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A moment problem on rational numbers

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Abstract. We give integral representations of positive and negative definite functions defined on an interval in a certain subsemigroup of the semigroup of rational numbers.

Key words: moment problem, positive definite function, semigroup.

1. Introduction

It was shown by D. V. Widder [8, Thereom A] that a real-valued function $f(x)$ defined on an open interval (a, b) in $\boldsymbol{R}$ has a form

$$f(x) = \int_{\boldsymbol{R}} e^{-xt}\, d\alpha(t), \quad a < x < b, \tag{1.1}$$

where $\alpha(t)$ is a non-decreasing function on $\boldsymbol{R}$ if and only if $f(x)$ is continuous and positive definite. Here the function $f(x)$ is said to be *positive definite* on (a, b) if

$$\sum_{i,j=1}^{n} c_i c_j f(x_i + x_j) \geq 0$$

for every $n \geq 1$ and for every $c_1, c_2, \ldots, c_n, x_1, \ldots, x_n \in \boldsymbol{R}$ such that $2x_i \in (a, b)$ for $i = 1, 2, \ldots, n$.

In this paper, we concern positive definite functions defined on a subset of the additive semigroup $\boldsymbol{Q}$ of rational numbers. Let $\overrightarrow{m} = \{m_n\}_{n=1}^{\infty}$ be a sequence of integers greater than or equal to 2, and let $S(\overrightarrow{m})$ be the subsemigroup of $\boldsymbol{Q}$ defined by

$$S(\overrightarrow{m}) = \left\{ \frac{k}{m_1 \cdots m_n} : k \in \boldsymbol{Z},\, n \geq 1 \right\},$$

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where $\boldsymbol{Z}$ denotes the set of all integers. For example, if $m_n = n+1$ for $n \geq 1$, we have $S(\overrightarrow{m}) = \boldsymbol{Q}$, and if $m_n = 2$ for $n \geq 1$, $S(\overrightarrow{m})$ is the set of all dyadic rational numbers.

Let I denote a finite or infinite interval in $\boldsymbol{R}$, and let $\varphi : I \cap S(\overrightarrow{m}) \to \boldsymbol{R}$ be a real-valued function on $I \cap S(\overrightarrow{m})$. We say φ is *positive definite* if

$$\sum_{i,j=1}^{n} c_i c_j \varphi(r_i + r_j) \geq 0$$

for all $n \geq 1$, $c_1, c_2, \ldots, c_n \in \boldsymbol{R}$ and $r_1, r_2, \ldots, r_n \in S(\overrightarrow{m})$ such that $2r_i \in I \cap S(\overrightarrow{m})$ for $i = 1, 2, \ldots, n$. The purpose of this paper is to show that every positive definite function on $I \cap S(\overrightarrow{m})$ has an integral representation such as (1.1) (see Section 2). The result we obtain will generalize the results of N. Sakakibara for the case $I = [0, \infty)$ ([7, Theorem 2.2]) and D. Atanasiu for the case $S(\overrightarrow{m}) = \boldsymbol{Q}$ ([2, Theorem 1, Proposition 1]). In Section 3, we extend the result of Section 2 to the case where φ is a mapping of $I \cap S(\overrightarrow{m})$ into the space of bounded linear operators on a complex Hilbert space. We also give a Lévy–Khinchin type formula for negative definite functions on $I \cap S(\overrightarrow{m})$ in Section 2.

2. Integral representations of positive and negative definite functions

First we consider the case where I is an open interval. Define the function χ on $S(\overrightarrow{m})$ as follows (cf. [7]):

If the sequence $\overrightarrow{m} = \{m_n\}_{n=1}^{\infty}$ contains no even numbers, put

$$\chi\left(\frac{k}{m_1 \cdots m_n}\right) = (-1)^k, \qquad \frac{k}{m_1 \cdots m_n} \in S(\overrightarrow{m});$$

If $\overrightarrow{m}$ contains only finitely many even numbers, we may suppose that $m_1, \ldots, m_p$ are even and m_q $(q > p)$ are odd. Then we put

$$\chi\left(\frac{k}{m_1 \cdots m_p m_{p+1} \cdots m_n}\right) = (-1)^k, \qquad \frac{k}{m_1 \cdots m_p m_{p+1} \cdots m_n} \in S(\overrightarrow{m}).$$

It is clear that χ is well-defined and multiplicative, i.e.

$$\chi(r_1 + r_2) = \chi(r_1)\chi(r_2), \qquad r_1, r_2 \in S(\overrightarrow{m}).$$

In fact, the functions $r \in S(\overrightarrow{m}) \mapsto e^{rx}$ and $r \in S(\overrightarrow{m}) \mapsto \chi(r)e^{rx}$, where $x \in \boldsymbol{R}$, are the semicharacters of $S(\overrightarrow{m})$ [7].

Throughout the paper, $E_+(I, A)$ denotes the set of all positive Radon measures μ on A, where A is an open or closed subset of $\boldsymbol{R}$, such that the function $x \mapsto e^{rx}$ is μ-integrable for all $r \in I$.

Theorem 2.1 *Let $a, b \in \boldsymbol{R} \cup \{-\infty, \infty\}$ such that $a < b$ and let $\overrightarrow{m} = \{m_n\}_{n=1}^{\infty}$ be a sequence of integers $m_n \geq 2$. Let φ be a positive definite function on $(a, b) \cap S(\overrightarrow{m})$.*

(1) *If the sequence $\overrightarrow{m}$ contains at most finitely many even numbers, then there exist positive Radon measures $\mu, \nu \in E_+((a, b), \boldsymbol{R})$ such that*

$$\varphi(r) = \int_{\boldsymbol{R}} e^{rx} d\mu(x) + \int_{\boldsymbol{R}} \chi(r)e^{rx} d\nu(x), \quad r \in (a, b) \cap S(\overrightarrow{m}).$$

Moreover, the pair (μ, ν) is uniquely determined by φ.

