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Projectively connected manifolds admitting groups of projective transformations of dimension n^2+n

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Introduction

Let M be a manifold of dimension n with a projective structure. It is well known that the group $\mathfrak{P}(M)$ of projective transformations of M is a Lie transformation group such that $\dim \mathfrak{P}(M) \leqq n^2+2n$ ([2], [3]).

The main purpose of this paper is to determine globally projectively connected manifolds admitting groups of projective transformations of the second largest dimension n^2+n.

Our main result is stated as follows;

THEOREM 7.11. *Let M be a connected manifold of dimension n $(n \geqq 3)$ with a projective structure. If M admits a group of projective transformations of dimension n^2+n, then M is projectively equivalent to one of the following spaces;*

(1) $P^n(\boldsymbol{R})$; *the real projective space,*
(2) S^n; *the universal covering space of* (1),
(3) $S^n \setminus \{$*one point*$\}$,
(4) $\boldsymbol{R}^n$; *the affine space,*
(5) $Q = P^n(\boldsymbol{R}) \setminus \{$*one point*$\}$,
(6) $\tilde{Q}$; *the universal covering space of* (5).

The local version of this theorem is obtained by S. Ishihara [1].

Our main emphasis is that the method, developed by the author [6], for Cartan connections associated with graded Lie algebras works equally well to the projective and conformal geometry.

Throughout this paper we always assume the differentiability of class C^∞. We use the notations and terminology in S. Kobayashi [2] without special references.

§ 1. Projective connection

In this section we will recall the notion of the normal projective connection and fix our terminology, following [2] and [3].

Let $P^n(\boldsymbol{R})=L/L_0$ be the real projective space of dimension n with its homogeneous coordinate $(x_0, x_1, \cdots, x_n)$, where

$L=PGL(n, \boldsymbol{R})=GL(n+1, \boldsymbol{R})/\text{center}$,

L_0; the isotropy subgroup of L at $o=(0, \cdots, 0, 1)\in P^n(\boldsymbol{R})$.

The Lie algebra $\mathfrak{l}$ of L has a gradation given by

$$\mathfrak{l}=\mathfrak{sl}(n+1, \boldsymbol{R}), \quad \mathfrak{l}_0=\mathfrak{g}_0+\mathfrak{g}_1,$$

$$\mathfrak{g}_{-1}=\left\{\begin{pmatrix}0 & v\\ 0 & 0\end{pmatrix}\right\}, \quad \mathfrak{g}_0=\left\{\begin{pmatrix}A & 0\\ 0 & -tr\ A\end{pmatrix}\right\}, \quad \mathfrak{g}_1=\left\{\begin{pmatrix}0 & 0\\ {}^t\xi & 0\end{pmatrix}\right\},$$

where v, $\xi\in\boldsymbol{R}^n$, $A\in\mathfrak{gl}(n, \boldsymbol{R})$. Moreover the graded Lie algebra $\mathfrak{l}$ can be described as follows. Let $V(=\boldsymbol{R}^n)$ be the n-dimensional vector space and V^* be the dual space of V. Then

$$\mathfrak{l}=V+\mathfrak{gl}(V)+V^*,$$

under the identification (p. 132 [2]);

$$\begin{pmatrix}0 & v\\ 0 & 0\end{pmatrix}\in\mathfrak{g}_{-1}\longmapsto v\in V, \quad \begin{pmatrix}0 & 0\\ {}^t\xi & 0\end{pmatrix}\in\mathfrak{g}_1\longmapsto \xi\in V^*,$$

$$\begin{pmatrix}A & 0\\ 0 & a\end{pmatrix}\in\mathfrak{g}_0\longmapsto A-aI_n\in\mathfrak{gl}(V).$$

The element $E\in\mathfrak{g}_0$, which defines the gradation of $\mathfrak{l}$, is given by $-\mathrm{id}_V\in\mathfrak{gl}(V)$.

Let $G^2(n)$ be the group of 2-frames at $0\in\boldsymbol{R}^n$. L_0 can be considered as a subgroup of $G^2(n)$ ([2], [3]). Let M be a manifold of dimension n and $P^2(M)$ be the bundle of 2-frames over M. Then a projective structure on M is, by definition, a subbundle P of $P^2(M)$ with structure group L_0. Let θ be the canonical form on $P^2(M)$. Then (P, ω) is called a projective connection if (P, ω) is a Cartan connection of type (L, L_0) (cf. Definition 1. 9 [6] I) and $\omega_{-1}+\omega_0$ coincides with the restriction of θ to P, where ω_i is the $\mathfrak{g}_i$-component of ω.

THEOREM A ([2], [3]). *Let M be a manifold of dimension n $(n\geqq 2)$. For each projective structure P of M, there exists a unique projective connection ω such that the curvature Ω satisfies the following condition*;

$$\Sigma K^i_{jil}=0, \quad \textit{where} \quad \Omega^i_j=\frac{1}{2}\Sigma K^i_{jkl}\omega^k\wedge\omega^l, \ \omega_{-1}=(\omega^i), \ \Omega_0=(\Omega^i_j).$$

This unique projective connection is called the normal projective connection.

Let $\mathfrak{P}(M)$ be the group of projective transformations of M. We consider the Lie algebra $\mathfrak{p}(M)$ of infinitesimal projective transformations of M that generate (global) 1-parameter subgroups of $\mathfrak{P}(M)$. $\mathfrak{p}(M)$ is naturally isomorphic with the Lie algebra of $\mathfrak{P}(M)$. Set $\mathfrak{p}(P)=\{X\in\mathfrak{X}(P)|L_X\omega=0,\ R_{a_*}X=X$ for $a\in L_0$, and X is complete$\}$. Then Theorem A implies that $\mathfrak{p}(P)$ is isomorphic with $\mathfrak{p}(M)$ under the bundle projection.

§ 2. Filtration of $\mathfrak{p}(M)$

In this section we will define a filtration of $\mathfrak{p}(M)$ at $x\in M$, following [6], and give an isomorphism of the associated graded Lie algebra of $\mathfrak{p}(M)$ (at x) into $\mathfrak{l}$.

First we set $\mathfrak{l}_{-1}=\mathfrak{l}$, $\mathfrak{l}_0=\mathfrak{g}_0+\mathfrak{g}_1$ and $\mathfrak{l}_1=\mathfrak{g}_1$. With respect to this filtration $\mathfrak{l}=\mathfrak{l}_{-1}$ becomes a filtered Lie algebra. Note that L_0 preserves this filtration.

Let M be a manifold of dimension n. And let (P,ω) be the normal porjective connection over M.

LEMMA 2.1. (Lemmas 2.2 and 2.3 [6] I). *For $X, Y\in\mathfrak{p}(P)$, and $u\in P$, we have*

(1) $\omega_u(X)\in\mathfrak{l}_0$ *if and only if* $\pi_{*_u}(X)=0$,

(2) $\Omega_u(X, Y)=0$ *if* $\pi_{*_u}(X)=0$ *or* $\pi_{*_u}(Y)=0$,

(3) $-\omega_u([X, Y])=[-\omega_u(X), -\omega_u(Y)]-2\Omega_u(X, Y)$,

where Ω is the curvature form of the connection and π is the bundle projection of P onto M.

The proof is immediate, hence is omitted.

Now let us fix a point x of M and choose a point u of the fibre $\pi^{-1}(x)$ over x. We set

$$\mathfrak{h}_k(x)=\mathfrak{p}(P)\cap\omega_u^{-1}(\mathfrak{l}_k),\quad \text{for}\quad k=-1,\quad 0 \text{ and } 1.$$

Note that this definition is independent of the choice of u in $\pi^{-1}(x)$. Hence the above defines a filtration of $\mathfrak{p}(M)$ at x. From Lemma 2.1 we have

PROPOSITION 2.2. *With respect to the above filtration, $\mathfrak{p}(M)$ becomes a filtered Lie algebra.*

Let $\tilde{\mathfrak{h}}(x)$ be the associated graded Lie algebra of the filtered Lie algebra $\mathfrak{h}_{-1}(x)=\mathfrak{p}(P)$. Setting $\tilde{\mathfrak{h}}_k=\mathfrak{h}_k/\mathfrak{h}_{k+1}$ for $k=-1$, 0 and 1, we have $\tilde{\mathfrak{h}}(x)=\tilde{\mathfrak{h}}_{-1}+\tilde{\mathfrak{h}}_0+\tilde{\mathfrak{h}}_1$.

