$ nodes, and
(iii) $\mathcal{C}$ is tangent to $\mathcal{D}$ at n smooth distinct points of $\mathcal{D}$.

Let us first recall what was done in [2] briefly. Let $f_{\mathcal{C}} : Z_{\mathcal{C}} \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ branched along $\mathcal{C}$. $Z_{\mathcal{C}} \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the covering involution σ_f is given by switching the coordinate component. Hence $\mathrm{Pic}(Z_{\mathcal{C}}) = \mathbb{Z} \oplus \mathbb{Z}$ and if we denote an element of $\mathrm{Pic}(Z_{\mathcal{C}})$ by a pair of integers (a, b), we have $\sigma_f(a, b) = (b, a)$. By the the condition (iii) as above, $f_{\mathcal{C}}^*\mathcal{D}$ splits into two irreducible components and we denote them by $f_{\mathcal{C}}^*\mathcal{D} = \mathcal{D}^+ + \mathcal{D}^-$. Note that if $\mathcal{D}^+ \sim (a, b)$, then $\mathcal{D}^- \sim (b, a)$. In the following, we may assume that $\mathcal{D}^+$ is always chosen so that $\mathcal{D}^+ \sim (a, b)$, $a \le b$. We here introduce a terminology.

Definition 1 Let $\mathcal{C}$ be a contact conic to $\mathcal{D}$. We say that $\mathcal{C}$ is of type (a, b) with respect to $\mathcal{D}$ if $\mathcal{D}^+ \sim (a, b)$

In [1], [2], we have

Proposition 1 ([1, Section 3.5], [2]) *Let $\mathcal{D}_i$ $(i = 1, 2)$ be nodal rational curves of the same degree. Let $\mathcal{C}_i$ $(i = 1, 2)$ be contact conics to $\mathcal{D}_i$ $(i = 1, 2)$, respectively. Put $f^*_{\mathcal{C}_i} D_i = D_i^+ + D_i^-$, $\mathcal{D}_1^+ \sim (a_1, b_1)$ and $\mathcal{D}_2^+ \sim (a_2, b_2)$. If $(a_1, b_1) \neq (a_2, b_2)$, then $(\mathbb{P}^2, \mathcal{C}_1 + \mathcal{D}_1)$ is not homeomorphic to $(\mathbb{P}^2, \mathcal{C}_2 + \mathcal{D}_2)$. In particular, if $\mathcal{C}_1 + \mathcal{D}_1$ and $\mathcal{C}_2 + \mathcal{D}_2$ have the same combinatorics, $(\mathcal{C}_1 + \mathcal{D}_1, \mathcal{C}_2 + \mathcal{D}_2)$ is a Zariski pair* (*see* [1] *for a Zariski pair and terminologies related with it*).

Nodal rational curves $\mathcal{D}_1$ and $\mathcal{D}_2$ satisfying the condition in Proposition 1 appear from the case of $\deg \mathcal{D}_i \geq 4$. Our purpose of this article is to study the case of $\deg \mathcal{D} = 4$ in more detail. In [2], we gave an example of a conic $\mathcal{C}$ and irreducible 3-nodal quartics $\mathcal{Q}_1$ and $\mathcal{Q}_2$ such that

(i) $\mathcal{C}$ is a contact conic to both of $\mathcal{Q}_1$ and $\mathcal{Q}_2$, and
(ii) $\mathcal{Q}_1^+ \sim (2, 2)$, $\mathcal{Q}_2^+ \sim (1, 3)$.

On the other hand, in this article, we fix one irreducible 3-nodal quartic $\mathcal{Q}$ and several contact conics $\mathcal{C}$ to $\mathcal{Q}$ at one time. In [13], [14], we studied geometry of irreducible quartics $\mathcal{Q}$ and their contact conics $\mathcal{C}$ via rational elliptic surfaces $S_{\mathcal{Q}, z_o}$ for $z_o \in \mathcal{C} \cap \mathcal{Q}$. In the case when $\mathcal{Q}$ is an irreducible 3-nodal quartic, by [13], we have the following table:

	$l_{z_o} \cap \mathcal{Q}$	$\sharp \mathrm{CC}_{z_o}$
(I)	s	4
(II)	b	1
(III)	sb	2

Here

- l_{z_o} is the tangent line of $\mathcal{Q}$ at z_o and $l_{z_o} \cap \mathcal{Q}$ shows how l_{z_o} meets $\mathcal{Q}$. We use the following notation to describe it.
  - s: $I_{z_o}(l_{z_o}, \mathcal{Q}) = 2$ or 3, and l_{z_o} meets $\mathcal{Q}$ transversely at other point(s).
  - b: l_{z_o} is either bitangent line through z_o or $I_{z_o}(l_{z_o}, Q) = 4$.
  - sb: $I_{z_o}(l_{z_o}, \mathcal{Q}) = 2$ and l_{z_o} passes through a double point of $\mathcal{Q}$.
- CC_{z_o}: the set of contact conics passing through z_o. $\sharp \mathrm{CC}_{z_o}$ denotes its cardinality.

Now our problem in this article can be formulated as follows:

Problem 1 Choose a smooth point z_o of $\mathcal{Q}$. For $\mathcal{C} \in \mathrm{CC}_{z_o}$, determine the type of $\mathcal{C}$ with respect to $\mathcal{Q}$. In particular, in the cases of (I) and (III), do there exist contact conics $\mathcal{C}_1, \mathcal{C}_2 \in \mathrm{CC}_{z_o}$ such that $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) is of type $(2,2)$ (resp. $(1,3)$) with respect to $\mathcal{Q}$?

Since any $\mathcal{C} \in \mathrm{CC}_{z_o}$ gives rise to sections $s^{\pm}_{\mathcal{C}}$ in $\mathrm{MW}(S_{\mathcal{Q},z_o})$, we can apply our results of geometry and arithmetic of sections of $S_{\mathcal{Q},z_o}$ to these $s^{\pm}_{\mathcal{C}}$. This is an essential step to consider Problem 1. Our answer to Problem 1 is the following:

Theorem 1 *With the same notation as before, we have the table below*:

	$l_{z_o} \cap \mathcal{Q}$	$\sharp\mathrm{CC}_{z_o}$ *of type* $(2,2)$	$\sharp\mathrm{CC}_{z_o}$ *of type* $(1,3)$
(I)	s	3	1
(II)	b	0	1
(III)	sb	2	0

Moreover, if we choose homogeneous coordinates $[T, X, Z]$ *of* $\mathbb{P}^2$ *such that* $z_o = [0, 1, 0]$, $l_{z_o} : Z = 0$, $\mathcal{Q} : F_{\mathcal{Q}}(T, X, Z) = 0$ *and* $\mathcal{C} : F_{\mathcal{C}}(T, X, Z) = 0$, *then there exist homogeneous polynomials* $F_i(T, X, Z)$, $G_i(T, X, Z)$ *of degree* i *such that*

$$F_{\mathcal{Q}} = F_1^2 F_{\mathcal{C}} + G_2^2 \quad \textit{if and only if } \mathcal{C} \textit{ is of type } (2,2)$$

$$Z^2 F_{\mathcal{Q}} = F_2^2 F_{\mathcal{C}} + G_3^2 \quad \textit{if and only if } \mathcal{C} \textit{ is of type } (1,3)$$

Remark 1 The two equations in Theorem 1 give quasi-toric relations for $\mathcal{C} + \mathcal{Q}$ (see [5] for a quasi-toric relation).

