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Weierstrass's function and chaos

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In this paper, we discuss the relationship between the famous Weierstrass's everywhere non-differentiable continuous function and the chaotic dynamical system. With this formulation, we can find an equation for which Weierstrass's function is a solution. This is a trial to combine two notions: Fractals and Chaos. B. B. Mandelbrot introduced the first object in his book [1] but he mentioned that he do not know how relate these two notions.* Of course, we owe many nice ideas to this Mandelbrot's book.

1. Introduction

It was proved by Weierstrass that the function:

$$W_{a,b}(x)=\sum_{n=0}^{\infty} a^n \cos(b^n \pi x) \tag{1.1}$$

where $0<a<1$, b is an odd integer and $ab>1+\frac{3}{2}\pi$, has no differential coefficient for any value of x. Weierstrass's result was generalized by G. H. Hardy [2], who has proved that $W_{a,b}(x)$ does not possess a finite differential coefficient at any point x in any case in which $0<a<1$, $b>1$ and $ab\geq 1$.

Our starting point is just in the representation of $\Psi^n(x)$, which is an n-fold iteration by $\Psi(x)=4x(1-x)$, that is,

$$\Psi^n(x)=\sin^2(2^n \operatorname{Arcsin}\sqrt{x}). \tag{1.2}$$

This function $\Psi(x)$ is called chaotic in the sense of Li-Yorke [10].
Combining (1.1) and (1.2), we have

$$\sum_{n=0}^{\infty} t^n \Psi^n(x)=\frac{1}{2(1-t)}-\frac{1}{2}W_{t,2}\left(\frac{2}{\pi}\operatorname{Arcsin}\sqrt{x}\right). \tag{1.3}$$

Perhaps one should think the left hand side of (1.3) as a generating function which generates the iterations of Ψ. More generally, it will be very interesting and important to investigate the function:

$$F(t,x)=\sum_{n=0}^{\infty} t^n g\big((\psi^n(x)\big) \tag{1.4}$$

for given mappings ψ: $J \to J$ and g: $J \to R$ where J is some closed interval.

Intuitively, the non-smoothness of the function $F(t, x)$ with respect to x is corresponding to the sensitive dependence of initial value x for the dynamical system ψ.

For example, if $\psi(x)=4x(1-x)$ and $g(x)=x$, then $F(t, x)$ is continuous and non-differentiable function with respect to x in any case in which $\frac{1}{2} \leq t < 1$ by Hardy's result. And the same holds if we choose

$$\psi(x)=\begin{cases} 2x & \left(0 \leq x < \frac{1}{2}\right) \\ 2(1-x) & \left(\frac{1}{2} \leq x \leq 1\right) \end{cases} \quad \text{and } g(x)=\cos \pi x .$$

In the case in which

$$\psi(x)=\begin{cases} 2x & \left(0 \leq x < \frac{1}{2}\right) \\ 2(1-x) & \left(\frac{1}{2} \leq x \leq 1\right) \end{cases} \quad \text{and } g(x)=x ,$$

the non-differentiablity of $F\left(\frac{1}{2}, x\right)$ follows from the result of T. Takagi [3]. (Also see van der Waerden [4].)

2. Functional equation.

In previous section, we introduce the function :

$$(2.1) \qquad F(t, x)=\sum_{n=0}^{\infty} t^n g\left(\psi^n(x)\right), \qquad (t, x) \in (-1, 1) \times J ,$$

where ψ: $J \to J$, g: $J \to R$ and J is a closed interval. One can easily find that the function (2.1) satisfies the functional equation :

$$(2.2) \qquad F(t, x)=t\, F\left(t, \psi(x)\right)+g(x) .$$

Conversely, let us consider this functional equation under suitable conditions. Since $F(0, x)=g(x)$, we call $g(x)$ an initial function for this equation. Then we will prove the following.

THEOREM 2.1. *Suppose that g: $J \to R$ is a bounded initial function and that ψ: $J \to J$ is a dynamical system. Then $F(t, x)$, which satisfies* (2.2) *and is bounded with respect to x for each t, is uniquely determined and expressed by* (2.1) *for* $-1<t<1$.