First observe that there exists an injective linear map ν_u^k of $\tilde{\mathfrak{h}}_k(x)$ into $\mathfrak{g}_k$ which satisfies the following commutative diagram

$$\begin{tikzcd} \mathfrak{h}_k(x) \arrow[r, "-\omega_u"] \arrow[d, "\mu_k"'] & \mathfrak{l} \arrow[d, "p_k"] \\ \tilde{\mathfrak{h}}_k(x) \arrow[r, "\nu_u^k"] & \mathfrak{g}_k \end{tikzcd}$$

where μ_k is the natural projection of $\mathfrak{h}_k$ onto $\tilde{\mathfrak{h}}_k = \mathfrak{h}_k/\mathfrak{h}_{k+1}$ and p_k is the projection of $\mathfrak{l}$ onto $\mathfrak{g}_k$ corresponding to $\mathfrak{l} = \Sigma \mathfrak{g}_k$. We define an injective linear map ν_u of $\tilde{\mathfrak{h}}(x)$ into $\mathfrak{l}$ by setting;

$$\nu_u = \nu_u^{-1} \times \nu_u^0 \times \nu_u^1 .$$

LEMMA 2.3. (Lemma 2.5. [6] I). *Notations being as above, ν_u is an isomorphism of $\tilde{\mathfrak{h}}(x)$ into $\mathfrak{l}$.*

This is immediate from Lemma 2.1. Hence setting $\tilde{\mathfrak{h}}(u) = \nu_u(\tilde{\mathfrak{h}}(x))$, we see that $\tilde{\mathfrak{h}}(u)$ is a graded subalgebra of $\mathfrak{l}$ such that $\dim \tilde{\mathfrak{h}}(u) = \mathrm{din}\, \mathfrak{p}(M)$.

REMARK 2.4. It is easily seen that the above filtration is nothing but the filtration in terms of jets (or Taylor expansions).

§ 3. Graded subalgebras of $\mathfrak{l}$

First recall that the bracket operation of $\mathfrak{l} = V + \mathfrak{gl}(V) + V^*$ can be described as follows (p. 133 [2]);

$$[v, v'] = 0, \quad [\xi, \xi'] = 0, \quad [U, v] = Uv, \quad [\xi, U] = U^*\xi,$$
$$[U, U'] = UU' - U'U, \quad [v, \xi] = v\xi + \langle \xi, v \rangle I_n,$$

where v, $v' \in V$, ξ, $\xi' \in V^*$, U, $U' \in \mathfrak{gl}(V)$ U^* is the adjoint linear map of U and $\langle \ , \ \rangle$ is the canonical pairing of V and V^*. Hence we have

$$\big[[v, \xi], v'\big] = \langle \xi, v' \rangle v + \langle \xi, v \rangle v', \tag{3.1}$$

$$\big[\xi', [v, \xi]\big] = \langle \xi', v \rangle \xi + \langle \xi, v \rangle \xi'. \tag{3.2}$$

Now we will consider a graded subalgebra $\mathfrak{k} = \mathfrak{k}_{-1} + \mathfrak{k}_0 + \mathfrak{k}_1$ of $\mathfrak{l}$. First, from $\mathfrak{g}_0 = [\mathfrak{g}_{-1}, \mathfrak{g}_1]$, we have

LEMMA 3.1. *If $\mathfrak{k}_{-1} = \mathfrak{g}_{-1}$ and $\mathfrak{k}_1 = \mathfrak{g}_1$, then $\mathfrak{k} = \mathfrak{l}$.*

We set $\mathfrak{b}(\mathfrak{k}_{-1}) = \mathfrak{k}_{-1} + \mathfrak{gl}(V, \mathfrak{k}_{-1}) + V^*$, where $\mathfrak{gl}(V, \mathfrak{k}_{-1}) = \{A \in \mathfrak{gl}(V) \mid A(\mathfrak{k}_{-1}) \subset \mathfrak{k}_{-1}\}$. Then we have

LEMMA 3.2. *$\mathfrak{b}(\mathfrak{k}_{-1})$ is a graded subalgebra of $\mathfrak{l}$ containing $\mathfrak{k}$ and* dim $\mathfrak{b}(\mathfrak{k}_{-1}) = r^2 - (n-1)r + n^2 + n$, *where* $r = \dim \mathfrak{k}_{-1}$.

PROOF. From (3.1), we have $[\mathfrak{k}_{-1}, V^*] \subset \mathfrak{gl}(V, \mathfrak{k}_{-1})$. Hence $\mathfrak{b}(\mathfrak{k}_{-1})$ is a graded subalgebra of $\mathfrak{l}$, which obviously contains $\mathfrak{k}$. Last assertion follows from $\dim \mathfrak{gl}(V, \mathfrak{k}_{-1}) = r^2 + n(n-r)$. q. e. d.

Similarly setting $\mathfrak{b}(\mathfrak{k}_1) = V + \mathfrak{gl}(V, \mathfrak{k}_1^*) + \mathfrak{k}_1$, where

$$\mathfrak{gl}(V, \mathfrak{k}_1^*) = \left\{A \in \mathfrak{gl}(V) \,\middle|\, A^*(\mathfrak{k}_1) \subset \mathfrak{k}_1\right\},$$

we have

LEMMA 3.3. *$\mathfrak{b}(\mathfrak{k}_1)$ is a graded subalgebra of $\mathfrak{l}$ containing $\mathfrak{k}$ and* dim $\mathfrak{b}(\mathfrak{k}_1)=r^2-(n-1)r+n^2+n$, *where* $r=\dim \mathfrak{k}_1$.

Take the natural base $\{e_i\}_{1\leqq i\leqq n}$ of $V=\boldsymbol{R}^n$. We denote by H (resp. W) the linear subspace of V spanned by the vectors $e_2, \cdots, e_n$ (resp. e_1). Let $W^\perp$ be the annihilator of W in V^*. We set

$$\mathfrak{b}_* = V+\mathfrak{gl}(V),$$
$$\mathfrak{b}_o = V+\mathfrak{gl}(V, W)+W^\perp,$$
$$\mathfrak{b}_{**} = H+\mathfrak{gl}(V, H)+V^*,$$

where $\mathfrak{gl}(V, W)=\{A\in\mathfrak{gl}(V)|A(W)\subset W\}$ and $\mathfrak{gl}(V, H)=\{A\in\mathfrak{gl}(V)|A(H)\subset H\}$. $\mathfrak{b}_*$, $\mathfrak{b}_o$ and $\mathfrak{b}_{**}$ are graded subalgebras of $\mathfrak{l}$. We set $G_0=\{\sigma\in L_0|\mathrm{Ad}(\sigma)(\mathfrak{g}_i)=\mathfrak{g}_i$ for $i=-1, 0, 1\}$ $(\cong GL(V))$.

Summarizing above discussion we obtain

PROPOSITION 3.4. *Let $\mathfrak{k}$ be a proper graded subalgebra of $\mathfrak{l}$. Then* $\dim\mathfrak{k}\leqq n^2+n$. *The equality holds if and only if there exists $\sigma\in G_0$ such that* $\mathrm{Ad}(\sigma)\mathfrak{k}=\mathfrak{b}_*$, $\mathfrak{b}_o$, $\mathfrak{b}_{**}$ *or* $\mathfrak{l}_0$.