(2) *If the sequence $\overrightarrow{m}$ contains infinitely many even numbers, then there exists a uniquely determined measure $\mu \in E_+((a, b), \boldsymbol{R})$ such that*

$$\varphi(r) = \int_{\boldsymbol{R}} e^{rx} d\mu(x), \qquad r \in (a, b) \cap S(\overrightarrow{m}).$$

Proof. (1) Fix $\alpha, \beta \in 2S(\overrightarrow{m}) = \{2r : r \in S(\overrightarrow{m})\}$ which satisfy $a < \alpha < \beta < b$. Using the above notation, we may put $\alpha = d_0(m_1 \cdots m_q)^{-1}, \beta = d_1(m_1 \cdots m_q)^{-1}$ where d_0 and d_1 are even numbers and $q > p$. For a fixed integer $n \geq 1$, put $M_n = (d_1 - d_0)m_{q+1} \cdots m_{q+n}$ and $L_n = m_1 \cdots m_{q+n}$, and define the sequence $\{s_k\}_{k=0}^{M_n}$ by $s_k = \varphi(\alpha + k/L_n), k = 0, 1, \ldots, M_n$. Then for $c_i \in \boldsymbol{R}$, $i = 0, 1, \ldots, M_n/2$, we have

$$\sum_{i,j=0}^{M_n/2} c_i c_j s_{i+j} = \sum_{i,j=0}^{M_n/2} c_i c_j \varphi\left(\left(\frac{\alpha}{2} + \frac{i}{L_n}\right) + \left(\frac{\alpha}{2} + \frac{j}{L_n}\right)\right) \geq 0,$$

because of the positive definiteness of φ. By [1, Theorem 2.6.3], there exists a finite positive Radon measure τ_n on $\boldsymbol{R}$ such that

$$\varphi\left(\alpha + \frac{k}{L_n}\right) = \int_{\boldsymbol{R}} t^k \, d\tau_n(t), \qquad k = 0, 1, \ldots, M_n - 1,$$

and $\varphi(\alpha + k/L_n) = \varphi(\beta) \geq \int_{\boldsymbol{R}} t^k d\tau_n(t)$ for $k = M_n$. Define the mappings f_n and g_n on $\underline{\boldsymbol{R}} = \boldsymbol{R} \cup \{-\infty\}$ by

$$f_n : \underline{\boldsymbol{R}} \to [0, \infty); \qquad f_n(x) = \exp \frac{x}{L_n},$$

$$g_n : \underline{\boldsymbol{R}} \to (-\infty, 0]; \quad g_n(x) = -\exp \frac{x}{L_n},$$

and let μ_n, ν_n be positive Radon measures on $\underline{\boldsymbol{R}}$ which satisfy

$$\mu_n \circ f_n^{-1} = \tau_n|_{[0,\infty)}, \qquad \nu_n \circ g_n^{-1} = \tau_n|_{(-\infty,0]},$$

respectively. Then we have

$$\varphi\left(\alpha + \frac{k}{L_n}\right) = \int_{\underline{\boldsymbol{R}}} \exp \frac{kx}{L_n} \, d\mu_n(x) + \int_{\underline{\boldsymbol{R}}} (-1)^k \exp \frac{kx}{L_n} \, d\nu_n(x).$$

Since $\mu_n(\underline{\boldsymbol{R}}) + \nu_n(\underline{\boldsymbol{R}}) = \varphi(\alpha) < +\infty$ for all $n \geq 1$, there exist subsequences $\{\mu_{n_i}\}_{i=1}^{\infty}$ and $\{\nu_{n_i}\}_{i=1}^{\infty}$ which converge vaguely to positive measures μ_0 and ν_0 respectively (see [2, Proposition 2.4.6, 2.4.10]). Put $s = k/L_n$ for $k = 0, 1, \ldots, M_n - 1$. If $n_i > n$, we have

$$\varphi(\alpha + s) = \varphi\left(\alpha + \frac{k m_{q+n+1} \cdots m_{q+n_i}}{m_1 \cdots m_{q+n_i}}\right) \tag{2.1}$$

$$= \int_{\underline{\boldsymbol{R}}} e^{sx} \, d\mu_{n_i}(x) + \int_{\underline{\boldsymbol{R}}} \chi(s) e^{sx} \, d\nu_{n_i}(x). \tag{2.2}$$

Using that for each nonnegative continuous function f on a locally compact space the integral $\int f \, d\mu$ is lower semicontinuous in μ with respect to the vague topology, we find that for $s = k/L_n$, $k = 0, 1, \ldots, M_n/2$,

$$\int_{\underline{\boldsymbol{R}}} e^{2sx} d\mu_0(x) \leq \varphi(\alpha + 2s), \quad \int_{\underline{\boldsymbol{R}}} e^{2sx} \, d\nu_0(x) \leq \varphi(\alpha + 2s).$$

Since $e^{sx} \leq (1 + e^{2sx})/2$, it follows that the function e^{sx} is μ_0- (and ν_0-) integrable for $s = k/L_n$, $k = 0, 1, \ldots, M_n - 1$. Define the function $h(x)$ by

$h(x) = 1 + e^{2(s+1)x}$. Then the sequence $\{h\mu_{n_i}\}_{i=1}^{\infty}$ converges to $h\mu_0$ vaguely and

$$\sup_{i\geq 1} \int_{\underline{\boldsymbol{R}}} h(x) d\mu_{n_i}(x) \leq \varphi(\alpha) + \varphi(\alpha + 2(s+1)).$$

Therefore, since $e^{sx}/h(x)$ is a continuous function on $\underline{\boldsymbol{R}}$ vanishing at infinity, we have

$$\begin{aligned}
\lim_{i\to\infty} \int_{\underline{\boldsymbol{R}}} e^{sx}\, d\mu_{n_i}(x) &= \lim_{i\to\infty} \int_{\underline{\boldsymbol{R}}} \frac{e^{sx}}{h(x)} h(x)\, d\mu_{n_i}(x) \\
&= \int_{\underline{\boldsymbol{R}}} \frac{e^{sx}}{h(x)} h(x)\, d\mu_0(x) \\
&= \int_{\underline{\boldsymbol{R}}} e^{sx}\, d\mu_0(x),
\end{aligned}$$

and similarly

$$\lim_{i\to\infty} \int_{\underline{\boldsymbol{R}}} e^{sx}\, d\nu_{n_i}(x) = \int_{\underline{\boldsymbol{R}}} e^{sx} d\nu_0(x).$$