PROOF. From (2. 2), we obtain

$$F(t, x)-\sum_{j=0}^{n} t^j g\left(\phi^j(x)\right)=t^{n+1}F\left(t, \phi^{n+1}(x)\right)$$

for any $n\in N$. Taking the limit as $n\to\infty$, we are done.

COROLLARY 2. 2. *Suppose that $f: J\to R$ is a bounded function and that $\phi: J\to J$ is a dynamical system. If we put*

$$g_s(x)=f(x)-sf\left(\phi(x)\right), \qquad |s|<1,$$

for $x\in J$, then we have

$$F(s, x)=f(x)$$

where $F(t, x)$ is a unique solution of (2. 2) with a dynamical system ϕ and an initial function g_s.

The proof is straightforward.

For example, if $J=[0, 1]$, $f(x)=2x-x^2$ and

$$\phi(x)=\begin{cases} 2x & \left(0\leq x<\dfrac{1}{2}\right) \\ 2(1-x) & \left(\dfrac{1}{2}\leq x\leq 1\right)\end{cases},$$

then

$$g_s(x)=(2-4s)\,x+(4s-1)\,x^2.$$

Therefore we obtain, using Theorem 2. 1 and Corollary 2. 2,

$$(2-4s)\sum_{n=0}^{\infty} s^n\phi^n(x)+(4s-1)\sum_{n=0}^{\infty} s^n\left(\phi^n(x)\right)^2=2x-x^2.$$

In particular, we have

$$\sum_{n=0}^{\infty}\frac{1}{4^n}\phi^n(x)=\sum_{n=0}^{\infty}\frac{1}{2^n}\left(\phi^n(x)\right)^2=2x-x^2, \qquad x\in[0, 1].$$

These are expansions of an analytic function by using iterations of piecewise-linear function ϕ.

Compare that the function:

$$\sum_{n=0}^{\infty}\frac{1}{2^n}\phi^n(x), \qquad x\in[0, 1]$$

is everywhere non-differentiable by the result of T. Takagi.

As a second example, if $J=[0, 1]$, $f(x)=x$ and $\phi_r(x)=rx \pmod 1$, $r>1$, then

$$g_s(x)=(1-sr)\,x+s[rx]\,,$$

where $[x]$ denotes the greatest integer which does not exceed x. In particular, if $s=\frac{1}{r}<1$, then we obtain, using Theorem 2.1 and Corollary 2.2,

$$\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\Big[r\psi_r^{\,n}(x)\Big]=x\,,\qquad x\in[0,1]\,.$$

In the special case $r=2$, we have

$$\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\chi_{[\frac{1}{2},1]}\Big(\psi_2^{\,n}(x)\Big)=x\,,\qquad x\in[0,1]\,,$$

where $\chi_{[\frac{1}{2},1]}(x)$ is a characteristic function of an interval $\left[\frac{1}{2},1\right]$. This formula is a well-known binary expansion.

Compare with the following formula:

$$\sum_{n=0}^{\infty}\frac{1}{2^{n-1}}\int_0^{\psi^n(x)}\chi_{[\frac{1}{2},1]}(y)\,dy=x\,,\qquad x\in[0,1]$$

obtained by the same method, where

$$\psi(x)=\begin{cases}2x & \left(0\le x<\frac{1}{2}\right)\\ 2(1-x) & \left(\frac{1}{2}\le x\le 1\right)\end{cases}.$$

As a final example, we put $J=[0,1]$ and

$$\psi(x)=\begin{cases}3x & \left(0\le x\le\frac{1}{3}\right)\\ 0 & \left(\frac{1}{3}<x<\frac{2}{3}\right)\\ 3x-2 & \left(\frac{2}{3}\le x\le 1\right)\end{cases}.$$

Note that

$$\bigcap_{n=0}^{\infty}\psi^{-n}\big([0,1]\big)$$

is well-known Cantor's ternary set. Then we can get Cantor's function $f(x)$, which is not constant and has null derivative almost everywhere, as an iteration expansion of ψ. In this case, by Corollary 2.2, we have

$$g_{\frac{1}{2}}(x)=\frac{1}{2}\chi_{[\frac{2}{3},1]}(x)$$

where $\chi_{[\frac{1}{3},1]}(x)$ is a characteristic function of $\left[\frac{1}{3}, 1\right]$. Hence we conclude that

$$\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \chi_{[\frac{1}{3},1]}\left(\psi^n(x)\right)$$

is Cantor's function.