Since the type of $\mathcal{C}$ does not depend on the choice of z_o, we have

Corollary 1 *Let* $\mathcal{C}$ *be a contact conic as in Theorem* 1.

(i) *If there exists a point* $z_o \in C \cap \mathcal{Q}$ *such that* l_{z_o} *is bitangent line to* $\mathcal{Q}$, *then the type of* $\mathcal{C}$ *with respect to* $\mathcal{Q}$ *is* $(1,3)$.
(ii) *If there exists a point* $z_o \in C \cap \mathcal{Q}$ *such that* l_{z_o} *passes through a node of* $\mathcal{Q}$, *then the type of* $\mathcal{C}$ *with respect to* $\mathcal{Q}$ *is* $(2,2)$.

Also by Proposition 1, we have the following corollary:

Corollary 2 *Let z_o be a general point of $\mathcal{Q}$. Then there exist contact conics $\mathcal{C}_1$ and $\mathcal{C}_2$ to $\mathcal{Q}$ such that* (i) $\mathcal{C}_i \in \mathrm{CC}_{z_o}$ $(i = 1, 2)$ *and* (ii) $(\mathbb{P}^2, \mathcal{C}_1 + \mathcal{Q})$ *is not homeomorphic to* $(\mathbb{P}^2, \mathcal{C}_2 + \mathcal{Q})$*. In particular, if $\mathcal{C}_1 + \mathcal{Q}$ and $\mathcal{C}_2 + \mathcal{Q}$ have the same combinatorics, then $(\mathcal{C}_1 + \mathcal{Q}, \mathcal{C}_2 + \mathcal{Q})$ is a Zariski pair.*

Note that the Zariski pair having the combinatorics in that of Corollary 2 can be found in [2]. In [2], we first consider a double cover $f_{\mathcal{C}} : Z_{\mathcal{C}} \to \mathbb{P}^2$ branched along a smooth conic $\mathcal{C}$. We then construct reduced curves $\mathcal{Q}_1^+$ and $\mathcal{Q}_2^+$ of types $(2, 2)$ and $(1, 3)$ on $Z_{\mathcal{C}}$, respectively. Two 3-nodal quartics $\mathcal{Q}_i$ $(i = 1, 2)$ such that $\mathcal{C}$ is a contact conic to both of $\mathcal{Q}_i$ $(i = 1, 2)$ are obtained as $\mathcal{Q}_i = f_{\mathcal{C}}(\mathcal{Q}_i^+)$ $(i = 1, 2)$. On the other hand, in this article, we consider $S_{\mathcal{Q},z_o}$ and contact conics are given by the image of sections of $S_{\mathcal{Q},z_o}$. Thus our construction is different. Also it would be an interesting question to determine whether the examples in Corollary 2 are deformation equivalent to those in [2] or not.

This paper consists of 5 section. In Section 1, we explain how to construct an irreducible 3-nodal quartic and give summary on various results on elliptic surfaces, which we need to prove Theorem 1. In Section 2, we study the structure of $S_{\mathcal{Q},z_o}$ and $\mathrm{MW}(S_{\mathcal{Q},z_o}) \cong E_{\mathcal{Q},z_o}(\mathbb{C}(t))$. In Section 3, we consider how we construct contact conics to $\mathcal{Q}$ via elementary arithmetic of $E_{\mathcal{Q},z_o}(\mathbb{C}(t))$. We prove Theorem 1 in Section 4 and give examples in Section 5 for cases (I), (II) and (III) in Theorem 1.

1. Preliminaries

1.1. Construction for irreducible 3-nodal quartics

Let $[T, X, Z]$ be homogeneous coordinates of $\mathbb{P}^2$. Let Q be the standard quadratic transformation or the standard Cremona transformation with respect to $\{T = 0\}, \{X = 0\}$ and $\{Z = 0\}$. We call $[0, 0, 1]$, $[0, 1, 0]$ and $[1, 0, 0]$, the fundamental points with respect to Q.

Lemma 1.1 (i) *Let C be a conic not tangent to any of three lines*: $\{T = 0\}, \{X = 0\}$ *and* $\{Z = 0\}$ *in* $\mathbb{P}^2$ *and passing through none of the three fundamental points. Then $Q(C)$ is a quartic whose singularities are only* 3 *nodes at* $[0, 0, 1], [0, 1, 0]$ *and* $[1, 0, 0]$.

(ii) *Let L be the line tangent to C at a point $P = [T_0, X_0, Z_0] \in C$, where $T_0X_0Z_0 \neq 0$. If L does not contain any of the fundamental points, then $Q(L)$ is a conic tangent to $Q(C)$ at $Q(P) = [X_0Z_0, T_0Z_0, T_0X_0]$*

and passes through $[0,0,1],[0,1,0]$ *and* $[1,0,0]$.

(iii) *Let* L *be the line tangent to* C *at a point* $P=[T_0,X_0,Z_0]\in C$*, where* $T_0X_0Z_0\neq 0$. *If* $[0,0,1]\in L$*, then* $Q(L)$ *is a line passing through* $[0,0,1]$.

(iv) *Let* L *be conic, that contains the fundamental points, then* $Q(L)$ *is a line.*

(v) *If* $x\in\mathbb{P}^2\setminus\{fundamental\ points\}$*, then* $I_x(C,L)=I_{Q(x)}(Q(C),Q(L))$.

Since all of these statements are well-known, we omit their proofs. We make use of Lemma 1.1 when we consider explicit examples in Section 5. Let $L_{Q(P)}$ be the tangent line to $Q(C)$ at $Q(P)$ and let Φ be a coordinate change such that $L_{Q(P)}$ is transformed into the line $Z=0$ and $Q(P)$ is mapped to $[0,1,0]$.

Then $\Phi(Q(C))$ has an affine equation of the form $x^3+b_2(t)x^2+b_3(t)x+b_4(t)=0$, where $t=T/Z, x=X/Z$, $b_i(t)\in\mathbb{C}[t]$ and $\deg_t b_i(t)\leq i$. Also $\Phi(Q(L))$ is given by an equation of the form $x-x_0(t)=0$, where $x_0(t)\in\mathbb{C}[t]$ and $\deg x_0(t)=2$.

1.2. Elliptic Surfaces

As for details on various results for elliptic surfaces, we refer to [3], [7], [16], [9], [10], [12], [14] and [15].

Throughout this article, an elliptic surface always means a smooth projective surface S with a fibration $\varphi: S\to C$ over a smooth projective curve, C, as follows:

(i) There exists non empty finite subset $\mathrm{Sing}(\varphi)\subset C$ such that $\varphi^{-1}(v)$ is a smooth curve of genus 1 for $v\in C\setminus\mathrm{Sing}(\varphi)$, while $\varphi^{-1}(v)$ is not a smooth curve of genus 1 for $v\in\mathrm{Sing}(\varphi)$.