REMARK 3.5. H and W being as above, we denote by S (resp. R) the linear subspace of V spanned by the vectors $e_3, \cdots, e_n$ (resp. e_1, e_2). Then using Proposition 5.7, Lemmas 3.2 and 3.3, we can obtain the following

PROPOSITION 3.6. *Let $\mathfrak{k}$ be a proper graded subalgebra of* $\mathfrak{l}=V+\mathfrak{gl}(V)+V^*$. *If* $\dim\mathfrak{k}\geqq n^2+2$ $(n\geqq 4)$, *then* $\dim\mathfrak{k}=n^2+n$, n^2+n-1 *or* n^2+2 *and there exists $\sigma\in G_0$ such that* $\mathrm{Ad}(\sigma)\mathfrak{k}$ *coincides with one of the following subalgebras of* $\mathfrak{l}$;

(1) $\dim\mathfrak{k}=n^2+n$ $\quad\mathfrak{b}_*$, $\mathfrak{b}_o$, $\mathfrak{b}_{**}$ *or* $\mathfrak{l}_0$,

(2) $\dim\mathfrak{k}=n^2+n-1$ $\quad V+\mathfrak{sl}(V)$, $V+[V, W^\perp]+W^\perp$, $H+[H, V^*]+V^*$ *or* $\mathfrak{sl}(V)+V^*$,

(3) $\dim\mathfrak{k}=n^2+2$ $\quad V+\mathfrak{gl}(V, H)+H^\perp$, $V+\mathfrak{gl}(V, R)+R^\perp$, $W+\mathfrak{gl}(V, W)+V^*$ *or* $S+\mathfrak{gl}(V, S)+V^*$.

§ 4. Structure of $\mathfrak{g}$

In this section we will consider a subalgebra $\mathfrak{g}$ of $\mathfrak{p}(M)$, and will determine the structure of $\mathfrak{g}$ with $\dim\mathfrak{g}\geqq n^2+n$, following the method of [6] I.

Let M be a manifold of dimension n and (P, ω) be the normal projective connection over M. We will consider a subalgebra $\mathfrak{g}$ of $\mathfrak{p}(M)$. We set $\hat{\mathfrak{g}}=\pi_*^{-1}(\mathfrak{g})\subset\mathfrak{p}(P)$.

Now let us fix a poit x of M. As in § 2, we introduce the filtration of $\mathfrak{p}(M)$ (hence of $\mathfrak{g}$) at x through the connection. We first consider the associated graded Lie algebra $\tilde{\mathfrak{g}}(x)$ of $\mathfrak{g}$ at x. Setting $\tilde{\mathfrak{g}}(u)=\nu_u(\tilde{\mathfrak{g}}(x))$, where $u\in\pi^{-1}(x)$, we have

LEMMA 4.1. (1) *If* $\dim \mathfrak{g}=n^2+2n$, *then* $\tilde{\mathfrak{g}}(u)=\mathfrak{l}$ *for any* $u\in\pi^{-1}(x)$,

(2) *If* $\dim \mathfrak{g}<n^2+2n$, *then we have* $\dim \mathfrak{g}\leqq n^2+n$. *The equality holds if and only if there exists* $u\in\pi^{-1}(x)$ *such that*

$$\tilde{\mathfrak{g}}(u)=\mathfrak{b}_*,\ \mathfrak{b}_o,\ \mathfrak{b}_{**}\quad or\quad \mathfrak{l}_0 .$$

This is immediate from Proposition 3.4 and $\dim \mathfrak{g}=\dim \tilde{\mathfrak{g}}(u)$.

In order to determine the structure of $\mathfrak{g}$, we have

LEMMA 4.2. (Lemma 5.5 [6] I). *If* $\tilde{\mathfrak{g}}(u')$ *contains* E *for some point* u' *of* $\pi^{-1}(x)$, *then there exists a point* u *of* $\pi^{-1}(x)$ *such that* $\mathfrak{g}(u)=\omega_u(\hat{\mathfrak{g}})$ *coincides with* $\tilde{\mathfrak{g}}(u')$ *as a vector subspace of* $\mathfrak{l}$, *where* E *is the element of* $\mathfrak{l}$ *which defines the gradation of* $\mathfrak{l}$.

LEMMA 4.3. (IV Theorem 3.2 [2]). *If* $\mathfrak{g}(u_0)$ *contains* E *for some point* u_0 *of* $\pi^{-1}(x)$, *then* $\Omega_u=0$ *for any* $u\in\pi^{-1}(x)$, *where* Ω *is the curvature form of the connection.*

For the proofs of these lemmas, see those of Lemma 5.5 and Proposition 5.6 [6] I.

Summarizing the above results we obtain

PROPOSITION 4.4. *Let* M *be a manifold of dimension* n *and* (P,ω) *be the normal projective connection over* M. *Let* $\mathfrak{g}$ *be a subalgebra of* $\mathfrak{p}(M)$. *Let* x *be an arbitrary point of* M.

(1) *If* $\dim \mathfrak{g}=n^2+2n$, *then* M *is projectively flat and* $\mathfrak{g}=\mathfrak{p}(M)$. *Moreover* $-\omega_u$ *is a Lie algebra isomorphism of* $\mathfrak{p}(P)$ $(\cong\mathfrak{p}(M))$ *onto* $\mathfrak{l}$ *for any* $u\in\pi^{-1}(x)$.

(2) *If* $\dim \mathfrak{g}<n^2+2n$, *then* $\dim \mathfrak{g}\leqq n^2+n$. *The equality holds if and only if* M *is projectively flat and there exists* $u\in\pi^{-1}(x)$ *such that* $-\omega_u$ *is a Lie algebra isomorphism of* $\hat{\mathfrak{g}}$ $(\cong\mathfrak{g})$ *onto* $\mathfrak{b}_*$, $\mathfrak{b}_o$, $\mathfrak{b}_{**}$ *or* $\mathfrak{l}_0$.

(1) is now well known ([1], [2]) and (2) is first obtained by S. Ishihara by a different method (cf. Theorem 1 and Remark 3 [1]).

§ 5. Model spaces

Let $\mathfrak{g}$ be a graded subalgebra of $\mathfrak{l}$ satisfying $\dim \mathfrak{g}\geqq n^2+n$. We will construct a model space for $\mathfrak{g}$.

5.1. The case $\dim \mathfrak{g}=n^2+2n$. We consider $P^n(\boldsymbol{R})=L/L_0$ as the model space for $\mathfrak{g}=\mathfrak{l}$. Let χ be the natural projection of $GL(n+1,\boldsymbol{R})$ onto $L=GL(n+1,\boldsymbol{R})/\boldsymbol{R}^\times$. We set

$$L_\Delta=\left\{\begin{pmatrix}A & 0\\ {}^t\xi & a\end{pmatrix}\in SL(n+1,\boldsymbol{R})\,\middle|\, a=\det A^{-1},\ A\in GL(n,\boldsymbol{R}),\ \xi\in\boldsymbol{R}^n\right\}$$

First we observe

LEMMA 5.1. *If n is even, we have*

(1) *L is isomorphic with $SL(n+1, \boldsymbol{R})$ under χ.*

(2) *L_0 is isomorphic with the group $A(n, \boldsymbol{R})$ of affine transformations of $\boldsymbol{R}^n$. Moreover L_0 is identified, under χ, with L_Δ.*

(3) *The center $Z(L)$ of L is reduced to the unit.*

(4) *The normalizer $N_L(B)$ of B in L coincides with L_0, where B is the identity component of L_0.*

PROOF. (1), (2) and (3) are elementary. In order to prove (4), we consider $L/B \approx S^n$. The action of $L=SL(n+1, \boldsymbol{R})$ on S^n is given through identifying S^n with $\boldsymbol{R}^{n+1}\backslash\{0\}/\boldsymbol{R}^+$. It is easily seen that the orbital decomposition of S^n by B consists of two fixed points and an open orbit (cf. Lemma 5.5 (1) and (3)). From this we conclude that the identity component of $N_L(B)$ coincides with B and the number of connected components $N_L(B)$ is at most two. On the other hand it is obvious $L_0 \subset N_L(B)$. Hence we must have $L_0=N_L(B)$. q. e. d.

LEMMA 5.2. *If n is odd, we have*

(1) *L has two connected components. Let L^0 be the identity compoenet of L. χ is a covering homomorphism of $SL(n+1, \boldsymbol{R})$ onto L^o with* Ker *$\chi=\boldsymbol{Z}_2$, where $\boldsymbol{Z}_2=\{I_n, -I_n\}$ is the center of $SL(n+1, \boldsymbol{R})$.*

(2) *L_0 is isomorphic with $A(n, \boldsymbol{R})$. Moreover $B=L^0 \cap L_0$ is connected and is identified, under χ, with the identity component L_Δ^+ of L_Δ.*

(3) *The center $Z(L^0)$ of L^o is reduced to the unit.*

(4) *The normalizer $N_{L^0}(B)$ of B in L^o coincides with B.*

PROOF. (1) and (2) are elementary. (3) and (4) can be proved quite analogously as in Proposition 6.7 [6] I, hence the proofs are omitted. q. e. d.