Thus by (2.2) we have

$$\varphi(\alpha + s) = \int_{\underline{\boldsymbol{R}}} e^{sx} d\mu_0(x) + \int_{\underline{\boldsymbol{R}}} \chi(s) e^{sx} d\sigma_0(x),$$

for $s = k/L_n$, $k = 0, 1, \ldots, M_n - 1$. Since this equality holds for every $n \geq 1$, putting $r = \alpha + s$ we see that

$$\varphi(r) = \int_{\boldsymbol{R}} e^{rx} e^{-\alpha x}\, d(\mu_0|_{\boldsymbol{R}})(x) + \int_{\boldsymbol{R}} \chi(r) e^{rx} e^{-\alpha x}\, d(\nu_0|_{\boldsymbol{R}})(x),$$
$$r \in (\alpha, \beta) \cap S(\overrightarrow{m}).$$

Consequently

$$\varphi(r) = \int_{\boldsymbol{R}} e^{rx}\, d\mu(x) + \int_{\boldsymbol{R}} \chi(r) e^{rx}\, d\nu(x), \quad r \in (\alpha, \beta) \cap S(\overrightarrow{m}), \tag{2.3}$$

where $d\mu(x) = e^{-\alpha x}\, d(\mu_0|_{\boldsymbol{R}})(x)$ and $d\nu(x) = e^{-\alpha x}\, d(\nu_0|_{\boldsymbol{R}})(x)$.

We now prove the uniqueness of μ and ν. Suppose that

$$\int_{\boldsymbol{R}} e^{rx}\, d\mu(x) + \int_{\boldsymbol{R}} \chi(r) e^{rx}\, d\nu(x) = \int_{\boldsymbol{R}} e^{rx}\, d\mu'(x) + \int_{\boldsymbol{R}} \chi(r) e^{rx}\, d\nu'(x),$$

for $r \in (\alpha, \beta) \cap S(\overrightarrow{m})$. If $r \in (\alpha, \beta) \cap 2S(\overrightarrow{m})$, we have

$$\int_{\boldsymbol{R}} e^{rx}\, d(\mu + \nu - \mu' - \nu')(x) = 0.$$

Define the holomorphic function $\Phi(z)$ on the strip $\alpha < \operatorname{Re} z < \beta$ in the complex plane by

$$\Phi(z) = \int_{\boldsymbol{R}} e^{zx}\, d(\mu + \nu - \mu' - \nu')(x).$$

Then the identity theorem ensures that $\Phi(z) \equiv 0$ for $\alpha < \operatorname{Re} z < \beta$ because the set $(\alpha, \beta) \cap 2S(\overrightarrow{m})$ is dense in the interval (α, β). Since for fixed $\alpha < \gamma < \beta$, the function $y \in \boldsymbol{R} \mapsto \Phi(\gamma + iy)$ is the Fourier transform of $e^{\gamma}(\mu + \nu - \mu' - \nu')$, it follows that $\mu + \nu - \mu' - \nu' = 0$. Using a similar argument for the equality

$$\int_{\boldsymbol{R}} e^{rx}\, d(\mu - \nu - \mu' + \nu')(x) = 0, \quad r \in (\alpha, \beta) \cap (S(\overrightarrow{m}) \setminus 2S(\overrightarrow{m})),$$

we have $\mu - \nu - \mu' + \nu' = 0$. Consequently $\mu = \mu'$ and $\nu = \nu'$. Since α and β are arbitrary, we conclude that (2.3) is valid not only for $r \in (\alpha, \beta) \cap S(\overrightarrow{m})$ but also for $r \in (a, b) \cap S(\overrightarrow{m})$, and that the pair (μ, ν) is uniquely determined.

(2) Suppose that $\overrightarrow{m} = \{m_n\}_{n=1}^{\infty}$ contains infinitely many even numbers. In this case, we have $2S(\overrightarrow{m}) = S(\overrightarrow{m})$. Fix $\alpha, \beta \in S(\overrightarrow{m})$ which satisfy $a < \alpha < \beta < b$, and put $\alpha = d_0(m_1 \cdots m_{p_0})^{-1}, \beta = d_1(m_1 \cdots m_{p_0})^{-1}$, where d_0, d_1 are even. For fixed $n \geq 1$, put $M_n = (d_1 - d_0) m_{p_0+1} \cdots m_{p_0+n}$ and $L_n = m_1 \cdots m_{p_0+n}$. Then the sequence $\{s_k\}_{k=0}^{M_n}$ defined by $s_k = \varphi(\alpha + k/L_n)$, $k = 0, 1, \ldots, M_n$, is a truncated Stieltjes moment sequence. To see this, pick a sufficiently large number n_0 such that $m_{p_0+n_0}$ is even. Then for $c_i \in \boldsymbol{R}$, $i = 0, 1, \ldots, M_n/2$, we have

$$\sum_{i,j=0}^{M_n/2} c_i c_j s_{i+j} = \sum_{i,j=0}^{M_n/2} c_i c_j \varphi\left(\alpha + \frac{i+j}{L_n}\right) \geq 0,$$

and

$$\sum_{i,j=0}^{M_n/2-1} c_i c_j s_{i+j+1}$$
$$= \sum_{i,j=0}^{M_n/2-1} c_i c_j \varphi\left(\alpha + \frac{i+j}{L_n} + 2\frac{m_{p_0+n+1}\cdots m_{p_0+n_0-1}\cdot \frac{m_{p_0+n_0}}{2}}{m_1\cdots m_{p_0+n_0}}\right) \geq 0.$$

Therefore there exists a finite positive Radon measure τ_n on $[0,\infty)$ such that

$$\varphi\left(\alpha + \frac{k}{L_n}\right) = \int_0^\infty t^k\, d\tau_n(t), \qquad k = 0, 1, \ldots, M_n - 1.$$

By an argument similar to that in the proof of (1), we find a unique measure $\mu \in E_+((a,b), \boldsymbol{R})$ such that

$$\varphi(r) = \int_{\boldsymbol{R}} e^{rx}\, d\mu(x), \qquad r \in (a,b) \cap S(\overrightarrow{m}).$$

Thus the proof is complete. □

For $\alpha \in S(\overrightarrow{m})$, let E_α denote the shift operator on $\boldsymbol{R}^{S(\overrightarrow{m})}$ defined by $E_\alpha\varphi(r) = \varphi(\alpha + r)$, $\varphi \in \boldsymbol{R}^{S(\overrightarrow{m})}$, $r \in S(\overrightarrow{m})$. In [3, Theorem 7.1.10], it is shown that a bounded function φ on a commutative semigroup S is completely monotone if and only if φ is completely positive definite on S. The following theorem gives an analogous result, which is a generalization of [2, Theorem 3].