(ii) There exists a section $O: C\to S$ (we identify O with its image in S).

(iii) there is no exceptional curve of the first kind in any fiber.

For $v\in\mathrm{Sing}(\varphi)$, we call $F_v=\varphi^{-1}(v)$ a singular fiber over v. As for the types of singular fibers, we use notation given by Kodaira ([7]). We denote the irreducible decomposition of F_v by

$$F_v=\Theta_{v,0}+\sum_{i=1}^{m_v-1}a_{v,i}\Theta_{v,i},$$

where m_v is the number of irreducible components of F_v and $\Theta_{v,0}$ is the irreducible component with $\Theta_{v,0}O = 1$. We call $\Theta_{v,0}$ *the identity component* of F_v. We also define a subset $\mathrm{Red}(\varphi)$ of $\mathrm{Sing}(\varphi)$ to be $\mathrm{Red}(\varphi) := \{v \in \mathrm{Sing}(\varphi) \mid F_v \text{ is reducible}\}$. For a section $s \in \mathrm{MW}(S)$, s is said to be *integral* if $sO = 0$.

Let $\mathrm{MW}(S)$ be the set of sections of $\varphi : S \to C$. By our assumption, $\mathrm{MW}(S) \neq \emptyset$. On a smooth fiber F of φ, by regarding $F \cap O$ as the zero element, we can consider the abelian group structure on F. Hence for $s_1, s_2 \in \mathrm{MW}(S)$, one can define the addition $s_1 \dot{+} s_2$ on $C \setminus \mathrm{Sing}(\varphi)$. By [7, Theorem 9.1], $s_1 \dot{+} s_2$ can be extended over C, and we can consider $\mathrm{MW}(S)$ as an abelian group. $\mathrm{MW}(S)$ is called the Mordell-Weil group. We also denote the multiplication-by-m map ($m \in \mathbb{Z}$) on $\mathrm{MW}(S)$ by $[m]s$ for $s \in \mathrm{MW}(S)$. Note that $[2]s$ is the double of s *with respect to the group law on* $\mathrm{MW}(S)$. On the other hand , we can regard the generic fiber $E := S_\eta$ of S as a curve of genus 1 over $\mathbb{C}(C)$, the rational function field of C. The restriction of O to E gives rise to a $\mathbb{C}(C)$-rational point of E, and one can regard E as an elliptic curve over $\mathbb{C}(C)$, O being the zero element. By considering the restriction to the generic fiber for each sections, $\mathrm{MW}(S)$ can be identified with the set of $\mathbb{C}(C)$-rational points $E(\mathbb{C}(C))$. For $s \in \mathrm{MW}(S)$, we denote the corresponding rational point by P_s. Conversely, for an element $P \in E(\mathbb{C}(C))$, we denote the corresponding section by s_P.

We also denote the addition and the multiplication-by-m map on $E(\mathbb{C}(C))$ by $P_1 \dot{+} P_2$ and $[m]P_1$ for $P_1, P_2 \in E(\mathbb{C}(C))$, respectively. Again, $[2]P$ is the double of P *with respect to the group law on* $E(\mathbb{C}(C))$.

For each singular fiber F_v, we associate it with finite abelian group $G_{F_v^\sharp}$, which is determined by irreducible components of F_v with $a_{v,i} = 1$ as follows:

Type of F_v	$G_{F_v^\sharp}$
I_b	$\mathbb{Z}/b\mathbb{Z}$
I_b^* (b: even)	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$
I_b^* (b: odd)	$\mathbb{Z}/4\mathbb{Z}$
II, II^*	$\{0\}$
III, III^*	$\mathbb{Z}/2\mathbb{Z}$
IV, IV^*	$\mathbb{Z}/3\mathbb{Z}$

We put $G_{\mathrm{Sing}(\varphi)} := \oplus_{v \in \mathrm{Sing}(\varphi)} G_{F_v^\sharp}$, and $\gamma : \mathrm{MW}(S) \to G_{\mathrm{Sing}(\varphi)}$ denotes the homomorphism as in [14, p. 83]. Note that for $s \in \mathrm{MW}(S)$, $\gamma(s)$ describes at which irreducible component s meets on F_v. For details, see [7, Section 9] or [14, pp. 81–83].

In [12], Shioda introduced a $\mathbb{Q}$-valued bilinear form on $E(\mathbb{C}(C))$ called the height pairing. We denote it by $\langle\, , \,\rangle$. It is known that $\langle P, P\rangle \geq 0$ for $\forall P \in E(\mathbb{C}(C))$ and the equality holds if and only if P is an element of finite order in $E(\mathbb{C}(C))$. For an explicit formula of $\langle P_1, P_2\rangle$ ($P_1, P_2 \in E(\mathbb{C}(C))$), see [12, Theorem 8.6].

We also remark double cover construction of an elliptic surface over $\mathbb{P}^1$. Let Σ_d be the Hirzebruch surface of degree d (d: even). Let $\mathfrak{f}$ be a fiber of $\Sigma_d \to \mathbb{P}^1$ and let Δ_0 and Δ be sections with self-intersection numbers $-d$ and d, respectively. Note that $\Delta \sim \Delta_0 + d\mathfrak{f}$ and $\Delta_0 \cap \Delta = \emptyset$.

Let $\mathcal{T}$ be a reduced divisor on Σ_d such that

- $\mathcal{T} \sim 3\Delta$, i.e., $\mathcal{T}$ is a tri-section with $\Delta_0 \cap \mathcal{T} = \emptyset$, and
- singularities of $\mathcal{T}$ are at worst simple (see [4] for simple singularities).

Since $\Delta_0 + \mathcal{T} \sim 2(2\Delta_0 + 3d/2\mathfrak{f})$, we have the double cover $f' : S' \to \Sigma_d$ with branch locus $\Delta_{f'} = \Delta_0 + \mathcal{T}$ (see [4, III, Section 7], for example). Let

$$\begin{tikzcd} S' \arrow[d, "f'"'] & S \arrow[l, "\mu"'] \arrow[d, "f"] \\ \Sigma_d & \widehat{\Sigma}_d. \arrow[l, "q"] \end{tikzcd}$$

denotes the diagram of the canonical resolution (see [6] for the canonical resolution). Namely, μ is the minimal resolution of singularities and q is a composition of blowing-ups so that the branch locus of f becomes smooth. Then the induced morphism $\varphi : S \to \Sigma_d \to \mathbb{P}^1$ gives rise to an elliptic vibration over $\mathbb{P}^1$, i.e., S is an elliptic surface over $\mathbb{P}^1$. Conversely it is known that any elliptic surface $\varphi : S \to \mathbb{P}^1$ is obtained this way ([9]).

An elliptic surface $\varphi : S \to \mathbb{P}^1$ is said to be rational if S is a rational surface. In the above diagram, we have an rational elliptic surface when $d = 2$. For a rational elliptic surface $\varphi : S \to \mathbb{P}^1$, if φ has a reducible singular fiber, $\widehat{\Sigma}_d$ in the above diagram can be blown down to $\mathbb{P}^2$ in such a way that $\mathcal{T}$ is transformed to a reduced quartic and O is mapped to a smooth point z_o on $\mathcal{Q}$. The induced morphism from $S \to \mathbb{P}^2$ is nothing but $f_{\mathcal{Q},z_o}$ explained in the Introduction.