Moreover we note

LEMMA 5.3. *If ϕ is an automorphism of L_0 satisfying $\phi_*=\mathrm{id}_{\mathfrak{l}_0}$, then $\phi=\mathrm{id}_{L_0}$.*

PROOF. Since $L_0 \cong A(n, \boldsymbol{R})$, we can replace L_0 by $A(n, \boldsymbol{R})$. We identify $A(n, \boldsymbol{R})$ with a closed subgroup of $GL(n+1, \boldsymbol{R})$ consisting of the matrices of the following form;

$$\begin{pmatrix} A & \xi \\ 0 & 1 \end{pmatrix} \quad A \in GL(n, \boldsymbol{R}),\ \xi \in \boldsymbol{R}^n .$$

Take an element $\sigma_o \in A(n, \boldsymbol{R})$, which does not belong to the identity component $A^+(n, \boldsymbol{R})$;

$$\sigma_o=\begin{pmatrix} J & 0\\ 0 & 1\end{pmatrix}, \quad J=\begin{pmatrix} -1 & 0\\ 0 & I_{n-1}\end{pmatrix}.$$

Then σ_o is characterized by the following relations;

(1) $\sigma_o^2=I_{n+1}$ and $\sigma_o \notin A^+(n, \boldsymbol{R})$,

(2) σ_o commutes with the following elements of $A^+(n, \boldsymbol{R})$;

$$\begin{pmatrix} I_n & e_i\\ 0 & 1\end{pmatrix} 1\leqq i\leqq n-1, \quad \begin{pmatrix} I & 0\\ 0 & 1\end{pmatrix}, \quad I=\begin{pmatrix} 2 & 0\\ 0 & I_{n-1}\end{pmatrix}.$$

Let ϕ be an automorphism of $A(n, \boldsymbol{R})$ satisfying $\phi_*=\mathrm{id}_{\mathfrak{a}(n,\boldsymbol{R})}$. Obviously we have $\phi|_{A^+(n,\boldsymbol{R})}=\mathrm{id}_{A^+(n,\boldsymbol{R})}$. Then $\phi(\sigma_o)$ also satisfies the above relations (1) and (2). Hence we get $\phi(\sigma_o)=\sigma_o$. Therefore we have $\phi=\mathrm{id}_{A(n,\boldsymbol{R})}$.

q. e. d.

Now we consider the normal projective connection over $P^n(\boldsymbol{R})$ or S^n. S^n has the natural projective structure induced by the covering projection; $p: S^n \rightarrow P^n(\boldsymbol{R})$ (p. 144 [2]). Let ω_L and ω_{SL} be the Maurer-Cartan form on L and $SL(n+1, \boldsymbol{R})$ respectively. Recall that the principal bundle L over $L/L_0=P^n(\boldsymbol{R})$ can be naturally identified with the projective structure on $P^n(\boldsymbol{R})$, and (L, ω_L) defines the normal projective connection on $P^n(\boldsymbol{R})$. Moreover the principal bundle $SL(n+1, \boldsymbol{R})$ over $S^n=SL(n+1, \boldsymbol{R})/L_\Delta^+$ can be identified with a connected component of the projective structure on S^n, and $(SL(n+1, \boldsymbol{R}), \omega_{SL})$ defines the normal projective connection on S^n ([2]).

5.2. The case $\dim \mathfrak{g}=n^2+n$. We will first consider the model space for $\mathfrak{b}_*$. Let B_* be the analytic subgroup of L corresponding to $\mathfrak{b}_*$. We consider the (open) orbit Q_* of B_* passing through $o\in P^n(\boldsymbol{R})$ as the model space corresponding to $\mathfrak{b}_*$.

LEMMA 5.4. (1) *B_* is isomorphic with $A^+(n, \boldsymbol{R})$.*

(2) *The orbital decomposition of $P^n(\boldsymbol{R})=L/L_0$ by B_* is given by;*

$$P^n(\boldsymbol{R})=Q_* \cup P^{n-1}(\boldsymbol{R}),$$

where $P^{n-1}(\boldsymbol{R})$ is the hyperplane defined by $x_n=0$.

(3) *Q_* is projective equivalent to the affine space $\boldsymbol{R}^n$.*

(4) *The center $Z(B_*)$ of B_* is reduced to the unit.*

(5) *The normalizer $N_{B_*}(C_*)$ of C_* in B_* coincides with C_*, where C_* is the isotropy subgroup of B_* at $o\in Q_*$.*

PROOF. (1) Since $\mathfrak{b}_*=\left\{\begin{pmatrix} A & v\\ 0 & -tr\, A\end{pmatrix}\in\mathfrak{sl}(n+1, \boldsymbol{R})\right\}$, we have *$B_*$=identity component of* $\left\{\begin{pmatrix} A & \xi\\ 0 & a\end{pmatrix}\in GL(n+1, \boldsymbol{R})\right\}\Big/\boldsymbol{R}^\times=\left\{\begin{pmatrix} A & \xi\\ 0 & a\end{pmatrix}\in GL^+(n+1, \boldsymbol{R})\,|\, a>0\right\}\Big/\boldsymbol{R}^+$.

From $\left\{\begin{pmatrix} A & \xi \\ 0 & a \end{pmatrix} \in GL^+(n+1, \boldsymbol{R}) | a>0\right\} = \boldsymbol{R}^+ \cdot A^+(n, \boldsymbol{R})$, we have

$$B_* = \boldsymbol{R}^+ \cdot A^+(n, \boldsymbol{R})/\boldsymbol{R}^+ \cong A^+(n, \boldsymbol{R}) .$$

(2), (3) and (4) are elementary. (5) can be shown quite analogously as in Proposition 6. 7 [6] I. q. e. d.

Let B_o be the analytic subgroup of L corresponding to $\mathfrak{b}_o$. Let Q be the (open) orbit of B_o passing through $o \in P^n(\boldsymbol{R})$ and C be the isotropy subgroup of B_o at o. Moreover let $\tilde{B}_o$ be the analytic subgroup of $SL(n+1, \boldsymbol{R})$ corresponding to $\mathfrak{b}_o (\subset \mathfrak{sl}(n+1, \boldsymbol{R}))$. Let $\tilde{Q}$ be the (open) orbit of $\tilde{B}_o$ passing through $e_n \in S^n$ and $\tilde{C}$ be the isotropy subgroup of $\tilde{B}_o$ at e_n.

LEMMA 5. 5. (1) *B_0 is the identity component of the isotropy subgroup of L at $o' = (1, 0, \cdots, 0) \in P^n(\boldsymbol{R})$. $\tilde{B}_o$ is isomorphic with B_o under χ.*

(2) *The orbital decomposition of $P^n(\boldsymbol{R})$ by B_0 is given by;*

$$P^n(\boldsymbol{R}) = Q \cup \{o'\} .$$

(3) *The orbital decomposition of $S^n = SL(n+1, \boldsymbol{R})/L_\Delta^+$ by $\tilde{B}_0$ is given by;*

$$S^n = \tilde{Q} \cup \{e_0\} \cup \{-e_0\} .$$

$\tilde{Q}$ is the (2-fold) universal covering space of Q $(n \geqq 3)$.