Theorem 2.2 *Let $a \in \boldsymbol{R}$ and let $\overrightarrow{m} = \{m_n\}_{n=1}^\infty$ be a sequence of integers $m_n \geq 2$. For a function $\varphi : (a,\infty) \cap S(\overrightarrow{m}) \to \boldsymbol{R}$, the following conditions are mutually equivalent*:

(1) *For any natural number p and for any $\alpha_1, \ldots, \alpha_p \in (0,\infty) \cap S(\overrightarrow{m})$,*

$$(E_0 - E_{\alpha_1}) \cdots (E_0 - E_{\alpha_p})\varphi(r) \geq 0$$

holds for $r \in (a, \infty) \cap S(\overrightarrow{m})$;

(2) *For any* $\alpha \in (0, \infty) \cap S(\overrightarrow{m})$, *the functions* $E_\alpha \varphi(r)$ *and* $(E_0 - E_\alpha)\varphi(r)$ *are both positive definite on* $(a, \infty) \cap S(\overrightarrow{m})$;

(3) *There exists a measure* $\mu \in E_+((-\infty, -a), [0, \infty))$ *such that*

$$\varphi(r) = \int_0^\infty e^{-rx} d\mu(x), \qquad r \in (a, \infty) \cap S(\overrightarrow{m}).$$

Proof. The proof is similar to that of [2, Theorem 3] and omitted. For the implication (2) $\Longrightarrow$ (3), see also [3, Lemma 7.3.8]. $\square$

Next we consider the case where $I = [a, b)$ is a half-open interval. Let $\delta_a(r)$ denote the function on $I \cap S(\overrightarrow{m})$ defined by $\delta_a(a) = 1$ and $\delta_a(r) = 0$ for $r \neq a$.

Theorem 2.3 *Let* $a \in 2S(\overrightarrow{m})$, $b \in \boldsymbol{R} \cup \{\infty\}$ *such that* $a < b$ *and let* $\overrightarrow{m} = \{m_n\}_{n=1}^\infty$ *be a sequence of integers* $m_n \geq 2$. *Let* φ *be a positive definite function on* $[a, b) \cap S(\overrightarrow{m})$.

(1) *If the sequence* $\overrightarrow{m}$ *contains at most finitely many even numbers, then there exist a nonnegative constant* ω *and* $\mu, \nu \in E_+([a, b), \boldsymbol{R})$ *such that*

$$\varphi(r) = \omega\delta_a(r) + \int_{\boldsymbol{R}} e^{rx} d\mu(x) + \int_{\boldsymbol{R}} \chi(r) e^{rx} d\nu(x), \qquad r \in [a, b) \cap S(\overrightarrow{m}). \tag{2.4}$$

Moreover the triple (ω, μ, ν) *is uniquely determined by* φ.

(2) *If the sequence* $\overrightarrow{m}$ *contains infinitely many even numbers, then there exist a nonnegative constant* ω *and* $\mu \in E_+([a, b), \boldsymbol{R})$ *such that*

$$\varphi(r) = \omega\delta_a(r) + \int_{\boldsymbol{R}} e^{rx} d\mu(x), \quad r \in [a, b) \cap S(\overrightarrow{m}). \tag{2.5}$$

The pair (ω, μ) *is uniquely determined by* φ.

Proof. (1) Since $a \in 2S(\overrightarrow{m})$, putting $\alpha = a$ in the proof of Theorem 2.1 (1), we have

$$\varphi(r) = \omega\delta_a(r) + \int_{\boldsymbol{R}} e^{rx}\, d\mu(x) + \int_{\boldsymbol{R}} \chi(r) e^{rx} d\nu(x), \quad r \in [a, \beta) \cap S(\overrightarrow{m}),$$

where $\omega = \mu_0(\{-\infty\}) + \nu_0(\{-\infty\}) \geq 0$. Then the same argument as in the proof of Theorem 2.1 (1) shows (2.4) and the uniqueness of μ and ν. The uniqueness of ω follows from the equality

$$\lim_{r\downarrow a, r\in 2S(\overrightarrow{m})} \varphi(r) = \int_{\boldsymbol{R}} e^{ax} d\mu(x) + \int_{\boldsymbol{R}} e^{ax} d\nu(x) = \varphi(a) - \omega.$$

The assertion (2) is proved analogously. □

Remark 2.1 If we put $S(\overrightarrow{m}) = \boldsymbol{Q}$ in Theorem 2.1(2) and Thoerem 2.2(2), we obtain [2, Theorem 1, Proposition 1]. If we put $[a, b) = [0, \infty)$ in Theorem 2.2, we obtain [7, Theorem 2.2].

A real-valued function ψ on $I \cap S(\overrightarrow{m})$ is said to be *negative definite* if

$$\sum_{i,j=1}^{n} c_i c_j \psi(r_i + r_j) \leq 0$$

for all $n \geq 2$, $c_1, c_2, \ldots, c_n \in \boldsymbol{R}$ such that $\sum_{i=1}^{n} c_i = 0$ and $r_1, \ldots, r_n \in S(\overrightarrow{m})$ such that $2r_i \in I \cap S(\overrightarrow{m})$ for $i = 1, 2, \ldots, n$. Using Theroem 2.1 and Theorem 2.2, we can obtain an integral representation of negative definite functions on $I \cap S(\overrightarrow{m})$.