Σ_d can be covered by 4 affine open sets U_i $(i = 1, 2, 3, 4)$ such that

- their local coordinates are

$$U_1 : (t, x), \quad U_2 : (s, x'), \quad U_3 : (t, u), \quad U_4 : (s, u').$$

- these coordinates are related by

$$s = 1/t, \quad x' = x/t^d, \quad u = 1/x, \quad u' = ut^d.$$

With these coordinates, Δ_0 is given by $u = 0$ and $u' = 0$ on U_3 and U_4, respectively. Also $\mathcal{T}$ is given by

$$f_{\mathcal{T}}(t, x) = x^3 + a_1(t)x^2 + a_2(t)x + a_3(t) = 0, \quad a_i \in \mathbb{C}[t], \deg a_i \leq id$$

on U_1 and $S'|_{f^{-1}}(U_1)$ is realized by

$$y^2 - f_{\mathcal{T}}(t, x) = 0 \subset \mathbb{C}^3.$$

We see that the covering map f' is the restriction of the projection $(t, x, y) \mapsto (t, x)$. The above equation can be regarded as a Weierstrass equation of the generic fiber, E, of $\varphi : S \to \mathbb{P}^1$, where $\mathbb{C}(\mathbb{P}^1)$ is identified with $\mathbb{C}(t)$, t being an inhomogeneous coordinate. Let $s \in \mathrm{MW}(S)$ be an integral section of S. Then we see that the coordinates of the corresponding rational point P_s are polynomial of degrees at most d (resp. $3d/2$) in the x-coordinate (resp. the y-coordinate). Conversely, for any point $P = (x(t), y(t)) \in E(\mathbb{C}(t))$ such that $x(t), y(t) \in \mathbb{C}[t]$ and $\deg x(t) \leq d, \deg y(t) \leq 3d/2$, s_P is an integral section. By an integral point, we mean a rational point corresponding to an integral section as above.

Choose an integral point $P_o = (x_o(t), y_o(t))$ of E with $y_o(t) \neq 0$ and let

$$y = l(t, x), \quad l(t, x) = m(t)(x - x_o(t)) + y_o(t), \quad m(t) = f_x(t, x_o(t))/2y_o(t)$$

be the tangent line at P_o.

Lemma 1.2 *If $[2]P_o$ is also integral, then $m(t) \in \mathbb{C}[t]$.*

See [16, Lemma 1.2] or [13, pp. 176–177].

Corollary 1.1 *Under the assumption of Lemma 1.2, if we put $[2]P_o = (x_1(t), y_1(t))$, then $f(t, x)$ has a decomposition*

$$f_{\mathcal{T}}(t,x) = (x - x_o(t))^2(x - x_1(t)) + \{l(t,x)\}^2.$$

2. Rational elliptic surface $S_{\mathcal{Q},z_o}$

Let $\mathcal{Q}$ be an irreducible 3-nodal quartic as before and let z_o be a smooth point on $\mathcal{Q}$. As we explain in the Introduction, we associate a rational elliptic surface with $\mathcal{Q}$ and z_o, which we denote by $\varphi : S_{\mathcal{Q},z_o} \to \mathbb{P}^1$. We also denote its generic fiber by $E_{\mathcal{Q},z_o}$.

The tangent line l_{z_o} gives rise to a singular fiber of φ whose type is determined by how l_{z_o} intersects with $\mathcal{Q}$ as follows:

(i)	I_2	l_{z_o} meets $\mathcal{Q}$ with two other distinct points.
(ii)	III	l_{z_o} is a 3-fold tangent point.
(iii)	I_3	l_{z_o} is a bitangent line.
(iv)	IV	l_{z_o} is a 4-fold tangent point.
(v)	I_4	l_{z_o} passes through a node of $\mathcal{Q}$

Other singular fibers are determined by how a line through z_o meets with $\mathcal{Q}$. Thus by taking [10, Table 6.2] into account and the above table, we have the following table for possible configurations of singular fibers of $S_{\mathcal{Q},z_o}$:

	Singular fibers	the position of l_{z_o}
1	$\{4\,\mathrm{I}_2, 4\,\mathrm{I}_1\}, \{4\,\mathrm{I}_2, 2\,\mathrm{I}_1, \mathrm{II}\}, \{4\,\mathrm{I}_2, 2\,\mathrm{II}\}$	(i)
2	$\{3\,\mathrm{I}_2, \mathrm{III}, 2\,\mathrm{I}_1\}, \{3\,\mathrm{I}_2, \mathrm{III}, \mathrm{II}\}$	(ii)
3	$\{\mathrm{I}_3, 3\,\mathrm{I}_2\, 3\,\mathrm{I}_1\}, \{\mathrm{I}_3, 3\,\mathrm{I}_2, \mathrm{I}_1, \mathrm{II}\}$	(iii)
4	$\{3\,\mathrm{I}_2, \mathrm{IV}, 2\,\mathrm{I}_1\}, \{3\,\mathrm{I}_2, \mathrm{IV}, \mathrm{II}\}$	(iv)
5	$\{\mathrm{I}_4, 2\,\mathrm{I}_2, 4\,\mathrm{I}_1\}, \{\mathrm{I}_4, 2\,\mathrm{I}_2, 2\,\mathrm{I}_1, \mathrm{II}\}, \{\mathrm{I}_4, 2\,\mathrm{I}_2, 2\,\mathrm{II}\}$	(v)

Note that cases 1, 2, cases 3, 4 and case 5 correspond to cases (I), (II) and (III) in Theorem 1, respectively. In our later argument, we need to know the structure of $E_{\mathcal{Q},z_o}(\mathbb{C}(t))$. We first note that $E_{\mathcal{Q},z_o}(\mathbb{C}(t))$ has no 2-torsion as $\mathcal{Q}$ is irreducible. Hence, by [11], the structure of $E_{\mathcal{Q},z_o}(\mathbb{C}(t))$ is as follows:

$$\text{(I) } (A_1^*)^{\oplus 4}, \quad \text{(II) } A_1^* \oplus \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \text{(III) } (A_1^*)^{\oplus 2} \oplus \langle 1/4 \rangle.$$

Also, since irreducible singular fibers and the difference between III (resp. IV) type and I_2 (resp. I_3) type do not affect the structure of $E_{\mathcal{Q},z_o}(\mathbb{C}(t))$ in these above cases, we may assume that the configurations of singular fibers are

$$\text{(I) } 4\,\mathrm{I}_2, 4\,\mathrm{I}_1, \quad \text{(II) } \mathrm{I}_3, 3\,\mathrm{I}_2, 3\,\mathrm{I}_1, \quad \text{(III) } \mathrm{I}_4, 2\,\mathrm{I}_2, 4\,\mathrm{I}_1\,.$$

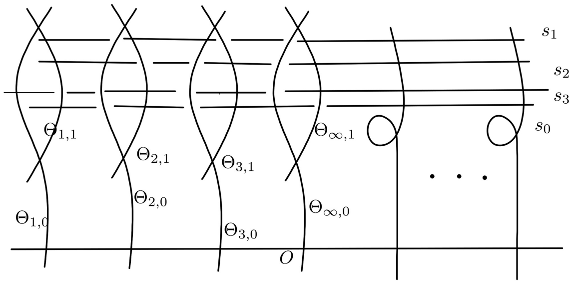
As we have seen in [13], an integral point P of $E_{\mathcal{Q},z_o}(\mathbb{C}(t))$ with $\langle P, P\rangle = 2$ gives rise to a contact conic to $\mathcal{Q}$. Hence we need to consider an integral element P with $\langle P, P\rangle = 2$ for each case. To this purpose, let us introduce some notation.