(4) *$\tilde{C}$ is isomorphic with the identity component C_o of C under χ. And C has two connected components.*

(5) *The center $Z(B_o)$ of B_o is reduced to the unit.*

(6) *The normalizer $N_{B_0}(C_o)$ of C_o in B_0 coincides with C.*

(7) *If ϕ is an automorphism of C satisfying $\phi_* = \mathrm{id}_{\mathfrak{c}}$, then $\phi = \mathrm{id}_C$.*

PROOF. (1) From $\mathfrak{b}_o = V + \mathfrak{gl}(V, W) + W^\perp$, we have more explicitly

$$\mathfrak{b}_o = \left\{\begin{pmatrix} A & v \\ {}^t\xi & -tr\, A \end{pmatrix} \in \mathfrak{sl}(n+1, \boldsymbol{R}) \middle| \xi = \begin{pmatrix} 0 \\ \xi' \end{pmatrix} \in \boldsymbol{R}^n,\ A = \begin{pmatrix} a & * \\ 0 & A' \end{pmatrix} \in \mathfrak{gl}(n, \boldsymbol{R})\right\},$$
$$= \left\{\begin{pmatrix} -trB & \eta \\ 0 & B \end{pmatrix} \in \mathfrak{sl}(n+1, \boldsymbol{R}) \middle| \eta \in \boldsymbol{R}^n,\ B \in \mathfrak{gl}(n, \boldsymbol{R})\right\} .$$

Hence B_o is the identity component of the isotropy subgroup of L at $o' \in P^n(\boldsymbol{R})$. Moreover from Lemma 5. 1 (2) and Lemma 5. 2 (2) we see that $\tilde{B}_o$ is isomorphic with B_o under χ.

(2), (3) and (4) are elementary. (5) can be proved quite analogously as in Proposition 6. 7 [6] I. In order to prove (6) we consider the orbital decomposition of $\tilde{Q} = \tilde{B}_o/\tilde{C}$ by $\tilde{C}$. From

$$\tilde{C}=\left\{\begin{pmatrix} a & {}^t v & 0 \\ 0 & B & 0 \\ 0 & {}^t\xi & b \end{pmatrix} \in GL(n+1, \boldsymbol{R}) \middle| a=(b\cdot\det B)^{-1}>0,\ b>0,\right.$$

$$\left. B\in GL^+(n-1, \boldsymbol{R}),\ v,\ \xi\in \boldsymbol{R}^{n-1}\right\},$$

we easily see that the orbital decomposition of $\tilde{Q}$ by $\tilde{C}$ is given by;

$$\tilde{Q}=W\cup R_1\cup R_2\cup R_3\cup R_4\cup\{e_n\}\cup\{-e_n\} \qquad (n\geqq 3),$$

where

$$W=\left\{(x_0, x', x_n)\in\tilde{Q}\subset S^n \middle| x_0, x_n\in\boldsymbol{R},\ x'\in\boldsymbol{R}^{n-1}\backslash\{0\}\right\},$$

$$R_1=\left\{(x_0, 0, x_n)\in\tilde{Q}\middle| x_0>0, x_n>0\right\},$$

$$R_2=\left\{(x_0, 0, x_n)\in\tilde{Q}\middle| x_0<0, x_n>0\right\},$$

$$R_3=\left\{(x_0, 0, x_n)\in\tilde{Q}\middle| x_0<0, x_n<0\right\},$$

and

$$R_4=\left\{(x_0, 0, x_n)\in\tilde{Q}\middle| x_0>0, x_n<0\right\}.$$

Hence as in the proof of Lemma 5. 1 (4), we get $N_{B_0}(C_o)=C$.

(7) can be proved quite analogously as in Lemma 5. 3, hence its proof is omitted. q. e. d.

It is obvious that B_*, B_o and $\tilde{B}_o$ are the identity component of the group of projective transformations of Q_*, Q and $\tilde{Q}$ respectively. Here Q_*, Q and $\tilde{Q}$ are endowed with the natural projective structure induced from those of $P^n(\boldsymbol{R})$ and S^n.

As for $\mathfrak{b}_{**}$ and $\mathfrak{l}_0$ we note

Lemma 5. 6. (1) *$\mathfrak{b}_*$, $\mathfrak{b}_{**}$, $\mathfrak{b}_o$ and $\mathfrak{l}_0$ are all isomorphic with $\mathfrak{a}(n, \boldsymbol{R})$.*

(2) *$\mathfrak{b}_*$ and $\mathfrak{b}_{**}$ are conjugate under an element of L.*

(3) *$\mathfrak{b}_o$ and $\mathfrak{l}_0$ are conjugate under an element of L.*

This is easily seen from the orbital decompositions of $P^n(\boldsymbol{R})$ by B_* and B_0 (cf. Proposition 3. 2 [6] II).

Moreover forgetting about the gradation of $\mathfrak{l}=\mathfrak{sl}(n+1, \boldsymbol{R})$, we have

Proposition 5. 7. *Let $\mathfrak{g}$ be a proper subalgebra of $\mathfrak{sl}(n+1, \boldsymbol{R})$ $(n\geqq 2)$. Then* $\dim\mathfrak{g}\leqq n^2+n$, *and the equality holds if and only if $\mathfrak{g}$ is conjugate to $\mathfrak{b}_*$ or $\mathfrak{b}_o$ under an inner automorphism of $\mathfrak{sl}(n+1, \boldsymbol{R})$.*

PROOF. If we identify (L, ω_L) with the normal projective connection over $P^n(\boldsymbol{R})$, $\mathfrak{p}(L)$ coincides with the Lie algebra of right invariant vector fields on L. Let $\hat{\mathfrak{g}}$ be the subalgebra of $\mathfrak{p}(L)$ correspoding to $\mathfrak{g}\subset\mathfrak{l}=\mathfrak{sl}(n+1, \boldsymbol{R})$. Let e be the unit element of L and set $\pi_L(e)=x$, where π_L is the bundle projection of L onto $P^n(\boldsymbol{R})$. Then from Proposition 4. 4, $\dim \mathfrak{g}\leqq n^2+n$ and the equality holds if and only if there exists $\sigma\in\pi_L^{-1}(x)=L_0$ such that $-\omega_\sigma$ is a Lie algebra isomorphism of $\hat{\mathfrak{g}}$ onto $\mathfrak{b}_*$, $\mathfrak{b}_o$, $\mathfrak{b}_{**}$ or $\mathfrak{l}_0$.

Let $A\in\hat{\mathfrak{g}}$ and set $X=-\omega_e(A)\in\mathfrak{g}\subset\mathfrak{l}$, $Y=-\omega_\sigma(A)$. Since A is a right invariant vector field we get

$$Y=-\omega_\sigma(A)=-R_\sigma^*\,\omega(A)=-\mathrm{Ad}\,(\sigma^{-1})\,\omega_e(A)=\mathrm{Ad}\,(\sigma^{-1})\,(X)\,.$$

Hence $\mathrm{Ad}\,(\sigma^{-1})\,\mathfrak{g}=\mathfrak{b}_*$, $\mathfrak{b}_o$, $\mathfrak{b}_{**}$ or $\mathfrak{l}_0$. Therefore from Lemma 5. 6, $\mathfrak{g}$ is conjugate to $\mathfrak{b}_*$ or $\mathfrak{b}_o$ under an element of $L^o=\mathrm{Int}\,(\mathfrak{sl}(n+1, \boldsymbol{R}))$. q. e. d.

§ 6. Transitive case

6. 1. Let M be a connected manifold of dimension n and (P, ω) be the normal projective connection over M. We denote by $\tilde{\sigma}$ the connection preserving bundle isomorphism of $P(M, L_0)$ induced by $\sigma\in\mathfrak{P}(M)$.

Let us fix a point $x\in M$ and take a point $u\in\pi^{-1}(x)$. And we define ι_u : $\mathfrak{P}(M)\to P$ by $\iota_u(\sigma)=\tilde{\sigma}(u)$, $\sigma\in\mathfrak{P}(M)$. Then it is well known ([2]) that ι_u is an imbedding of $\mathfrak{P}(M)$ as a closed submanifold of P.