Theorem 2.4 *Let $a, b \in \boldsymbol{R} \cup \{-\infty, \infty\}$ such that $a < b$ and let $\overrightarrow{m} = \{m_n\}_{n=1}^{\infty}$ be a sequence of integers $m_n \geq 2$. Let ψ be a negative definite function on $(a, b) \cap S(\overrightarrow{m})$. Let $\alpha \in 2S(\overrightarrow{m})$ such that $a < \alpha < b$ and let $\beta \in S(\overrightarrow{m})$ such that $\beta > 0$ and $a < \alpha + 2\beta < b$.*

(1) *If the sequence $\overrightarrow{m}$ contains at most finitely many even numbers, then ψ has a representation of the form*

$$\psi(r) = A + Br - Cr^2 + \int_{\boldsymbol{R}\setminus\{0\}} \left(e^{\alpha x} - e^{rx} - \frac{r-\alpha}{\beta} e^{\alpha x}(1 - e^{\beta x}) \right) d\mu(x) - \int_{\boldsymbol{R}} \chi(r) e^{rx} d\nu(x)$$

for $r \in (a, b) \cap S(\overrightarrow{m})$, where A, B, C are real constants such that $C \geq 0$ and μ, ν are positive Radon measures such that

$$\int_{0<|x|\le 1} x^2 d\mu(x) < +\infty,$$

$$\int_{|x|\ge 1} e^{rx} d\mu(x) < +\infty \quad and \quad \int_{\boldsymbol{R}} e^{rx} d\nu(x) < +\infty$$

for $r \in (a,b) \cap S(\overrightarrow{m})$. *Moreover, the quintuple* (A,B,C,μ,ν) *is uniquely determined by* ψ, α *and* β.

(2) *If the sequence* $\overrightarrow{m}$ *contains infinitely many even numbers, then* ψ *has a representation of the form*

$$\psi(r) = A + Br - Cr^2 + \int_{\boldsymbol{R}\setminus\{0\}} \left(e^{\alpha x} - e^{rx} - \frac{r-\alpha}{\beta} e^{\alpha x}(1 - e^{\beta x}) \right) d\mu(x)$$

for $r \in (a,b) \cap S(\overrightarrow{m})$, *where* A, B, C *are real constants such that* $C \ge 0$ *and* μ *is a positive Radon measure such that*

$$\int_{0<|x|\le 1} x^2 d\mu(x) < +\infty \quad and \quad \int_{|x|\ge 1} e^{rx} d\mu(x) < +\infty$$

for $r \in (a,b) \cap S(\overrightarrow{m})$. *Moreover, the quadruple* (A,B,C,μ) *is uniquely determined by* ψ, α *and* β.

Proof. We prove only (1). Replacing ψ by $\psi - \psi(\alpha)$ if necessary, we may suppose that $\psi(\alpha) = 0$. By Theorem 2.1 (1) and [3, Theorem 3.2.2], we have

$$e^{-t\psi(r)} = \int_{\boldsymbol{R}} e^{(r-\alpha)x} d\mu_t(x) + \int_{\boldsymbol{R}} \chi(r) e^{(r-\alpha)x} d\nu_t(x),$$
$$\text{for } r \in (a,b) \cap S(\overrightarrow{m}),\ t > 0,$$

where μ_t and ν_t are finite positive Radon measures on $\boldsymbol{R}$ such that $\mu_t(\boldsymbol{R}) + \nu_t(\boldsymbol{R}) = 1$. For $r \in (a,b) \cap S(\overrightarrow{m})$, we have

$$\int_{\boldsymbol{R}} \left(1 - e^{(r-\alpha)x} - \frac{r-\alpha}{\beta}(1 - e^{\beta x}) \right) d\mu_t(x)$$
$$+ \int_{\boldsymbol{R}} \left(1 - \chi(r) e^{(r-\alpha)x} - \frac{r-\alpha}{\beta}(1 - \chi(\beta) e^{\beta x}) \right) d\nu_t(x)$$

$$= 1 - e^{-t\psi(r)} - \frac{r-\alpha}{\beta}(1 - e^{-t\psi(\alpha+\beta)}),$$

so

$$\frac{1}{t}\int_{\boldsymbol{R}}\left(1 - e^{(r-\alpha)x} - \frac{r-\alpha}{\beta}(1 - e^{\beta x})\right)d\mu_t(x)$$
$$+ \frac{1}{t}\int_{\boldsymbol{R}}\left(1 - \chi(r)e^{(r-\alpha)x} - \frac{r-\alpha}{\beta}(1 - \chi(\beta)e^{\beta x})\right)d\nu_t(x)$$

converges to $\psi(r) - \dfrac{r-\alpha}{\beta}\psi(\alpha+\beta)$ as $t \to 0$. Similarly, if $r \in S(\overrightarrow{m})$ satisfies $a < \alpha + 2r < b$, we have

$$\lim_{t\to 0}\frac{1}{t}\left(\int_{\boldsymbol{R}}(1 - e^{rx})^2 d\mu_t(x) + \int_{\boldsymbol{R}}(1 - \chi(r)e^{rx})^2 d\nu_t(x)\right) = 2\psi(\alpha+r) - \psi(\alpha+2r),$$

which implies that

$$\sup_{0<t\leq 1}\int_{\boldsymbol{R}}\frac{1}{t}(1 - e^{rx})^2 d\mu_t(x) \leq A_r,$$
$$\sup_{0<t\leq 1}\int_{\boldsymbol{R}}\frac{1}{t}(1 - \chi(r)e^{rx})^2 d\nu_t(x) \leq A_r \tag{2.6}$$

for some constant $A_r > 0$ depending on r. Fix $\beta' \in S(\overrightarrow{m}) \setminus 2S(\overrightarrow{m})$ such that $\beta < \beta'$ and $a < \alpha + 2\beta' < b$. By (2.6), there exist finite positive Radon measures σ, τ and a sequence $\{t_j\}$ which tends to 0 such that

$$\lim_{j\to\infty}\frac{1}{t_j}(1 - e^{\beta x})^2\mu_{t_j} = \sigma, \tag{2.7}$$

$$\lim_{j\to\infty}\frac{1}{t_j}(1 + e^{\beta' x})^2\nu_{t_j} = \lim_{j\to\infty}\frac{1}{t_j}(1 - \chi(\beta')e^{\beta' x})^2\nu_{t_j} = \tau \tag{2.8}$$

in vague topology.