Let $\mathcal{L}_i$ $(i = 1, 2, 3)$ be three lines passing through two of the three nodes of $\mathcal{Q}$. For the cases (I) and (II), we denote a smooth conic tangent to $\mathcal{Q}$ at z_o and passing through the three nodes by $\overline{\mathcal{C}}$. Note that there is no smooth conic such as $\overline{\mathcal{C}}$ for the case (III), as l_{z_o} is also tangent to $\mathcal{Q}$ at z_o and passes through one of 3 nodes.

Then by our construction of $S_{\mathcal{Q},z_o}$, $\mathcal{L}_i$ $(i = 1, 2, 3)$ and $\overline{\mathcal{C}}$ give rise to sections $s^{\pm}_{\mathcal{L}_i}$ $(i = 1, 2, 3)$ and $s^{\pm}_{\overline{\mathcal{C}}}$. In the following, we put $s_i = s^{+}_{\mathcal{L}_i}$ $(i = 1, 2, 3)$ and $s_0 = s^{+}_{\overline{\mathcal{C}}}$. We denote the corresponding element to s_i in $E_{\mathcal{Q},z_o}(\mathbb{C}(t))$ by P_i for simplicity. We also write $[2]s_i$ $(i = 0, 1, 2, 3)$ for sections corresponding to $[2]P_i$ $(i = 0, 1, 2, 3)$, respectively.

Case (I). We label irreducible components of singular fibers of type I_2 in such a way that $\Theta_{i,1}$ $(i = 1, 2, 3)$ are those arising from the nodes of $\mathcal{Q}$ and $\Theta_{\infty,1}$ is the one from l_{z_o}. By our construction of $S_{\mathcal{Q},z_o}$, we may assume that s_i $(i = 0, 1, 2, 3)$ meet each singular fiber as in the figure below.

By [12, Theorem 8.6], we have



Case (I)

$$\langle P_i, P_i \rangle = \frac{1}{2},\ i = 0, 1, 2, 3, \quad \langle P_i, P_j \rangle = 0\ (i \neq j).$$

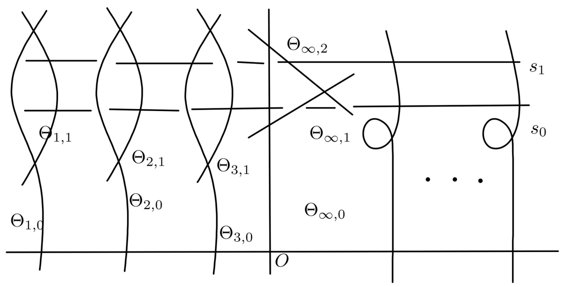
This means that $E_{\mathcal{Q},z_o}(\mathbb{C}(t))$ is generated by P_i $(i = 0, 1, 2, 3)$. As $\gamma([2]s_i) = 0$ and $\langle [2]P_i, [2]P_i \rangle = 2$ for each i, $[2]s_i$ is integral with $\langle [2]P_i, [2]P_i \rangle = 2$ and this means that $f_{\mathcal{Q},z_o}([2]s_i)$ is a contact conic to $\mathcal{Q}$ through z_o by [13, Lemma 2.1]. Conversely, for any contact conic $\mathcal{C} \in \mathrm{CC}_{z_o}$, the closure of $f^{-1}_{\mathcal{Q},z_o}(\mathcal{C} \setminus \{z_o\})$ consists of two integral sections $s^{\pm}_{\mathcal{C}}$ which intersect the identity component at each singular fiber, i.e., $\langle P_{s^{\pm}_{\mathcal{C}}}, P_{s^{\pm}_{\mathcal{C}}} \rangle = 2$. This means that the set of integral sections with height 2 up to $\pm$ are in one to one correspondence with CC_{z_o}. Thus we have four contact conics $\mathcal{C}_i$ $(i = 0, 1, 2, 3)$ in CC_{z_o} such that $\mathcal{C}_i = f_{\mathcal{Q},z_o}([2]s_i)$ $(i = 0, 1, 2, 3)$.

Case (II). We label irreducible components of singular fibers of type I_2 in the same way as in Case (I) and those of type I_3 such that $\Theta_{\infty,1}, \Theta_{\infty,2}$ are irreducible components from l_{z_o}. By our construction, s_i $(i = 1, 2, 3)$ meet two of $\Theta_{1,1}, \Theta_{2,1}$ and $\Theta_{3,1}$ at I_2 fibers and either $\Theta_{\infty,1}$ or $\Theta_{\infty,2}$, while s_0 meets $\Theta_{i,1}$ $(i = 1, 2, 3)$ at I_2 fibers and $\Theta_{\infty,0}$ at the I_3 fiber. In the figure below, we only draw s_1 and s_3 and assume that s_1 meets $\Theta_{\infty,1}$. By [12, Theorem 8.6], this means that

$$\langle P_i, P_i \rangle = \frac{1}{3},\ i = 1, 2, 3, \quad \langle P_0, P_0 \rangle = \frac{1}{2}.$$

As $\gamma([2]s_0) = 0$ and $\langle [2]P_0, [2]P_0 \rangle = 2$, $[2]P_0$ is integral and $f_{\mathcal{Q},z_o}([2]s_0)$ is a unique contact conic $\mathcal{C}_0$ to $\mathcal{Q}$ through z_o by [13, Lemma 2.1]. Hence the contact conic is obtained as $f_{\mathcal{Q},z_o}([2]s_0)$.

Case (III). Let z_1 be the node on l_{z_o}, and we may assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ pass through z_1. We label irreducible components of singular fibers

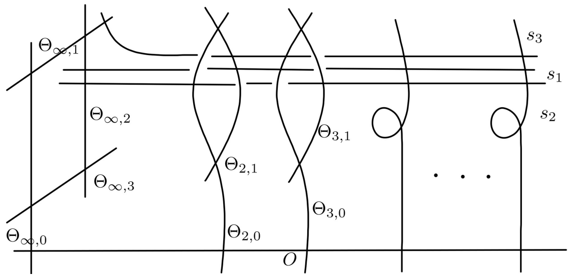


Case (II)

of type I_2 in the same way as in Case (I) and those of type I_4 so that $\Theta_{\infty,1}, \Theta_{\infty,3}$ are irreducible components from l_{z_o} and $\Theta_{\infty,2}$ is the one from the node z_1. Then we see that s_1 and s_2 meet one of $\Theta_{1,1}$ and $\Theta_{2,1}$ at I_2 fibers and $\Theta_{\infty,2}$ at the I_4 fiber, while s_3 meet $\Theta_{1,1}$ and $\Theta_{2,1}$ at I_2 fibers and either $\Theta_{\infty,1}$ or $\Theta_{\infty,3}$. In the figure below, we assume that s_3 meets $\Theta_{\infty,1}$. By [12, Theorem 8.6], this means that

$$\langle P_i, P_i \rangle = \frac{1}{2},\ i = 1, 2, \quad \langle P_3, P_3 \rangle = \frac{1}{4}.$$

As $\gamma([2]s_i) = 0$ and $\langle [2]P_i, [2]P_i \rangle = 2$ $(i = 1, 2)$, $[2]P_i$ is integral and this means that $f_{\mathcal{Q},z_o}([2]s_i)$ $(i = 1, 2)$ are contact conics to $\mathcal{Q}$ through z_o by [13, Lemma 2.1]. Hence we have two contact conics $f_{\mathcal{Q},z_o}([2]s_i)$ $(i = 1, 2)$.