Let $\mathfrak{P}_x(M)$ be the isotropy subgroup of $\mathfrak{P}(M)$ at $x\in M$. Obviously we have

$$\iota_u\big(\mathfrak{P}_x(M)\big)\subset\pi^{-1}(x)\,.$$

On the other hand the fiber $\pi^{-1}(x)$ of $P(M, L_0)$ is diffeomorphic with L_0 via a diffeomorphism γ_u of L_0 onto $\pi^{-1}(x)$, where $\gamma_u(a)=ua$, $a\in L_0$. Therefore the composite map $\rho_u=\gamma_u^{-1}\cdot\iota_u$ is an imbedding of $\mathfrak{P}_x(M)$ into L_0 and $\rho_u(\mathfrak{P}_x(M))$ is closed in L_0. Moreover we have

LEMMA 6. 1. *ρ_u ; $\mathfrak{P}_x(M)\to L_0$ is an injective homomorphism. And $\rho_u(\mathfrak{P}_x(M))$ is a closed subgroup of L_0. Moreover $(\rho_{u*})_e=\omega_u\cdot(\iota_{u*})_e$, where e is the unit of $\mathfrak{P}_x(M)$.*

If we assume that $\mathfrak{P}(M)$ acts transitively on M, $\mathfrak{P}(M)$ is a principal $\mathfrak{P}_x(M)$-bundle over M. Then we have

LEMMA 6. 2. *The imbedding ι_u ; $\mathfrak{P}(M)\to P$ is an injective bundle homomorphism of $\mathfrak{P}(M)\,(M, \mathfrak{P}_x(M))$ into $P(M, L_0)$ corresponding to ρ_u; $\mathfrak{P}_x(M)\to L_0$, which preserves the base space M.*

When the curvature of the normal projective connection vanishes, we have

PROPOSITION 6.3. *Suppose that the curvature form Ω of the normal projective connection vanishes identically. Then the linear map $\iota_u^*\omega$; $\mathfrak{p}(M)\rightarrow \mathfrak{l}$ is a Lie algebra isomorphism of $\mathfrak{p}(M)$ into $\mathfrak{l}$. Hence $\mathfrak{h}(u)=\iota_u^*\omega(\mathfrak{p}(M))$ $(=\omega_u(\mathfrak{p}(P)))$ is a subalgebra of $\mathfrak{l}$ which is isomorphic with $\mathfrak{p}(M)$. Moreover if we identify $\mathfrak{p}(M)$ with $\mathfrak{h}(u)$, $\iota_u^*\omega$ is the Maurer-Cartan form of $\mathfrak{P}(M)$.*

For the proofs of above lemmas and proposition, see those of Lemma 3.1, Proposition 3.2 and Proposition 3.4 of [6] I.

Now we will consider an equivalence of two projectively connected homogeneous manifolds. Let M (resp. M') be a connected manifold of dimension n with the normal projective connection (P, ω) (resp. (P, ω')). We assume that $\mathfrak{P}(M)$ (resp. $\mathfrak{P}(M')$) acts transitively on M (resp. M'). We denote by $\mathfrak{P}^0(M)$ the identity component of $\mathfrak{P}(M)$, and set $\mathfrak{P}_x^0(M)=\mathfrak{P}^0(M)\cap\mathfrak{P}_x(M)$. Note that the identity component $\mathfrak{P}^0(M)$ acts transitively on M.

PROPOSITION 6.4. *Notations being as above, let $x\in M$ and $x'\in M'$. Suppose that for points, $u\in\pi^{-1}(x)$, $u'\in\pi^{-1}(x')$ suitably chosen, there exists a group isomorphism ϕ of $\mathfrak{P}^0(M))$ onto $\mathfrak{P}^0(M')$ satisfying* i), ii);

i) $\phi(\mathfrak{P}_x^0(M))=\mathfrak{P}_{x'}^0(M')$ *and* $\rho_u=\rho_{u'}\cdot\phi|_{\mathfrak{P}_x^0(M)}$,

ii) $\phi^*\iota_{u'}^*\omega'=\iota_u^*\omega$.

Then the bundle isomorphism ϕ of $\mathfrak{P}^0(M)$ $(M, \mathfrak{P}_x^0(M))$ onto $\mathfrak{P}^0(M')$ $(M', \mathfrak{P}_{x'}^0(M'))$ induces a projective isomorphism of M onto M'.

For the proof, see that of Proposition 3.5 [6] I.

6.2. In this paragraph we will determine projectively connected manifolds M with $\dim\mathfrak{P}(M)=n^2+2n$. Though the sketch of the proof of the following theorem is already given in [2], we will give another proof for the sake of completeness.

THEOREM 6.5. (cf. Theorem 6.2 [2], Theorem 3 [1]). *Let M be a connected manifold of dimension n $(n\geqq 2)$ with a projective structure. Let $\mathfrak{P}(M)$ be the group of projective transformations of M. If* $\dim\mathfrak{P}(M)=$ *n^2+2n, then M is projectively equivalent to the real projective space $P^n(\boldsymbol{R})$ or its universal covering space S^n.*

PROOF. From Proposition 4.4 (1), it is obvious that $\mathfrak{P}^0(M)$ acts transitively on M. Let (P, ω) be the normal projective connection over M. Let us fix a point $x\in M$ and take a point $u\in\pi^{-1}(x)$. Then from Proposition 4.4 and Proposition 6.3, we see that $\iota_u^*\omega$ is a Lie algebra isomorphism of $\mathfrak{p}(M)$ onto $\mathfrak{l}$, where $\mathfrak{p}(M)$ is the Lie algebra of $\mathfrak{P}(M)$. In particular we have $\iota_u^*\omega(\mathfrak{p}_x(M)=\mathfrak{l}_0$.

Now we compare $\mathfrak{P}^0(M)$ with L^o. Since L^o is connected and $Z(L^o)=$ $\{e\}$, the adjoint representation Ad_{L^o} of L^o is a isomorphism of L^o onto the adjoint group $\mathrm{Int}(\mathfrak{l})$. On the other hand the adjoint representation $\mathrm{Ad}_{\mathfrak{P}^0(M)}$ of $\mathfrak{P}^0(M)$ is a homomorphism of $\mathfrak{P}^0(M)$ onto $\mathrm{Int}(\mathfrak{p}(M))$. Set $h=\iota_u^*\omega$. Then since h is a Lie algebra isomorphism of $\mathfrak{p}(M)$ onto $\mathfrak{l}$, h naturally induces a group isomorphism $\tilde{h}$ of $\mathrm{Int}(\mathfrak{p}(M))$ onto $\mathrm{Int}(\mathfrak{l})$. More precisely we set $(\tilde{h}(\tau))(X)=h\cdot\tau\cdot h^{-1}(X)$ for $\tau\in\mathrm{Int}(\mathfrak{p}(M))$, $X\in\mathfrak{l}$. Then we have $\tilde{h}_*\cdot\mathrm{ad}_{\mathfrak{p}(M)}=\mathrm{ad}_{\mathfrak{l}}\cdot h$. We set $\phi=(\mathrm{Ad}_L)^{-1}\cdot\tilde{h}\cdot\mathrm{Ad}_{\mathfrak{P}^0(M)}$. Then ϕ is a covering homomorphism, of $\mathfrak{P}^0(M)$ onto L^0 such that $\phi_*=h$.

In the following we divide the proof according as n is even or odd.

(1) The case n is even. From Lemma 5.1, we identify $SL(n+1, \boldsymbol{R})$ with L through χ. Let ω_L be the Maurer-Cartan form on L. Then (L, ω_L) can be identified with the normal projective connection over $P^n(\boldsymbol{R})=L/L_0$. Moreover (L, ω_L) can be identified with a connected component of the normal projective connection over $S^n=L/B$, where B is the identity component of L_0.

Let $(\mathfrak{P}_x(M))^0$ be the identity component of $\mathfrak{P}_x(M)$. Since $\phi_*=\iota_u^*\omega$ as a Lie algebra isomorphism, we have $\phi((\mathfrak{P}_x(M))^0)=B$, i. e. $(\mathfrak{P}_x(M))^0\subset\phi^{-1}(B)$. On the other hand we have $\mathfrak{P}^0(M)/\phi^{-1}(B)\approx L/B\approx S^n$. Since S^n is simply connected, $\phi^{-1}(B)$ is connected. Hence we have $(\mathfrak{P}_x(M))^0=\phi^{-1}(B)$. In particular $\mathrm{Ker}\,\phi\subset\mathfrak{P}_x^0(M)=\mathfrak{P}^0(M)\cap\mathfrak{P}_x(M)$. Hence $\mathrm{Ker}\,\phi$ is a normal subgroup of $\mathfrak{P}^0(M)$ contained in $\mathfrak{P}_x^0(M)$. Since $\mathfrak{P}^0(M)$ acts effectively on $M=\mathfrak{P}^0(M)/\mathfrak{P}_x^0(M)$, we conclude that $\mathrm{Ker}\,\phi$ is trivial, i. e. ϕ is an isomorphism of $\mathfrak{P}^0(M)$ onto L.