For a fixed $r \in (a,b) \cap S(\overrightarrow{m})$, choose $\delta, \gamma \in S(\overrightarrow{m}) \setminus 2S(\overrightarrow{m})$ satisfying $\delta < 0$, $\beta < \gamma$ and

$$a < \alpha + 2\delta < r < \alpha + 2\gamma < b.$$

Then it follows from (2.6), (2.7) and [3, Proposition 2.4.4] that

$$\lim_{j\to\infty}\frac{1}{t_j}\int_{\boldsymbol{R}} f(x)(1-e^{\delta x})^2 d\mu_{t_j}(x)=\int_{\boldsymbol{R}} f(x)\biggl(\frac{1-e^{\delta x}}{1-e^{\beta x}}\biggr)^2 d\sigma(x), \tag{2.9}$$

$$\lim_{j\to\infty}\frac{1}{t_j}\int_{\boldsymbol{R}} f(x)(1-e^{\gamma x})^2 d\mu_{t_j}(x)=\int_{\boldsymbol{R}} f(x)\biggl(\frac{1-e^{\gamma x}}{1-e^{\beta x}}\biggr)^2 d\sigma(x) \tag{2.10}$$

for every continuous function f on $\boldsymbol{R}$ vanishing at infinity. Using (2.9) and (2.10), we have

$$\begin{aligned}
&\lim_{j\to\infty}\frac{1}{t_j}\int_{\boldsymbol{R}}\biggl(1-e^{(r-\alpha)x}-\frac{r-\alpha}{\beta}(1-e^{\beta x})\biggr)d\mu_{t_j}(x)\\
&=\lim_{j\to\infty}\frac{1}{t_j}\int_{\boldsymbol{R}}\frac{1-e^{(r-\alpha)x}-\frac{r-\alpha}{\beta}(1-e^{\beta x})}{(1-e^{\delta x})^2+(1-e^{\gamma x})^2}\bigl\{(1-e^{\delta x})^2+(1-e^{\gamma x})^2\bigr\}d\mu_{t_j}(x)\\
&=\int_{\boldsymbol{R}}\frac{1-e^{(r-\alpha)x}-\frac{r-\alpha}{\beta}(1-e^{\beta x})}{(1-e^{\beta x})^2}d\sigma(x)\\
&=\frac{(r-\alpha)(\alpha+\beta-r)}{2\beta^2}\sigma(\{0\})\\
&\quad+\int_{\boldsymbol{R}\setminus\{0\}}\biggl(e^{\alpha x}-e^{rx}-\frac{r-\alpha}{\beta}e^{\alpha x}(1-e^{\beta x})\biggr)d\mu(x),
\end{aligned}$$

where $\mu=(e^{-\alpha x}/(1-e^{\beta x})^2)\sigma|_{\boldsymbol{R}\setminus\{0\}}$. Similarly,

$$\begin{aligned}
&\frac{1}{t_j}\int_{\boldsymbol{R}}\biggl(1-\chi(r)e^{(r-\alpha)x}-\frac{r-\alpha}{\beta}(1-\chi(\beta)e^{\beta x})\biggr)d\nu_{t_j}(x)\\
&\qquad=\frac{1}{t_j}\int_{\boldsymbol{R}}\frac{1-\chi(r)e^{(r-\alpha)x}-\frac{r-\alpha}{\beta}(1-\chi(\beta)e^{\beta x})}{(1+e^{\delta x})^2+(1+e^{\gamma x})^2}\\
&\qquad\qquad\times\bigl\{(1+e^{\delta x})^2+(1+e^{\gamma x})^2\bigr\}d\nu_{t_j}(x)
\end{aligned}$$

converges to

$$\int_{\boldsymbol{R}}\frac{1-\chi(r)e^{(r-\alpha)x}-\frac{r-\alpha}{\beta}(1-\chi(\beta)e^{\beta x})}{(1+e^{\beta' x})^2}d\tau(x)$$

$$= \int_{\boldsymbol{R}} \left(e^{\alpha x} - \chi(r) e^{rx} - \frac{r-\alpha}{\beta} e^{\alpha x} (1 - \chi(\beta) e^{\beta x}) \right) d\nu(x)$$

as $j \to \infty$, where $\nu = (e^{-\alpha x}/(1+e^{\beta' x})^2)\tau$. Thus we get

$$\begin{aligned}\psi(r) &= \frac{r-\alpha}{\beta}\psi(\alpha+\beta) + \frac{(r-\alpha)(\alpha+\beta-r)}{2\beta^2}\sigma(\{0\}) \\ &+ \int_{\boldsymbol{R}\setminus\{0\}} \left(e^{\alpha x} - e^{rx} - \frac{r-\alpha}{\beta} e^{\alpha x}(1-e^{\beta x}) \right) d\mu(x) \\ &+ \int_{\boldsymbol{R}} \left(e^{\alpha x} - \chi(r) e^{rx} - \frac{r-\alpha}{\beta} e^{\alpha x}(1-\chi(\beta)e^{\beta x}) \right) d\nu(x).\end{aligned}$$

By (2.6), we have

$$\int_{\boldsymbol{R}\setminus\{0\}} \left(\frac{1-e^{rx}}{1-e^{\beta x}} \right)^2 d\sigma(x) \le A_r, \quad \int_{\boldsymbol{R}} \left(\frac{1-\chi(r)e^{rx}}{1+e^{\beta' x}} \right)^2 d\tau(x) \le A_r$$

for $r \in S(\overrightarrow{m})$ satisfying $a < \alpha + 2r < b$, and it follows that μ and ν have the asserted properties. Moreover, since we have

$$\begin{aligned}2\psi(r+\beta) - \psi(r) - \psi(r+2\beta) &= 2C\beta^2 + \int_{\boldsymbol{R}\setminus\{0\}} e^{rx}(e^{\beta x}-1)^2 d\mu(x) \\ &+ \int_{\boldsymbol{R}} \chi(r) e^{rx} (\chi(\beta)e^{\beta x} - 1)^2 d\nu(x) \\ &= \int_{\boldsymbol{R}} e^{rx} d\tilde{\mu}(x) + \int_{\boldsymbol{R}} \chi(r) e^{rx} d\tilde{\nu}(x)\end{aligned}$$

for $r \in (a, b-2\beta) \cap S(\overrightarrow{m})$, where $\tilde{\mu} = 2C\beta^2\delta_0 + (e^{\beta x}-1)^2\mu$ and $\tilde{\nu} = (\chi(\beta)e^{\beta x} - 1)^2\nu$, it follows from Theorem 2.1 that C, μ, ν, A and B are uniquely determined. □

In the case of half-open intervals, we can prove the following theorem. The proof can be done in a similar way as that of Theorem 2.4.