Case (III)

3. Contact conics to 3-nodal quartic

Let $\mathcal{Q}$ be a 3-nodal quartic as before. Let z_1, z_2 and z_3 be the nodes of $\mathcal{Q}$ and let $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ be the lines through $\{z_1, z_2\}$, $\{z_1, z_3\}$ and $\{z_2, z_3\}$, respectively. Let z_o be the distinguished smooth point on $\mathcal{Q}$. For the cases (I) and (II), $\overline{\mathcal{C}}$ is the smooth conic tangent to $\mathcal{Q}$ at z_o and passes through z_1, z_2 and z_3.

Now we choose homogeneous coordinates $[T, X, Z]$ of $\mathbb{P}^2$ such that $z_o = [0, 1, 0]$ and $Z = 0$ is the tangent line of $\mathcal{Q}$ at z_o. Then we may assume that $\mathcal{Q}$ is given by a homogeneous polynomial $F_{\mathcal{Q}}(T, X, Z)$ of the form

$$F_{\mathcal{Q}}(T, X, Z) = ZX^3 + b_2(T, Z)X^2 + b_3(T, Z)X + b_4(T, Z).$$

Then the affine part of $\mathcal{Q}$, i.e., the part with $Z \neq 0$ is given by

$$F_{\mathcal{Q}}(t,x,1) = x^3 + b_2(t,1)x^2 + b_3(t,1)x + b_4(t,1).$$

For simplicity, we denote $b_i(t,1)$ by $b_i(t)$. By our choice of coordinates, the affine part of $\mathcal{L}_i$ is given by an equation of the form

$$x - x_i(t), \quad x_i(t) \in \mathbb{C}[t], \quad \deg x_i(t) = 1,$$

and that of $\overline{\mathcal{C}}$ is given by

$$x - x_0(t), \quad x_0(t) \in \mathbb{C}[t], \quad \deg x_0(t) = 2.$$

From our observation in Section 2, we have the following facts:

Case (I). There exist four integral points P_i $(i = 0,1,2,3)$ in $E_{\mathcal{Q},z_o}(\mathbb{C}(t))$ as follows:

(i) The x-coordinate of P_i $(i = 0,1,2,3)$ are $x_i(t)$ $(i = 0,1,2,3)$ as above, respectively.
(ii) $[2]P_i$ $(i = 0,1,2,3))$ are also integral.
(iii) Put $[2]P_i = (\tilde{x}_i(t), \tilde{y}_i(t))$. Then $\deg \tilde{x}_i(t) = 2$ and the conics given by $x - \tilde{x}_i(t) = 0$ $(i = 0,1,2,3)$ are contact conics to $\mathcal{Q}$ through z_o.

Case (II). There exists an integral point P_0 in $E_{\mathcal{Q},z_o}(\mathbb{C}(t))$ as follows:

(i) The x-coordinate of P_0 is $x_0(t)$ as above.
(ii) $[2]P_0$ are also integral.
(iii) Put $[2]P_0 = (\tilde{x}_0(t), \tilde{y}_0(t))$. Then $\deg \tilde{x}_0(t) = 2$ and the conics given by $x - \tilde{x}_0(t) = 0$ is the unique contact conic to $\mathcal{Q}$ through z_o.

Case (III). Suppose that the tangent line at l_{z_o} passes through z_1. There exist two integral points P_i $(i = 1,2)$ in $E_{\mathcal{Q},z_o}(\mathbb{C}(t))$ as follows:

(i) The x-coordinate of P_i $(i = 1,2)$ are $x_i(t)$ $(i = 1,2)$ as above, respectively.
(ii) $[2]P_i$ $(i = 1,2)$ are also integral.
(iii) Put $[2]P_i = (\tilde{x}_i(t), \tilde{y}_i(t))$ $(i = 1,2)$. Then $\deg \tilde{x}_i(t) = 2$ and the conics given by $x - \tilde{x}_i(t) = 0$ $(i = 1,2)$ are contact conics to $\mathcal{Q}$ through z_o.

We here introduce another terminology to describe these two kinds of contact conics as above:

Definition 3.1 Let $\mathcal{C}$ be a contact conic appeared in Proposition 3.1, we call $\mathcal{C}$ *a duplicated line* (*resp. a duplicated conic*) if $\deg(x - x_i(t)) = 1$ (resp. $= 2$).

By Corollary 1.1, we have decompositions as follows:

Proposition 3.1 *Let $\mathcal{Q}$, z_o and l_{z_o} be as in the Introduction, and we choose homogeneous coordinates $[T, X, Z]$ as in the Introduction. Put $t = T/Z$, $x = X/Z$.*

Case (I). *There exist 4 contact conics $\mathcal{C}_i$ $(i = 0, 1, 2, 3)$ to $\mathcal{Q}$ through z_o. We may assume that $\mathcal{C}_0$ is a duplicated conic, while $\mathcal{C}_i$ $(i = 1, 2, 3)$ are duplicated lines. For each $\mathcal{C}_i$, we have the following decomposition*:

$$F_{\mathcal{Q}}(t, x, 1) = (x - x_i(t))^2(x - \tilde{x}_i(t)) + \{l_i(t, x)\}^2, \ (i = 0, 1, 2, 3).$$

Case (II). *There exists a unique contact conic $\mathcal{C}_0$ to $\mathcal{Q}$ through z_o. $\mathcal{C}_0$ is a duplicated line and we have the following decomposition*:

$$F_{\mathcal{Q}}(t, x, 1) = (x - x_0(t))^2(x - \tilde{x}_0(t)) + \{l_0(t, x)\}^2.$$

Case (III). *There exist two contact conics $\mathcal{C}_i$ $(i = 1, 2)$ to $\mathcal{Q}$ through z_o. Both of $\mathcal{C}_i$ $(i = 1, 2)$ are duplicated lines and we have the following decompositions*:

$$F_{\mathcal{Q}}(t, x, 1) = (x - x_i(t))^2(x - \tilde{x}_i(t)) + \{l_i(t, x)\}^2, \ (i = 1, 2).$$

Note that, for each case as above, $\mathcal{C}_i$ is given by $x - \tilde{x}_i(t) = 0$ and $l_i(t, x)$ is a polynomial in $\mathbb{C}[t, x]$ such that $y = l_i(t, x)$ gives an equation of the tangent line of $E_{\mathcal{Q}, z_o}$ at P_i as above.