From Lemma 5.1 (4), we know that $N_L(B)=L_0$. Hence Lie subgroups of L with Lie algebra $\mathfrak{l}_0\subset\mathfrak{l}$ are B and L_0. Then it follows that $\phi(\mathfrak{P}_x^0(M))$ coincides with B or L_0.

(1.1) In case $\phi(\mathfrak{P}_x^0(M))=B$. ϕ is a bundle isomorphism, of $\mathfrak{P}^0(M)$ $(M, \mathfrak{P}_x^0(M))$ onto $L(S^n, B)$. Moreover from Lemma 6.1 and $\phi_*=\iota_u^*\omega$, we have $\rho_u=\phi|_{\mathfrak{P}_x^0(M)}$. Therefore from Proposition 6.4, we conclude that M is projectively equivalent to S^n.

(1.2) In case $\phi(\mathfrak{P}_x^0(M))=L_0$. ϕ is a bundle isomorphism, of $\mathfrak{P}^0(M)$ $(M, \mathfrak{P}_x^0(M))$ onto $L(P^n(\boldsymbol{R}), L_0)$. Moreover from Lemma 5.3, Lemma 6.1 and $\phi_*=\iota_u^*\omega$, we have $\rho_u=\phi|_{\mathfrak{P}_x^0(M)}$. Therefore from Proposition 6.4, M is projectively equivalent to $P^n(\boldsymbol{R})$.

(2) The case n is odd. Recall from Lemma 5.2 that χ is a covering homorphism of $SL(n+1, \boldsymbol{R})$ onto L^o with $\mathrm{Ker}\,\chi=\boldsymbol{Z}_2$ (the center of $SL(n+1, \boldsymbol{R})$). And χ induces an isomorphism of L_Δ^+ onto B. Let ω_{L^o} and ω_{SL} be the Maurer-Cartan form on L^o and $SL(n+1, \boldsymbol{R})$. Then (L^o, ω_{L^o}) (resp. $(SL(n+1, \boldsymbol{R}), \omega_{SL})$) can be identified with a connected component of the normal projective connection over $P^n(\boldsymbol{R})=L^o/B$ (resp. $S^n=SL(n+1, \boldsymbol{R})/L_\Delta^+$).

From $N_{L^o}(B)=B$ ((4) of Lemma 5.2) and the connectedness of B, we see that B is the only Lie subgroup of L^o with Lie algebra $\mathfrak{l}_0=\phi_*(\mathfrak{p}_x(M))$. Hence we have $\phi(\mathfrak{P}^0_x(M))=B$. Let ϕ' be the restriction of ϕ to $\mathfrak{P}^0_x(M)$. Since $\mathrm{Ker}\,\phi'=\mathrm{Ker}\,\phi\cap\mathfrak{P}^0_x(M)$ is a central subgroup of $\mathfrak{P}^0(M)$, the effectiveness of the action of $\mathfrak{P}^0(M)$ on M implies that $\mathrm{Ker}\,\phi'$ is trivial, i. e. ϕ' is an isomorphism of $\mathfrak{P}^0_x(M)$ onto B. Moreover from Lemma 5. 3, Lemma 6. 1 and $\phi_*=\iota_u^*\omega$, we have $\rho_u=\phi'$. Since $\mathfrak{P}^0_x(M)$ is the identity component of $\phi^{-1}(B)$, $M=\mathfrak{P}^0(M)/\mathfrak{P}^0_x(M)$ is a covering space over $\mathfrak{P}^0(M)/\phi^{-1}(B)\approx L^0/B=P^n(\boldsymbol{R})$. From $\pi_1(P^n(\boldsymbol{R}))\cong\boldsymbol{Z}_2$ $(n\geqq 2)$, we see that $\phi^{-1}(B)$ has at most two connected components, i. e. $\mathrm{Ker}\,\phi=\{e\}$ or $\boldsymbol{Z}_2$.

(2. 1) In case $\mathrm{Ker}\,\phi=\{e\}$. ϕ induces a bundle isomorphism of $\mathfrak{P}^0(M)$ $(M,\mathfrak{P}^0_x(M))$ onto $L^o(P^n(\boldsymbol{R}),B)$. Therefore M is projectively equivalent to $P^n(\boldsymbol{R})$.

(2. 2) In case $\mathrm{Ker}\,\phi=\boldsymbol{Z}_2$. $M=\mathfrak{P}^0(M)/\mathfrak{P}^0_x(M)$ is homeomorphic with S^n. Hence the natural inclusion ι of $\mathfrak{P}^0_x(M)$ into $\mathfrak{P}^0(M)$ induces a homomorphism ι_* of $\pi_1(\mathfrak{P}^0_x(M),e)$ onto $\pi_1(\mathfrak{P}^0(M),e)$. Then we have

$$\phi_*\Big(\pi_1\big(\mathfrak{P}^0(M),e\big)\Big)=\phi'_*\Big(\pi_1\big(\mathfrak{P}^0_x(M),e\big)\Big)=\pi_1(B,e)\,.$$

Similarly we have

$$\chi_*\Big(\pi_1\big(SL(n+1,\boldsymbol{R}),I_n\big)\Big)=\pi_1(B,e)\,.$$

Hence we get

$$\phi_*\Big(\pi_1\big(\mathfrak{P}^0(M),e\big)\Big)=\chi_*\Big(\pi_1\big(SL(n+1,\boldsymbol{R}),I_n\big)\Big)$$

From this there exists a unique isomorphism $\tilde{\phi}$ of $\mathfrak{P}^0(M)$ onto $SL(n+1,\boldsymbol{R})$ satisfying $\phi=\chi\cdot\tilde{\phi}$. Then $\tilde{\phi}$ induces a bundle isomorphism of $\mathfrak{P}^0(M)$ $(M,\mathfrak{P}^0_x(M))$ onto $SL(n+1,\boldsymbol{R})\,(S^n,L^+_\Delta)$. Therefore M is projectively equivalent to S^n. q. e. d.

6. 3. In this paragraph we will determine projectively connected homogeneous manifolds M with $\dim\mathfrak{P}(M)=n^2+n$.

THEOREM 6. 6. *Let M be a connected manifold of dimension n $(n\geqq 3)$ with a projective structure. Let $\mathfrak{P}(M)$ be the group of projective transformations of M. If $\dim\mathfrak{P}(M)<n^2+2n$, then $\dim\mathfrak{P}(M)\leqq n^2+n$. Moreover if $\dim\mathfrak{P}(M)=n^2+n$ and $\mathfrak{P}(M)$ acts transitively on M, then M is projectively equivalent to the affine space $\boldsymbol{R}^n$, Q or $\tilde{Q}$, where $Q=P^n(\boldsymbol{R})\backslash\{o\}$ and $\tilde{Q}=S^n\backslash(\{e\}\cup\{-e\})$ (the universal covering space of Q).*

PROOF. First assertion is clear from Proposition 4. 4. Let (P,ω) be the normal projective connection over M. Let us fix a point x of M. Then from Proposition 4. 4 and Proposition 6. 3, there exists $u\in\pi^{-1}(x)$ such that

$\iota_u^* \omega$ is a Lie algebra isomorphism of $\mathfrak{p}(M)$ onto $\mathfrak{b}_*$ or $\mathfrak{b}_o$.