Theorem 2.5 *Let $\overrightarrow{m} = \{m_n\}_{n=1}^{\infty}$ be a sequence of integers $m_n \ge 2$ and let $a \in 2S(\overrightarrow{m})$, $b \in \boldsymbol{R} \cup \{\infty\}$ such that $a < b$. Let ψ be a negative definite function on $[a, b) \cap S(\overrightarrow{m})$ and let $\beta \in S(\overrightarrow{m})$ such that $a < a + 2\beta < b$.*

(1) *If the sequence $\overrightarrow{m}$ contains at most finitely many even numbers, then ψ has a representation of the form*

$$\begin{aligned}\psi(r) &= A + Br - Cr^2 - D\delta_a(r) \\ &\quad + \int_{\boldsymbol{R}\setminus\{0\}} (e^{ax} - e^{rx} - \frac{r-a}{\beta} e^{ax}(1 - e^{\beta x}))d\mu(x) \\ &\quad - \int_{\boldsymbol{R}} \chi(r) e^{rx} d\nu(x)\end{aligned}$$

for $r \in [a, b) \cap S(\overrightarrow{m})$, where A, B, C, D are real constants such that $C, D \geq 0$ and μ, ν are positive Radon measures such that

$$\int_{0<|x|\leq 1} x^2\, d\mu(x) < +\infty,$$

$$\int_{|x|\geq 1} e^{rx} d\mu(x) < +\infty \quad \textit{and} \quad \int_{\boldsymbol{R}} e^{rx} d\nu(x) < +\infty$$

for $r \in (a, b) \cap S(\overrightarrow{m})$. Moreover, the sextuple (A, B, C, D, μ, ν) is uniquely determined by ψ and β.

(2) *If the sequence $\overrightarrow{m}$ contains infinitely many even numbers, then ψ has a representation of the form*

$$\begin{aligned}\psi(r) &= A + Br - Cr^2 - D\delta_a(r) \\ &\quad + \int_{\boldsymbol{R}\setminus\{0\}} \left(e^{ax} - e^{rx} - \frac{r-a}{\beta} e^{ax}(1 - e^{\beta x})\right) d\mu(x)\end{aligned}$$

for $r \in [a, b) \cap S(\overrightarrow{m})$, where A, B, C, D are real constants such that $C, D \geq 0$ and μ is a positive Radon measure such that

$$\int_{0<|x|\leq 1} x^2 d\mu(x) < +\infty \quad \textit{and} \quad \int_{|x|\geq 1} e^{rx} d\mu(x) < +\infty$$

for $r \in (a, b) \cap S(\overrightarrow{m})$. Moreover, the quintuple (A, B, C, D, μ) is uniquely determined by ψ and β.

Remark 2.2 If we put $S(\overrightarrow{m}) = \boldsymbol{Q}$ in Theorem 2.4 (2) and Thoerem 2.5 (2), we obtain [2, Theorem 4, Proposition 2]. If we put $[a, b) = [0, \infty)$ and

$\beta = 1$ in Theorem 2.5, we obtain [7, Theorem 2.4].

3. Integral representations of operator-valued functions

In this section, we consider the case of operator-valued functions. Let $\mathcal{H}$ be a complex Hilbert space, $\langle \cdot, \cdot \rangle$ the inner product on $\mathcal{H}$, $B(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$, and $B(\mathcal{H})_+$ the set of all positive operators on $\mathcal{H}$. A function $\varphi : I \cap S(\overrightarrow{m}) \to B(\mathcal{H})$ is said to be *positive definite* if

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \langle \varphi(r_i + r_j)\xi, \xi \rangle \geq 0$$

for all $n \geq 1, c_1, c_2, \ldots, c_n \in \boldsymbol{C}, r_1, r_2, \ldots, r_n \in S(\overrightarrow{m})$ such that $2r_i \in I \cap S(\overrightarrow{m})$ for $i = 1, 2, \ldots, n$ and $\xi \in \mathcal{H}$, and *of positive type* if

$$\sum_{i,j=1}^{n} \langle \varphi(r_i + r_j)\xi_i, \xi_j \rangle \geq 0$$

for all $n \geq 1, r_1, r_2, \ldots, r_n \in S(\overrightarrow{m})$ such that $2r_i \in I \cap S(\overrightarrow{m})$ for $i = 1, 2, \ldots, n$ and $\xi_1, \xi_2, \ldots, \xi_n \in \mathcal{H}$.

If φ is a function of positive type, then φ is positive definite, and the converse is true if $\dim \mathcal{H} = 1$. Furthermore, it is known that a positive definite function defined on a perfect $*$-semigroup is necessarily of positive type ([5, Theorem 3.1, Proposition 1.1]). But there exists a positive definite function defined on a semiperfect $*$-semigroup which is not of positive type ([4, Theorem 1], [5, Theorem 3.7]).

Let us denote by $\mathcal{B}(\boldsymbol{R})$ the σ-algebra of all Borel subsets of $\boldsymbol{R}$, and by $E_+(I, \boldsymbol{R}, \mathcal{H})$ the set of all functions $F : \mathcal{B}(\boldsymbol{R}) \to B(\mathcal{H})_+$ satisfying $\langle F(\cdot)\xi, \xi \rangle \in E_+(I, \boldsymbol{R})$ for all $\xi \in \mathcal{H}$.