4. Proof of Theorem 1

Let $\mathcal{C}$ be a contact conic to $\mathcal{Q}$ and let $f_{\mathcal{C}} : Z_{\mathcal{C}} \to \mathbb{P}^2$ be the double cover branched along $\mathcal{C}$. $Z_{\mathcal{C}} \cong \mathbb{P}^1 \times \mathbb{P}^1$ and more explicitly, $Z_{\mathcal{C}}$ is a quadric surface in $\mathbb{P}^3$ given by

$$W^2 - (XZ - Z^2\tilde{x}(T/Z)) = 0,$$

where $x - \tilde{x}(t)$ is a defining equation of the affine part of $\mathcal{C}$. $f_{\mathcal{C}}$ is given by the restriction of the projection $\mathbb{P}^3 \setminus \{[0, 0, 0, 1]\} \to \mathbb{P}^2$ and the covering transformation is given by $[T, X, Z, W] \mapsto [T, X, Z, -W]$. Now we have the following proposition:

Proposition 4.1 *Let $\mathcal{C}$ be a contact conic to $\mathcal{Q}$.*

- *$\mathcal{C}$ is a duplicated line if and only if $\mathcal{C}$ is $(2,2)$ type with respect to $\mathcal{Q}$.*
- *$\mathcal{C}$ is a duplicated conic if and only if $\mathcal{C}$ is $(1,3)$ type with respect to $\mathcal{Q}$.*

Proof. Since any contact conic to $\mathcal{Q}$ is either a duplicated line or a duplicated conic, it is enough to show the following two statements:

- If $\mathcal{C}$ is a duplicated line, then $\mathcal{C}$ is $(2,2)$ type with respect to $\mathcal{Q}$.
- If $\mathcal{C}$ is a duplicated conic, $\mathcal{C}$ is $(1,3)$ type with respect to $\mathcal{Q}$.

We write the corresponding decomposition with respect to $\mathcal{C}$ given in Proposition 3.1:

$$F_{\mathcal{Q}}(t, x, 1) = (x - x(t))^2(x - \tilde{x}(t)) + \{l(t, x)\}^2, \tag{1}$$

where the affine part of $\mathcal{C}$ is given by $x - \tilde{x}(t) = 0$.

The case when $\mathcal{C}$ is a duplicated line. Since $\deg(x - x(t)) = 1$ and $\deg(x - \tilde{x}(t)) = 2$, $\deg(l(t, x)) \leq 2$. Hence by homogenizing the decomposition (1), we have

$$F_{\mathcal{Q}}(T, X, Z) = (X - Zx(T/Z))^2(XZ - Z^2\tilde{x}(T/Z)) + \{Z^2 l(T/Z, X/Z)\}^2.$$

Put $f_{\mathcal{C}}^*\mathcal{Q} = \mathcal{Q}^+ + \mathcal{Q}^-$. As $Z_{\mathcal{C}}$ is defined by $W^2 - (XZ - Z^2\tilde{x}_i(T/Z)) = 0$, we may assume that

$$\mathcal{Q}^{\pm} = Z_{\mathcal{C}} \cap \left\{(X - Zx(T/Z))W \pm \sqrt{-1}Z^2 l(T/Z, X/Z) = 0\right\}.$$

Since a divisor on $Z_{\mathcal{C}}$ cut out by a quadric surface is of type $(2,2)$, we have the assertion.

The case when $\mathcal{C}$ is a duplicated conic. In this case, $\deg(x - x(t)) = 2$ and $\deg(x - \tilde{x}(t)) = 2$. By homogenizing the decomposition (1), we have

$$\begin{aligned} Z^2 F_{\mathcal{Q}}(T, X, Z) = {} & (XZ - Z^2 x(T/Z))^2(XZ - Z^2\tilde{x}(T/Z)) \\ & + \{Z^3 l(T/Z, X/Z)\}^2. \end{aligned} \tag{2}$$

Put $x(t) = c_0t^2 + c_1t + c_2, \tilde{x}(t) = d_0t^2 + d_1t + d_2, c_0d_0 \neq 0$, and $l(t,x)$ is of the form

$$(a_0t + a_1)x + (b_0t^3 + b_1t^2 + b_2t + b_3) \quad a_i, b_j \in \mathbb{C},\ b_0 \neq 0$$

by comparing monomials appearing the both hand of (2).

Since l_{z_o} is given by $Z = 0$ and $W^2 - (XZ - Z^2\tilde{x}(T/Z)) = 0$, we have

$$f_{\mathcal{C}}^* l_{z_o} = l^+ \cup l^-, \quad l^\pm = \big\{[T, X, 0, \pm\sqrt{-d_0}T] \in \mathbb{P}^3\big\}.$$

Hence from the decomposition (2), we have

$$2(l^+ + l^-) + (\mathcal{Q}^+ + \mathcal{Q}^-) = D^+ + D^-,$$

where $D^\pm$ are divisors scheme-theoretically given by

$$Z_{\mathcal{C}} \cap \big\{(XZ - Z^2x(T/Z))W \pm \sqrt{-1}Z^3 l(T/Z, X/Z) = 0\big\},$$

respectively. Since $D^\pm \sim (3,3)$, we may assume either (a) or (b) below holds:

(a) $l^+ + l^- + \mathcal{Q}^+ = D^+$, or
(b) $2l^+ + \mathcal{Q}^+ = D^+$,

We show that the case (a) does not occur. Choose a point $[T, X, 0, \sqrt{-d_0}T] \in l^+ \subset D^+, T \neq 0$. If the case (a) happens, $[T, X, 0, -\sqrt{-d_0}T] \in l^- \subset D^+$. On the other hand, if $[T, X, 0, \sqrt{-d_0}T] \in l^+$, as $l^+ \subset D^+$, we have

$$-c_0T^2\big(\sqrt{-d_0}T\big) + \sqrt{-1}b_0T^3 = \big(-c_0\sqrt{-d_0} + \sqrt{-1}b_0\big)T^3 = 0.$$

Hence we have

$$c_0T^2\big(\sqrt{-d_0}T\big) + \sqrt{-1}b_0T^3 = \big(c_0\sqrt{-d_0} + \sqrt{-1}b_0\big)T^3 \neq 0.$$

This means that $[T, X, 0, -\sqrt{-d_0}T] \notin D^+$ for $T \neq 0$. This leads us to a contradiction. $\square$

From Propositions 3.1 and 4.1, Theorem 1 follows.

5. Examples

We end this paper by giving explicit example for an irreducible 3-nodal quartic and its contact conics observed so far, by which we have some examples of Zariski pairs. As for homogeneous coordinates of $\mathbb{P}^2$ we keep our previous notation, $[T, X, Z]$. In order to give curves by explicit equations, we make use of our observation in Section 1, 1.1.

We first consider the case (I) of Theorem 1. Let C be a conic given by the equation $XZ - T^2 = 0$. Let Q denote the standard quadratic transformations with respect to three lines: $-3T + X + 2Z = 0, 3T + X + 2Z = 0$, $X - 2Z = 0$. Let $l : Z = 0$ be the tangent line at $p = [0, 1, 0]$ to C.

