Hokkaido Mathematical Journal Vol. 12 (1973) p. 293-297

An example of a globally hypo-elliptic operator

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(Received November 1, 1982)

§ 1. Introduction

Let $T^2=R^2/2\pi\mathbf{Z}^2$ be the 2-dimmensional torus. A function $f(x, y)$ of $(x, y)\in R^2$ is identified with a function on the torus $\mathbf{T}^2$ if and only if it is doubly periodic, i.e.,

(1) $\quad f(x+2n\pi, y+2m\pi)=f(x, y)$ for any n and m in $\mathbf{Z}$.

We consider a linear partial differential operator of the second order

(2) $$L=-\frac{\partial^2}{\partial x^2}-\phi(x)^2\frac{\partial^2}{\partial y^2},$$

where $\phi(x)$ is a real-valued function of x of class C^∞. We assume that

(3) $$\begin{aligned}\phi(x) &= 1 \text{ for } |x|<\frac{\pi}{2},\\ &= 0 \text{ for } \frac{3}{4}\pi\leqq|x|\leqq\pi\end{aligned}$$

and that $\phi(x)$ is periodic, i.e., $\phi(x)=\phi(x+2\pi)$.

The aim of this note is to show the following

THEOREM. *The operator L is hypo-elliptic. That is, if a distribution $u\in\mathscr{D}'(\mathbf{T}^2)$ satisfies*

(4) $\quad Lu=f$

and if $f\in C^\infty(\mathbf{T}^2)$, then $u\in C^\infty(\mathbf{T}^2)$.

REMARK. Let U be an open set outside the support of the function $\phi(x)$. Then the restriction of L to U coincides with $-\left(\frac{\partial}{\partial x}\right)^2$. This means that the operator L is not locally hypo-elliptic. Let $X_1=\frac{\partial}{\partial x}$ and $X_2=\phi(x)\frac{\partial}{\partial y}$. Then these vector fields do not satisfy Fefferman-Phong condition [2]. However they are controlable in the sense of Amano [1].

§ 2. Proof.

We shall begin with the following

Proposition 1. *Suppose that* $f \in \mathscr{D}'(\mathbf{T}^2)$ *and*

$$(4) \qquad Lu = f$$

for some $u \in \mathscr{D}'(\mathbf{T}^2)$. *Then*

$$(5) \qquad \langle f, 1 \rangle = 0\,,$$

where $\langle\ ,\ \rangle$ *denotes the canonical bilinear map of* $\mathscr{D}'(\mathbf{T}^2) \times \mathscr{D}(\mathbf{T}^2)$ *to* C.

Proof is ommitted.

We assume from now on that f is of class C^∞ and that it satisfies condition (5). Let

$$(6) \qquad f(x, y) = \sum_{n=-\infty}^{\infty} f_n(x) e^{iny}\,.$$

be the partial fourier expansion of $f(x, y)$ with respect to y. The condition (5) implies that

$$(7) \qquad f_0(x) = 0\,.$$

For any pair of positive integers N and m there exists a constant $C > 0$ such that

$$\left|\left(\frac{\partial}{\partial x}\right)^m f_n(x)\right| \leqq C(1+|n|)^{-N}\,,$$

because $f(x, y)$ is of class C^∞.

Let u_n be the distribution of one variable defined by

$$(8) \qquad \langle u_n\, \phi \rangle = \langle u, \phi \times e^{-iny} \rangle\,, \qquad \text{for any } \phi \in \mathscr{D}(\mathbf{T}^1)\,.$$

Then the partial fourier expansion of u with respect to y is

$$(9) \qquad u = \sum_{n=-\infty}^{\infty} u_n(x) e^{iny}\,.$$

Proposition 2. *Assume that* $f \in C^\infty(\mathbf{T}^2)$ *and that* u *satisfies* (4). *Then for each* n $u_n(x)$ *is a* C^∞ *function of* x *in* T^1 *and it satisfies the equation*

$$(10) \qquad \left\{-\left(\frac{d}{dx}\right)^2 + n^2 \phi(x)^2\right\} u_n(x) = f_n(x)\,, \qquad \textit{if } n \neq 0\,,$$

$$u_0(x) = \text{const}\,.$$

Proof. For any $\phi(x)$ in $C^\infty(\mathbf{T}^1)$, then

$$\begin{aligned} \langle f_n, \phi \rangle &= \langle f, \phi \times e^{-iny} \rangle \\ &= \langle Lu, \phi \times e^{-iny} \rangle \end{aligned}$$

$$= -\left\langle u, \left\{\left(\frac{\partial}{\partial x}\right)^2 - n^2\phi(x)^2\right\} \psi(x) \times e^{-iny}\right\rangle$$
$$= \left\langle \left\{-\left(\frac{d}{dx}\right)^2 + n^2\phi(x)^2\right\} u_n(x), \psi\right\rangle .$$

Hence we have (10). Since f_n is of class $C^\infty(\mathbf{T}^1)$ and ordinary differential operators are hypo-elliptic, $u_n(x) \in C^\infty(\mathbf{T}^1)$. Proposition 2 is proved.

In what follows we shall majorize $u_n(x)$.