Theorem 3.1 *Let $a, b \in \boldsymbol{R} \cup \{-\infty, \infty\}$ such that $a < b$ and let $\overrightarrow{m} = \{m_n\}_{n=1}^{\infty}$ be a sequence of integers $m_n \geq 2$ which contains at most finitely many even numbers. Let $\varphi : (a, b) \cap S(\overrightarrow{m}) \to B(\mathcal{H})$ be a function on $(a, b) \cap S(\overrightarrow{m})$. Then the following conditions are mutually equivalent*:

(1) *φ is of positive type*;
(2) *φ is positive definite*;
(3) *For any fixed $\alpha \in (a, b) \cap 2S(\overrightarrow{m})$, there exist functions $F_1, F_2 : \mathcal{B}(\boldsymbol{R}) \to$*

$B(\mathcal{H})_+$ such that $e^{-\alpha x}F_1, e^{-\alpha x}F_2 \in E_+((a,b),\boldsymbol{R},\mathcal{H})$ and

$$\langle\varphi(r)\xi,\eta\rangle = \int_{\boldsymbol{R}} e^{(r-\alpha)x} d\langle F_1(x)\xi,\eta\rangle + \int_{\boldsymbol{R}} \chi(r)e^{(r-\alpha)x} d\langle F_2(x)\xi,\eta\rangle,$$

for $r \in (a,b)\cap S(\overrightarrow{m})$, $\xi,\eta \in \mathcal{H}$.

Moreover, the pair (F_1,F_2) is uniquely determined by φ and α.

Proof. The implication (1) $\Longrightarrow$ (2) is clear, while the implication (3) $\Longrightarrow$ (1) is proved by a similar way as the proof of [5, Proposition 1.1]. Suppose that (2) holds and fix $\alpha \in (a,b)\cap 2S(\overrightarrow{m})$. By the proof of Theorem 2.1 (1), for each $\xi \in \mathcal{H}$ there exist finite positive Radon measures μ_ξ, ν_ξ on $\boldsymbol{R}$ such that

$$\langle\varphi(r)\xi,\xi\rangle = \int_{\boldsymbol{R}} e^{(r-\alpha)x} d\mu_\xi(x) + \int_{\boldsymbol{R}} \chi(r)e^{(r-\alpha)x} d\nu_\xi(x), \quad r \in (a,b)\cap S(\overrightarrow{m}).$$

For $\xi,\eta \in \mathcal{H}$, define the signed measures $\mu_{\xi,\eta}$, $\nu_{\xi,\eta}$ by

$$\mu_{\xi,\eta} = \frac{1}{4}\left\{\mu_{\xi+\eta} - \mu_{\xi-\eta} + i\mu_{\xi+i\eta} - i\mu_{\xi-i\eta}\right\},$$
$$\nu_{\xi,\eta} = \frac{1}{4}\left\{\nu_{\xi+\eta} - \sigma_{\xi-\eta} + i\nu_{\xi+i\eta} - i\nu_{\xi-i\eta}\right\}.$$

Then

$$\langle\varphi(r)\xi,\eta\rangle = \int_{\boldsymbol{R}} e^{(r-\alpha)x} d\mu_{\xi,\eta}(x) + \int_{\boldsymbol{R}} \chi(r)e^{(r-\alpha)x} d\nu_{\xi,\eta}(x), \quad r \in (a,b)\cap S(\overrightarrow{m}).$$

By Theorem 2.1(1), we can see that for each $B \in \mathcal{B}(\boldsymbol{R})$ the mappings

$$(\xi,\eta) \mapsto \mu_{\xi,\eta}(B), \quad (\xi,\eta) \mapsto \nu_{\xi,\eta}(B)$$

are sesqui-linear forms on $\mathcal{H}\times\mathcal{H}$ respectively. Furthermore, for $\xi \in \mathcal{H}$ we have

$$0 \le \mu_{\xi,\xi}(B) \le \mu_{\xi,\xi}(\boldsymbol{R}) \le \langle\varphi(\alpha)\xi,\xi\rangle,$$

so that $0 \le \mu_{\xi,\xi}(B) \le \|\varphi(\alpha)\|\,\|\xi\|^2$. Therefore there exists a unique operator $F_1(B) \in B(\mathcal{H})_+$ such that $\mu_{\xi,\eta}(B) = \langle F_1(B)\xi,\eta\rangle$. Similarly $\nu_{\xi,\eta}(B) =$

$\langle F_2(B)\xi,\eta\rangle$ with $F_2(B)\in B(\mathcal{H})_+$. Then we have

$$\langle\varphi(r)\xi,\eta\rangle=\int_{\boldsymbol{R}}e^{(r-\alpha)x}d\langle F_1(x)\xi,\eta\rangle+\int_{\boldsymbol{R}}\chi(r)e^{(r-\alpha)x}d\langle F_2(x)\xi,\eta\rangle.$$

Thus the condition (3) holds. □

We can obtain a result analogous to Theorem 3.1 for the case where $\overrightarrow{m}$ contains infinitely many even numbers. We also obtain the following theorem:

Theorem 3.2 *Let $\overrightarrow{m}$ be a sequence of integers $m_n\geq 2$ which contains at most finitely many even numbers, and let $a\in 2S(\overrightarrow{m})$, $b\in\boldsymbol{R}\cup\{\infty\}$ such that $a<b$. Let $\varphi:[a,b)\cap S(\overrightarrow{m})\to B(\mathcal{H})$ be a function on $[a,b)\cap S(\overrightarrow{m})$. Then the following conditions are mutually equivalent*:

(1) *φ is of positive type*;
(2) *φ is positive definite*;
(3) *There exist a positive operator $T\in B(\mathcal{H})$ and functions $F_1,F_2:\mathcal{B}(\boldsymbol{R})\to\mathcal{B}(\mathcal{H})_+$ such that $e^{-ax}F_1$, $e^{-ax}F_2\in E_+([a,b),\boldsymbol{R},\mathcal{H})$ and*

$$\langle\varphi(r)\xi,\eta\rangle=\delta_a(r)\langle T\xi,\eta\rangle+\int_{\boldsymbol{R}}e^{(r-a)x}d\langle F_1(x)\xi,\eta\rangle$$
$$+\int_{\boldsymbol{R}}\chi(r)e^{(r-a)x}d\langle F_2(x)\xi,\eta\rangle,$$

for $r\in[a,b)\cap S(\overrightarrow{m})$, $\xi,\eta\in\mathcal{H}$.

Moreover the triple (T,F_1,F_2) is uniquely determined by φ.

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