Let us denote $\mathcal{Q} := Q(C)$, $\overline{\mathcal{C}} := Q(l)$ and $z_o := Q(p)$. Let l_{z_o} be the tangent line to $\mathcal{Q}$ at z_o. Then we have the equations of $\mathcal{Q}$, $\overline{\mathcal{C}}$ and l_{z_o} as follows:

$$
\begin{aligned}
F_{\mathcal{Q}} &= 36T^2X^2 - T^2Z^2 - 34TXZ^2 - X^2Z^2 \\
F_{\overline{\mathcal{C}}} &= 2TX - TZ - XZ \\
z_o &= [1, 1, 1] \\
F_{l_{z_o}} &= T + X - 2Z
\end{aligned}
$$

We see that, $\mathcal{Q}$ is a quartic and $\overline{\mathcal{C}}$ is a conic passing through 3 nodes and tangent to $\mathcal{Q}$ at z_o. Also l_{z_o} meets $\mathcal{Q}$ with two other distinct points.

Let $E := E_{\mathcal{Q},z_o}$ be a generic fiber of rational elliptic surface $S_{\mathcal{Q},z_o}$ and $E(\mathbb{C}(t))$ be the set of rational points and the point at infinity O. Let Φ be a coordinate change such that l_{z_o} is transformed into the line $Z = 0$ and z_o is mapped to $[0, 1, 0]$. Then $\Phi(\mathcal{Q})$ and $\Phi(\overline{\mathcal{C}})$ are given by the affine equations as follows:

$$
\begin{aligned}
F_{\Phi(\mathcal{Q})} &= x^3 + \frac{5}{36}(8t^2 + 8t - 7)x^2 + (-2t^2 - 2t)x - t^2(t + 1)^2 = 0 \\
F_{\Phi(\overline{\mathcal{C}})} &= x + 2t^2 + 2t = 0,
\end{aligned}
$$

where $t = T/Z$ and $x = X/Z$.

Note that $\Phi(\mathcal{Q})$ has 3 nodes at $[0, 0, 1], [-1/2, 1/2, 1]$ and $[-1, 0, 1]$. Three lines passing through two of the 3 nodes together with $\Phi(\overline{\mathcal{C}})$ correspond to rational points in $E(\mathbb{C}(t))$ as shown in the table below:

Equations	Rational points
$x+2t^2+2t=0$	$P_0^{\pm}=\left(-2t^2-2t, \pm\frac{2\sqrt{-2}}{3}t(1+2t)(1+t)\right)$
$x+t=0$	$P_1^{\pm}=\left(-t, \pm\frac{t(1+2t)}{6}\right)$
$x=0$	$P_2^{\pm}=\left(0, \pm\sqrt{-1}(t+1)t\right)$
$x-t-1=0$	$P_3^{\pm}=\left(t+1, \pm\frac{(2t+1)(t+1)}{6}\right)$

Since we have $\langle P_i^+, P_j^+\rangle = (1/2)\delta_{ij}$ for $i, j = 0, 1, 2, 3$, we assume that $E(\mathbb{C}(t))$ is generated by P_0^+, P_1^+, P_2^+ and P_3^+. Also we have the following table for $[2]P_i^+, (i = 0, 1, 2, 3)$:

Duplicated points of $P_i^+, (i=0,1,2,3)$
$[2]P_0^+=\left(-\frac{9}{8}t^2-\frac{9}{8}t-\frac{1}{32}, -\frac{\sqrt{-2}}{768}(72t^3+108t^2+70t+17)\right)$
$[2]P_1^+=\left(10t^2+2t+1, \frac{100}{3}t^3+\frac{34}{3}t^2+\frac{11}{3}t+\frac{1}{6}\right)$
$[2]P_2^+=\left(-\frac{10}{9}t^2-\frac{10}{9}t-\frac{1}{36}, \frac{\sqrt{-1}(4t^2+4t+1)}{36}\right)$
$[2]P_3^+=\left(10t^2+18t+9, -\frac{100}{3}t^3-\frac{266}{3}t^2-81t-\frac{51}{2}\right)$

Note that, if we denote $\mathcal{C}_0 : 32x+36t^2+36t+1 = 0$, $\mathcal{C}_1 : x-10t^2-2t-1 = 0$, $\mathcal{C}_2 : 36x + 40t^2 + 40t - 71 = 0$, $\mathcal{C}_3 : x - 10t^2 - 18t - 9 = 0$, then $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ are duplicated lines, while $\mathcal{C}_{1,0}$ is a duplicated conic. By Proposition 3.2 $\mathcal{C}_j, (j = 1, 2, 3)$ are $(2, 2)$ type and $\mathcal{C}_0$ is $(1, 3)$ type with respect to $\mathcal{Q}$. Also we have for $i = 0, 1, 3$, the number of tangent points of $\mathcal{C}_i$ to $\mathcal{Q}$ is equal to 4, while the number of tangent points of $\mathcal{C}_2$ to $\mathcal{Q}$ is equal to 2. This means that $\mathcal{C}_j + \mathcal{Q}$ $(j = 1, 3)$ and $\mathcal{C}_0 + \mathcal{Q}$ have the same combinatorics. Hence, by Corollary 2, $(\mathcal{C}_j + \mathcal{Q}, \mathcal{C}_0 + \mathcal{Q}), (j = 1, 3)$ are Zariski pairs.

Similarly, we have explicit examples for the cases (II) and (III). We end this section by giving explicit equations of $\mathcal{Q}$ and contact conics to $\mathcal{Q}$ for both cases:

Case (II). Let $\mathcal{Q}$ be a quartic and let l be a line as follows:

$$\begin{aligned} \mathcal{Q}:\ & X^3Z + \left(6T^2 - 6TZ + \frac{7}{6}Z^2\right)X^2 + \left(-24T^3 + 15T^2Z - \frac{7}{3}TZ^2\right)X \\ & \quad + 24T^4 - 16T^3Z + \frac{8}{3}T^2Z^2 = 0 \\ l:\ & Z = 0 \\ z_o:\ & [0,1,0]. \end{aligned}$$

Then l is a bitangent line to $\mathcal{Q}$ at z_o. By Theorem 1 we have only one contact conic of $(1,3)$ type which is given by the equation: $48XZ - 36T^2 - 60TZ + 7Z^2 = 0$.

Case (III). Let $\mathcal{Q}$ be a quartic, and let l be a line as follows:

$$\begin{aligned} \mathcal{Q}:\ & 2X^3Z + (T^2 + TZ + 4Z^2)X^2 + (-2T^3 - T^2Z + 3TZ^2)X \\ & \quad + T^4 - 2T^3Z + T^2Z^2 = 0 \\ l:\ & Z = 0 \\ z_o:\ & [0,1,0]. \end{aligned}$$

Then l is tangent to $\mathcal{Q}$ at z_o and pass through one of nodes at $[1,1,0]$. By Theorem 1 we have two contact conics of $(2,2)$ type which are given by the equations: $64XZ - 17T^2 + 14TZ + 7Z^2 = 0$ and $16XZ - 17T^2 + 20TZ - 4Z^2 = 0$.

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