3. Connection with ergodic theory

In this section, we will discuss the connection between the functional equation (2. 2) and ergodic theory. Let $(J, \mathscr{F}, P)$ be a probability space and ψ be a transformation of J into itself. We assume that ψ is measure preserving. Then ergodic theorem states as follows.

THEOREM 3. 1. (Billingsley [5], p. 13) *If $g \in L^1(J)$ and ψ is ergodic, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g\left(\psi^j(x)\right) = \int_J g\, dP \quad a.e.$$

On the other hand, G. H. Hardy has shown the following Tauberian theorem.

THEOREM 3. 2. (Hardy [6], p. 154) *Suppose that $a_n = O(1)$. If*

$$\lim_{t\to 1-0} (1-t) \sum_{n=0}^{\infty} a_n t^n = A\,,$$

then it follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} a_j = A\,.$$

Furthermore, the converse is true.

Combining above two theorems, we have the following.

THEOREM 3. 3. *Suppose that g: $J \to R$ is a bounded function and ψ: $J \to J$ is ergodic. Then we have*

$$\lim_{t\to 1-0} (1-t)\, F(t, x) = \int_J g\, dP \quad a.e.$$

where $F(t, x)$ is a unique solution of (2. 2) with an initial function g and a dynamical system ψ.

As a example,

$$\psi(x) = \begin{cases} 2x & \left(0 \le x < \frac{1}{2}\right) \\ 2(1-x) & \left(\frac{1}{2} \le x \le 1\right) \end{cases}$$

is a measure preserving ergodic transformation on $J=[0,1]$ with respect to Lebesgue measure. Applying Theorem 3.3, we obtain for Weierstrass's function $W_{t,2}(x)$,

$$\lim_{t\to 1-0}(1-t)\sum_{n=0}^{\infty} t^n \cos(2^n \pi x)=\int_0^1 \cos \pi x\, dx=0 \quad \text{a.e.}$$

On the other hand, computation shows that

$$\lim_{t\to 1-0}(1-t)\sum_{n=0}^{\infty} t^n \cos(2^n \pi x)$$
$$=\begin{cases} 1 & \text{if } x=\dfrac{q}{2^p} \qquad (q, p\in N) \\ -\dfrac{1}{2} & \text{if } x=\dfrac{2}{3}+\dfrac{s}{2^r} \qquad (s, r\in N). \end{cases}$$

Note that this limit function still keeps the sensitivity of initial value for the dynamical system ψ, since both $\left\{\frac{q}{2^p}\right\}$ and $\left\{\frac{2}{3}+\frac{s}{2^r}\right\}$ are dense sets in $[0,1]$.

4. Operator theoretical approach.

In this section, we will deal with the functional equation (2.2) from an operator theoretical point of view. Let E be a complex Banach space of all complex-valued bounded functions on a closed interval J with uniform norm. And let $L(E_1, E_2)$ denote a Banach space of all bounded linear operators which map E_1 into E_2 where E_1 and E_2 are some closed subspaces of E. Then, for a dynamical system $\psi: J\to J$, we will introduce a linear operator $T_\psi: E\to E$ defined by

$$T_\psi(f)=f\cdot\psi \qquad \text{for } f\in E,$$

where $f\cdot\psi$ denotes the composition $(f\cdot\psi)(x)=f(\psi(x))$. Plainly it follows that $T_\psi\in L(E, E)$. More precisely, we have the following.

