On the nilpotency index of the radical of a group algebra

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Throughout the present note, K will represent *an algebraically closed field of characteristic* $p>0$. In case G is a p-solvable group of order $p^a m$ $(a\geqq 1, p\nmid m)$, concerning the nilpotency index $t(G)$ of the radical $J(KG)$ of the group algebra KG, D. S. Passman [4; Th. 1.6], Y. Tsushima [5; Th .2] and D. A. R. Wallace [7; Th. 3.3] have obtained the following:

$$p^a \geqq t(G) \geqq a(p-1)+1 .$$

In §§1 and 2 of the present note, we shall investigate when $t(G)=p^a$ or $t(G)=a(p-1)+1$, where G is a p-solvable group of order $p^a m (a\geqq 1, p\nmid m)$. Furthermore, as an application of Th. 1, we shall present a characterization of a finite group G with $t(G)=[J(KG): K]+1$ (Th. 2).

1. We shall begin our study with the following:

THEOREM 1. *If G is a p-group of order p^a, then there holds the following*:

(1) $t(G)=a(p-1)+1$ *if and only if G is elementary abelian.*

(2) $t(G)=p^a$ *if and only if G is cyclic.*

PROOF. (1) Following [3], we consider the $\mathfrak{K}$-series of G:

$$G=\mathfrak{K}_1 \supseteq \mathfrak{K}_2 \supseteq \cdots \supseteq \mathfrak{K}_{t(G)}=1 ,$$

where $\mathfrak{K}_\lambda=\{x\in G \mid 1-x \in J(KG)^\lambda\}$. Then, every $\mathfrak{K}_\lambda$ is a characteristic subgroup of G and $\mathfrak{K}_\lambda/\mathfrak{K}_{\lambda+1}$ is an elementary abelian group of order p^{d_λ}. By [3; Th. 3.7], we have $t(G)=\sum_\lambda \lambda d_\lambda(p-1)+1$. If $t(G)=a(p-1)+1$ then $\sum_\lambda \lambda d_\lambda=a$. Combining this with $\sum_\lambda d_\lambda=a$, we readily obtain $d_1=a$ and $d_\lambda=0 (\lambda\neq 1)$, namely, G is elementary abelian. The converse is obvious by [3; Th. 6.2].

(2) Suppose $t(G)=p^a$. If $\Phi(G)$ is the Frattini subgroup of G, then [7; Th. 2.4] yields $|G|=t(G)\leqq t(\Phi(G))\cdot t(G/\Phi(G))\leqq |\Phi(G)|\cdot|G/\Phi(G)|=|G|$, whence it follows $t(G/\Phi(G))=|G/\Phi(G)|=p^b (b\leqq a)$. Since $G/\Phi(G)$ is elementary abelian, $t(G/\Phi(G))=b(p-1)+1$ by (1). Hence, $p^b=|G/\Phi(G)|=t(G/\Phi(G))=b(p-1)+1$, which means $b=1$ and $G/\Phi(G)$ is cyclic. Now, as is well-known, G is cyclic. Concerning the converse, there is nothing to prove.

In what follows, G_p will represent a Sylow p-subgroup of G.

COROLLARY 1. *Let G be a p-solvable group of order $p^a m (a \geqq 1, p \nmid m)$. Then there holds the following*:

(1) *If G has p-length* 1 *and $t(G)=a(p-1)+1$ then G_p is elementary abelian, and conversely.*

(2) *If G has p-length* 1 *and $t(G)=p^a$ then G_p is cyclic, and conversely.*

PROOF. Since G is a p-solvable group of p-length 1, G_p is normal or G contains a normal p-nilpotent subgroup H with $p \nmid (G:H)$. In either case, we have $t(G)=t(G_p)$ by [2; Th. 2]. Our assertions are therefore obvious by Theorem 1.

The next contains [7; Th. 3. 4].

COROLLARY 2. *Let G be a p-solvable group of order $p^a m (a \geqq 1, p \nmid m)$. If either $p^a=3$ or $p^a=4$ and G_2 is elementary abelian, then $t(G)=3$, and conversely.*

2. Throughout the present section, G *will represent the symmetric group of degree* 4, *and K an algebraically closed field of characteristic* 2. Obviously, G is a solvable group whose 2-length>1 and whose any Sylow 2-subgroup is not elementary abelian. However, the proposition stated below says that $t(G)=4(=a(p-1)+1)$.

Let G_i be the stabilizer of a letter i and $\hat{S}=\sum_{x \in S} x (\in KG)$ for any subset S of G.

LEMMA 1. $\hat{G}_i x \hat{G}_j = 0$ *for every* $x \in G$.

PROOF. Since $\hat{G}_i x \hat{G}_j x^{-1} = \hat{G}_i \hat{G}_{x(j)}$, it suffices to prove that $\hat{G}_i \hat{G}_j = 0$. In case $i=j$, $\hat{G}_i^2 = 6\hat{G}_i = 0$. While, if $i \neq j$ and $\{1, 2, 3, 4\} = \{i, j\} \cup \{k, l\}$, then $\hat{G}_i = \hat{G}_i(k, l)$, $\hat{G}_i(k, i) = \hat{G}_i(k, l, i)$ and $\hat{G}_i(l, i) = \hat{G}_i(k, i, l)$. Hence $\hat{G}_i \hat{G}_j = \hat{G}_i(1 + (k, l) + (k, i) + (k, l, i) + (l, i) + (k, i, l)) = 2\hat{G}_i + 2\hat{G}_i(k, i) + 2\hat{G}_i(l, i) = 0$.