DEFINITION. *For any function $v(x)$ in $C^\infty(\mathbf{T}^1)$ we define three norms:*

$$\|v\|_\phi = \left\{\int_{-\pi}^{\pi} \left(\frac{d}{dx} v(x)\right)^2 dx + \int_{-\pi}^{\pi} \phi(x)^2 \left|v(x)\right|^2 dx\right\}^{1/2}, \tag{11}$$

$$\|v\| = \left\{\int_{-\pi}^{\pi} \left|v(x)\right|^2 dx\right\}^{1/2}, \tag{12}$$

$$\|v\|_1 = \left\{\int_{-\pi}^{\pi} \left\{\left(\frac{d}{dx} v(x)\right)^2 + v(x)^2\right\} dx\right\}^{1/2}. \tag{13}$$

LEMMA 3. *There exists a positive constant C such that for any function v in $C^\infty(\mathbf{T}^1)$*

$$\left|v(x)\right| \leqq C\|v\|_\phi \quad \textit{for any } x \textit{ in } \mathbf{T}^1, \tag{14}$$

$$\|v\|_1 \leqq C\|v\|_\phi , \tag{15}$$

$$\|v\| \leqq C\|v\|_\phi . \tag{16}$$

PROOF. Let $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then for any $x \in (-\pi, \pi)$

$$v(x) = v(t) + \int_t^x \left(\frac{d}{ds}\right) v(s)\, ds .$$

Hence

$$\begin{aligned}\left|v(x)\right|^2 &\leqq 2\left|v(t)\right|^2 + 2\left\{\int_t^x \left|\left(\frac{d}{ds}\right) v(s)\right| ds\right\}^2 \\ &\leqq 2\left|v(t)\right|^2 + 4\pi \int_{-\pi}^{\pi} \left|\left(\frac{d}{dx}\right) v(s)\right|^2 ds .\end{aligned}$$

Integrating both sides of this with respect to $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have

$$\begin{aligned}\pi\left|v(x)\right|^2 &\leqq 2\int_{-\pi/2}^{\pi/2} \left|v(t)\right|^2 dt + 4\pi^2 \int_{-\pi}^{\pi} \left|\left(\frac{d}{dx}\right) v(s)\right|^2 ds \\ &\leqq 2\int_{-\pi}^{\pi} \phi(t)^2 \left|v(t)\right|^2 dt + 4\pi^2 \int_{-\pi}^{\pi} \left|\left(\frac{d}{dx}\right) v(s)\right|^2 ds \\ &\leqq 4\pi^2 \|v\|_\phi^2 .\end{aligned} \tag{17}$$

Thus (14) has been proved. Estimates (15) and (16) follow from this.

PROPOSITION 4. *Assume that the function $f(x,y)$ satisfies* (5) *and u is the solution of* (4). *Then there exists a constant C independent of n such that*

(18) $\qquad ||u_n||_\phi \leqq ||f_n|| \qquad$ *for* $n \neq 0$.

PROOF. Multiply (10) by $u_n(x)$ and integrate with respect to x. Then

$$(19)\qquad \begin{aligned} ||u_n||_\phi^2 &\leqq \int_{-\pi}^{\pi} \left|\left(\frac{d}{dx}\right) u_n(x)\right|^2 dx + n^2 \int_{-\pi}^{\pi} \phi(x)^2 \left|u_n(x)\right|^2 dx \\ &= \int_{-\pi}^{\pi} f_n(x)\, u_n(x)\, dx \\ &\leqq ||f_n|| \, ||u_n||_\phi . \end{aligned}$$

Using (16), we have (19).

Now we can prove

THEOREM. *Assume that $f \in C^\infty(\mathbf{T}^2)$ and that u satisfies the equation*

(4) $\qquad Lu = f$.

Then $u \in C^\infty(\mathbf{T}^2)$.

PROOF. By Proposition 2, we may assume $u_0(x)=0$. Since u satisfies (4), its partial fourier coefficients $u_n(x)$ satisfy estimate (18). Combining this with (14), we have for any positive intseger N and for any $x \in [-\pi, \pi]$

$$\left|u_n(x)\right| \leqq C||u_n||_\phi \leqq C||f_n|| \leqq C(1+|n|)^{-N} \qquad (n \neq 0).$$

This implies that the partial fourier series

$$\sum_{n \neq 0} u_n(x)\, e^{iny} \text{ and } \sum_{n \neq 0} n\, u_n(x)\, e^{iny}$$

converge absolutely and uniformly with respect to x and y. Therefore $u(x,y)$ and $\left(\frac{\partial}{\partial y}\right) u(x,y)$ are continuous. The function $v_n(x) = \frac{d}{dx} u_n(x)$ satisfies the equation

$$-\left(\frac{d}{dx}\right)^2 v_n(x) + n^2 \phi(x)^2 v_n(x) = \frac{d}{dx} f_n(x) - \left(\frac{d}{dx} \phi(x)\right)^2 n^2 u_n(x).$$

Since $|2\phi(x)\,\phi'(x)\, n^2 u_n(x)| < Cn^{-N+2}$, we have

(20) $\qquad \left|v_n(x)\right| \leqq Cn^{2-N}$.

As we can choose N in (20) very large,

$$\left(\frac{\partial}{\partial x}\right) u(x,y) = \sum_{n \neq 0} v_n(x)\, e^{iny}$$

converges uniformly in x and y. Thus $\left(\frac{\partial}{\partial x}\right)u(x,y)$ is continuous. Therefore $u(x,y)$ is of class $C^1(\mathbf{T}^2)$. Similar discussion proves that $u(x,y)\in C^\infty(\mathbf{T}^2)$. Theorem has been proved.

References

[1] AMANO, K.: A necessary condition for hypoellipticity of degenerate elliptic-parabolic operators. Tokyo J. Math. Vol. 2 111–120, (1979).

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