PROPOSITION 4.1. *Suppose that T_ψ maps E_0 into itself, where E_0 is a closed subspace of E. Then*

$$\sigma(T_\psi|_{E_0})\subset\left\{z;\ |z|\leq 1\right\}$$

where $\sigma(T_\psi|_{E_0})$ is a spectrum of T_ψ: $E_0\to E_0$. Moreover, if E_0 contains constant functions, or if ψ maps J onto itself, then

$$\sup\left|\sigma(T_\psi|_{E_0})\right|=1.$$

PROOF. Plainly we have, for any $n\in N$,

$$\left\|(T_\phi|_{E_0})^n\right\| = \sup_{\substack{\|f\|=1 \\ f\in E_0}} \sup_{x\in J} \left|f\left(\phi^n(x)\right)\right| \leq 1\,.$$

Conversely, if $f^* \equiv 1 \in E_0$, then

$$\left\|(T_\phi|_{E_0})^n\right\| \geq \|f^* \cdot \phi^n\| = 1\,.$$

Or if $\phi(J)=J$, then

$$\left\|(T_\phi|_{E_0})^n\right\| \geq \|f \cdot \phi^n\| = 1$$

for any $f \in E_0$ such that $\|f\|=1$. This completes the proof.

By the above proposition, there exists the resolvent operator:

$$R(\lambda, T_\phi) = (\lambda Id - T_\phi)^{-1} \in L(E, E) \qquad \text{for} \quad |\lambda|>1\,,$$

and is expressed in Laurent expansion at the origin as follows:

$$R(\lambda, T_\phi) = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} T_\phi{}^n\,.$$

Again this proves Theorem 2.1 in section 2, since the functional equation (2.2) takes the form:

$$\left(\frac{1}{t} Id - T_\phi\right) F(t, x) = \frac{1}{t} g(x)$$

and therefore

$$F(t, x) = \left(\frac{1}{t} Id - T_\phi\right)^{-1} \left(\frac{1}{t} g(x)\right) = \sum_{n=0}^{\infty} t^n g\left(\phi^n(x)\right),$$

as required. So we have the following.

PROPOSITION 4.2.

$$F(t, x) = \frac{1}{t} R\left(\frac{1}{t}, T_\phi\right) g \qquad \textit{for} \quad |t|<1 \ \textit{and} \ g \in E,$$

where $F(t, x)$ is a unique solution of (2.2) with an initial function g and a dynamical system ϕ.

PROPOSITION 4.3. *Suppose that ϕ: $J \to J$ is an onto and one-to-one mapping. Then*

$$T_{\phi^{-1}} = T_\phi{}^{-1} \in L(E, E) \quad \textit{and} \quad \sigma(T_\phi|_{E_0}) \subset \left\{z\,;\ |z|=1\right\}$$

where E_0 is any closed invarient subspace of E under T_ϕ.

PROOF. Evidently we have

$$T_{\phi^{-1}}\cdot T_{\phi}(f)=T_{\phi}\cdot T_{\phi^{-1}}(f)=f \qquad \text{for any } f\in E.$$

Also we have, for $|\lambda|<1$,

$$R(\lambda, T_\phi)=-\frac{1}{\lambda}T_\phi{}^{-1}\cdot R\left(\frac{1}{\lambda}, T_\phi{}^{-1}\right)=-\sum_{n=0}^{\infty}\lambda^n T_\phi{}^{-n-1}.$$

This completes the proof.

Let K be a closed cone of all non-negative real-valued functions in E. Then it should be noted that T_ϕ is a positive operator which maps K into itself. The dual operator $T_\phi{}^*$ is also a positive operator which maps dual cone K^* into itself and, roughly speaking, is corresponding to Perron-Frobenius operator. (See Y. Takahashi [7].)

In general, T_ϕ is not a completely continuous operator in E. In fact, we have the following.

Proposition 4. 4. *Suppose that E_C is the set of all continuous functions on J. If a dynamical system $\phi: J\to J$ satisfies the following two conditions*:

(1) $\phi\in E_C$

(2) *there exists a sequence $\{p_n\}_{n\geq 0}\subset J$ such that $\phi(p_{n+1})=p_n$ for any $n\geq 0$ and $p_2\neq p_1=p_0$, then*

$$\sigma(T_\phi|_{E_C})=\left\{z;\ |z|\leq 1\right\}.$$

Proof. Assume that there exists a solution $f\in E_C$ of the equation:

$$f\big(\phi(x)\big)=\lambda f(x)+g(x)$$

for any $g\in E_C$ and for some fixed $0<|\lambda|<1$. Then inductively we have, for any $m\in N$,

$$\lambda^m\left\{f(p_m)-f(p_0)\right\}=\sum_{k=1}^{m}\lambda^{k-1}\left\{g(p_0)-g(p_k)\right\}.$$