PROPOSITION. (1) $J(KG) = K\hat{G}_1 \oplus J(KV)KG$, *where V is the Klein's four group contained in* G.

(2) $t(G)=4$.

PROOF. (1) Since V is a normal 2-subgroup of G, $J(K(G/V)) \cong J(KG)/J(KV)KG$. Now, G/V (naturally isomorphic to G_1) is isomorphic to the symmetric group of degree 3, and then $J(K(G/V)) = K\widehat{G/V}$ by [6; Th. 2]. Moreover, noting that $\hat{G}_1$ is an element of $J(KG)$ not contained in $J(KV)KG$, we readily obtain (1).

(2) Since $J(KV)^2 = K\hat{V}$, we have $J(KG)^2 = (K\hat{G}_1 + J(KV)KG)^2 = (K\hat{G}_1)^2 + J(KV)^2 KG + \hat{G}_1 J(KV) + J(KV)\hat{G}_1 = \hat{V}KG + \hat{G}_1 J(KV) + J(KV)\hat{G}_1$. Noting further that $\hat{G}_1 J(KV) \hat{G}_1 = 0$, $\hat{G}_1^2 = 0$ (Lemma 1), $\hat{V} J(KV) = 0$ and that $\hat{V}$ is a central element of KG, we obtain $J(KG)^4 = (\hat{V}KG + \hat{G}_1 J(KV) + J(KV)\hat{G}_1)^2$

$= (\hat{V}KG)^2 + (\hat{G}_1 J(KV))^2 + (J(KV)\hat{G}_1)^2 + \hat{V}\hat{G}_1 J(KV) + (\hat{G}_1 J(KV)\hat{V})KG + \hat{G}_1 J(KV)^2 \hat{G}_1 + J(KV)\hat{G}_1^2 J(KV) + \hat{V}KG(J(KV)\hat{G}_1) + (J(KV)\hat{G}_1\hat{V})KG = 0$. Hence, $t(G) \leqq 4 = 3(2-1)+1$, whence it follows $t(G)=4$.

3. Let G be an arbitrary finite group such that p is a divisor of $|G|$, and $\{e_{ij} | 1 \leqq i \leqq s, 1 \leqq j \leqq f(i)\}$ a set of orthogonal primitive idempotents of KG with $1 = \sum_{i,j} e_{ij}$ such that $KGe_{ij} \cong KGe_{i'j'}$ if and only if $i=i'$. Let $e_i = e_{i1}$ $(1 \leqq i \leqq s)$, $KGe_1/J(KG)e_1$ a trivial KG-module, and $C=(c_{kl})$ the Cartan matrix of G. In this section, we shall investigate when $t(G)=[J(KG): K]+1$. To our end, a couple of lemmas will be needed.

LEMMA 2. $t(G) \leqq \max_k \{\sum_l c_{kl}\} \leqq \max_k \{[J(KG)e_k : K]+1\} \leqq [J(KG): K]+1$.

PROOF. Since $\sum_l c_{kl}$ coincides with the length of the composition series of an indecomposable KG-module KGe_k, $\sum_l c_{kl} \leqq [J(KG)e_k : K]+1$. Thus, $t(G) \leqq \max_k \{\sum_l c_{kl}\} \leqq \max_k \{[J(KG)e_k : K]+1\} \leqq [J(KG): K]+1$ (cf. [7; Lemma 4.2]).

LEMMA 3. *The following conditions are equivalent*:

(1) $[J(KG): K] = \max_k \{\sum_l c_{kl}\} - 1$.
(2) $C = \mathrm{diag}(p^a, 1, \cdots, 1)$.
(3) $[J(KG): K] = p^a - 1$.
(4) *G is either a p-group or a Frobenius group with a complement G_p.*

PROOF. (2), (3) and (4) are equivalent by the proof of [6; Th. 2]. Hence, it remains only to prove that (1) implies (2). Assume that $[J(KG): K] = \max_k \{\sum_l c_{kl}\} - 1$. Then, by Lemma 2, $1 + \sum_{k,j} [J(KG)e_{kj}: K] = 1 + [J(KG): K] = \max_k \{[J(KG)e_k : K]+1\}$. Since $[J(KG)e_1 : K] \geqq p^a - 1$ (cf. [1; p. 562]), it follows that $J(KG) = J(KG)e_1$ and $J(KG)e_k = 0$ for $k \neq 1$. Therfore, $C = \mathrm{diag}(c_{11}, 1, \cdots, 1)$. This means that the first block contains only one irreducible modular character, and hence $c_{11} = [J(KG)e_1 : K] = p^a$ (cf. [1; p. 587]).

Now, we shall conclude our study with the following:

THEOREM 2. $t(G) = [J(KG): K]+1$ *if and only if G is either a cyclic p-group or a Frobenius group with a cyclic complement G_p.*

PROOF. If G is either a cyclic p-group or a Frobenius group with a cyclic complement G_p, then $t(G) = t(G_p) = |G_p| = [J(KG): K]+1$ (cf. [2; Th. 2] and [6; Th. 2]). Conversely, if $t(G) = [J(KG): K]+1$ then, by Lemmas 2 and 3, G is either a p-group or a Frobenius group with a complement G_p. Moreover, we have $t(G_p) = t(G) = [J(KG): K]+1 = |G_p|$, so that

G_p is cyclic by Theorem 1.

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