Hence, taking the limit as $m\to\infty$, we get

$$\sum_{k=1}^{\infty}I_k(g)\equiv\sum_{k=1}^{\infty}\lambda^k\left\{g(p_0)-g(p_k)\right\}=0 \qquad \text{for any } g\in E_C.$$

Now let g_M denote a function on $\{p_1, p_2, \cdots, p_{m-1}\}$ such that

$$g_M(p_1)=0,\ g_M(p_k)=-e^{-ik\theta} \qquad (2\leq k\leq M-1)$$

and

$$r^{M-1}<\frac{1}{2}$$

for sufficiently large integer M where $\lambda=re^{i\theta}$. By Tietze's extension theorem, we get $\bar{g}_M \in E_C$, which is an extension of g_M to the interval J such that $|\bar{g}_M(J)| \leq 1$. Then

$$\sum_{k=1}^{M-1} I_k(\bar{g}_M) = \sum_{k=1}^{M-1} r^k = \frac{r-r^M}{1-r}$$

and

$$\left|\sum_{k=M}^{\infty} I_k(\bar{g}_M)\right| \leq \sum_{k=M}^{\infty} r^k = \frac{r^M}{1-r} .$$

Hence

$$\left|\sum_{k=1}^{\infty} I_k(\bar{g}_M)\right| \geq \frac{r-r^M}{1-r} - \frac{r^M}{1-r} = \frac{r-2r^M}{1-r} > 0 .$$

This contradiction completes the proof.

Finally, in connection with Theorem 3. 3, we will discuss the following limit:

$$\lim_{t \to 1-0} (1-t) \sum_{n=0}^{\infty} t^n T_\phi^n \tag{4.1}$$

where T_ϕ maps E_0 into itself and E_0 is a closed subspace of E. From Proposition 4. 2, one can easily find that if $R(z, T_\phi|_{E_0})$ has a pole at $z=1$, then the limit (4. 1) exists in $L(E_0, E_0)$ with operator norm and is equal to the residue of $R(z, T_\phi|_{E_0})$ at $z=1$, that is,

$$\lim_{t \to 1-0} (1-t) \sum_{n=0}^{\infty} t^n T_\phi^n = \frac{1}{2\pi i} \int_C R(z, T_\phi|_{E_0})\, dz \text{ in } L(E_0, E_0)$$

where C is a sufficiently small circle about $z=1$. Note that the order of a pole of $R(z, T_\phi|_{E_0})$ at $z=1$ must be equal to unity, since we have

$$\left\|R(z, T_\phi|_{E_0})\right\| \leq \frac{1}{|z|-1} \qquad \text{for} \quad |z|>1 .$$

In this case, E_0 is decomposed as follows:

$$E_0 = \left\{f \in E_0 ; f = T_\phi(f)\right\} \oplus (Id - T_\phi)(E_0)$$

and the limit operator (4. 1) is a projection to the fixed points set of T_ϕ in E_0.

As a example, we take $J=[0, 1]$ and $\phi(x)=0$. In this case, we have

$$T_\phi(f) = f(0) \qquad \text{for} \quad f \in E$$

and one may think that T_ϕ is Dirac's δ-function. Then we obtain, for $\lambda \neq 0, 1$,

$$R(\lambda, T_\phi) = \frac{1}{\lambda(\lambda-1)} T_\phi + \frac{1}{\lambda} Id \in L(E, E)$$

and $R(\lambda, T_{\varphi})$ has two poles at $\lambda=0$ and $\lambda=1$.

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Note:

We could see Mandelbrot's new book "The Fractal Geometry of Nature" (1982) just after having sent our manuscript to the editor. There, we found that Mandelbrot write about some relation between his Fractals and Chaos (Strange Attractor) Chapt. 20.

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