(1) The case $\iota_u^* \omega(\mathfrak{p}(M))=\mathfrak{b}_*$. From $Z(B_*)=\{e\}$ ((4) of Lemma 5.4), we get a covering homomorphism ϕ of $\mathfrak{P}^0(M)$ onto B_* satisfying $\phi_*=\iota_u^* \omega$, as in the proof of Theorem 6.5. From $N_{B_*}(C_*)=C_*$ ((5) of Lemma 5.4) and the connectedness of C_*, we have $\phi(\mathfrak{P}_x^0(M))=C_*$. On the other hand $\mathfrak{P}^0(M))/\phi^{-1}(C_*)$ is homeomorphic with $B_*/C_*=Q_*\approx\boldsymbol{R}^n$. Since $\boldsymbol{R}^n$ is simply connected we see that $\phi^{-1}(C_*)$ is connected. Hence we have $\mathfrak{P}_x^0(M)=\phi^{-1}(C_*)$. In particular $\operatorname{Ker}\phi\subset\mathfrak{P}_x^0(M)$. Then the effectiveness of the action of $\mathfrak{P}^0(M)$ on M implies that $\operatorname{Ker}\phi$ is trivial, i.e. ϕ is an isomorphism of $\mathfrak{P}^0(M)$ onto B_*. Therefore ϕ induces a bundle isomorphism of $\mathfrak{P}^0(M)$ $(M, \mathfrak{P}_x^0(M))$ onto $B_*(\boldsymbol{R}^n, C_*)$ such that $\phi_*=\iota_u^* \omega$. From Proposition 6.4, we conclude that M is projectively equivalent to the affine space $\boldsymbol{R}^n$.

(2) The case $\iota_u^* \omega(\mathfrak{p}(M))=\mathfrak{b}_o$. From $Z(B_o)=\{e\}$ ((5) of Lemma 5.5), we get a covering homomorphism ϕ of $\mathfrak{P}^0(M)$ onto B_o satisfying $\phi_*=\iota_u^* \omega$. Let $(\mathfrak{P}_x(M))^0$ be the identity component of $\mathfrak{P}_x(M)$. Then we have $\phi(\mathfrak{P}_x(M))^0)=C_o$. On the other hand $\mathfrak{P}^0(M)/\phi^{-1}(C_o)$ is homeomorphic with $B_o/C_o\approx\tilde{Q}$ (Lemma 5.5). Since $\tilde{Q}$ is simply connected ($n\geqq 3$), we see that $\phi^{-1}(C_o)$ is connected. Hence we have $\phi^{-1}(C_o)=(\mathfrak{P}_x(M))^0$. In particular $\operatorname{Ker}\phi\subset\mathfrak{P}_x^0(M)$. From this we see that ϕ is an isomorphism of $\mathfrak{P}^0(M)$ onto B_o. From $N_{B_0}(C_o)=C$ ((6) of Lemma 5.5), we have $\phi(\mathfrak{P}_x^0(M))=C_o$ or C.

(2.1) In case $\phi(\mathfrak{P}(_x^0 M))=C_o$. ϕ is a bundle isomorphism of $\mathfrak{P}^0(M)$ $(M, \mathfrak{P}_x^0(M))$ onto $B_o(\tilde{Q}, C_o)$ ((1), (4) of Lemma 5.5). Moreover from Lemma 6.1 and $\phi_*=\iota_u^* \omega$, we have $\rho_u=\phi|_{\mathfrak{P}_x^0(M)}$. Therefore M is projectively equivalent to $\tilde{Q}$.

(2.2) In case $\phi(\mathfrak{P}_x^0(M))=C$. ϕ is a bundle isomorphism of $\mathfrak{P}^0(M)$ $(M, \mathfrak{P}_x^0(M))$ onto $B_0(Q, C)$. Moreover from Lemma 5.5 (7), Lemma 6.1 and $\phi_*=\iota_u^* \omega$, we have $\rho_u=\phi|_{\mathfrak{P}_x^0(M)}$. Therefore M is projectively equivalent to Q.

q. e. d.

§ 7. Intransitive case

In this section we will determine n-dimensional projectively connected manifolds admitting groups of projective transformations of dimension n^2+n.

7.1. Let M be a connected manifold of dimension n ($n\geqq 3$) and (P, ω) be the normal projective connection over M. We assume that M admits a group of projective transormations of dimension n^2+n. Then without loss of generality we may assume that there exists a connected Lie subgroup G of $\mathfrak{P}(M)$ of dimension n^2+n. Let $\mathfrak{g}$ be the subalgebra of $\mathfrak{p}(M)$ corresponding to $G\subset\mathfrak{P}(M)$. Let us fix a point $x\in M$. From Proposition 4.4 and Proposition 6.3, we have

(1) M is projectively fiat,

(2) There exists a point $u\in\pi^{-1}(x)$ such that $\iota_u^*\omega$ is a Lie algebra isomorphism of $\mathfrak{g}$ onto one of the following four subalgebras of $\mathfrak{l}$;

(a) $\mathfrak{h}_*=V+\mathfrak{gl}(V)$,

(b) $\mathfrak{h}_o=V+\mathfrak{gl}(V, W)+W^\perp$,

(c) $\mathfrak{h}_{**}=H+\mathfrak{gl}(V, H)+V^*$,

(d) $\mathfrak{l}_0=\mathfrak{gl}(V)+V^*$.

Hence the orbit of G passing through x is an open orbit (in case $\iota_u^*\omega(\mathfrak{g})=\mathfrak{h}_*$ or $\mathfrak{h}_o$), a hyperorbit (in case $\iota_u^*\omega(\mathfrak{g})=\mathfrak{h}_{**}$) or a fixed point (in case $\iota_u^*\omega(\mathfrak{g})=\mathfrak{l}_0$). We say that an open orbit O is of type (a) (resp. of type (b)), if $\iota_u^*\omega(\mathfrak{g})=\mathfrak{h}_*$ (resp. $=\mathfrak{h}_0$) for $x\in O$.

As for open orbits we have

LEMMA 7. 1. (1) *The open orbit of G of type* (a) *is projectively equivalent to the affine space* $\boldsymbol{R}^n$.

(2) *The open orbit of G of type* (b) *is projectively equivalent to Q or $\tilde{Q}$.*

PROOF. It is easily seen that G acts effectively and transitively on the open orbit O. Hence O is a projectively connected homogeneous manifold with $\dim\mathfrak{P}(O)\geqq n^2+n$. On the other hand from Lemma 5. 4, Lemma 5. 5 and Proposition 5. 7 it is easily seen that a connected Lie subgroup of L (resp. $SL(n+1, \boldsymbol{R})$) of dimension n^2+n never acts transitively on $P^n(\boldsymbol{R})$ (resp. on S^n). Hence we get $\dim\mathfrak{P}(O)=n^2+n$. Then the lemma follows from Theorem 6. 6. q. e. d.

7. 2. Now we will recall the notion of the (projective) normal coordinates of M. Let $L(M)$ be the linear frame bundle over M. Let $\bar{l}$ be the bundle homomorphism of P onto $L(M)$ corresponding to the linear isotropy representation l of L_0 onto $GL(n, \boldsymbol{R})$ (cf. § 5 of Chapter IV [2]). l can be identified with the homomorphism of L_0 onto $GL(\mathfrak{g}_{-1})$ defined by the following commutative diagram;

$$\begin{array}{ccc} \mathfrak{l} & \xrightarrow{\mathrm{Ad}(a)} & \mathfrak{l} \\ p\big\downarrow & & \big\downarrow p \\ \mathfrak{g}_{-1} & \xrightarrow{l(a)} & \mathfrak{g}_{-1} \end{array} \qquad a\in L_0,$$

where p is the projection corresponding to $\mathfrak{l}=\mathfrak{g}_{-1}+\mathfrak{l}_0$. We set $G_0=\{a\in L_0 | \mathrm{Ad}(a)$ preserves the gradation of $\mathfrak{l}\}$. Then l induces an isomorphism of G_0 onto $GL(\mathfrak{g}_{-1})$.

Let us fix a point u of P. Let U be a sufficiently small neighbourhood of $T_x(M)$ around 0, where $x=\pi(u)$. For a vector $X\in U$, we consider the horizontal vector field $B(\xi)$ such that $X=\bar{l}(u)\,\xi$. Let ϕ_t^ξ be the (local) 1-parameter subgroup generated by $B(\xi)$. Then the exponential map $\exp_u$ of U into M is defined by

$$\exp_u X=\pi\left(\phi_1^\xi(u)\right).$$

It is clear that $\exp_u$ is a local diffeomorphism around $0\in T_x(M)$. $(U, \exp_u)$ is called the normal ocordinate relative to u (cf. § 5 [1] or § 7 [3]).

LEMMA 7.2. *Notations being as above, we have*

(1) $\sigma\cdot\exp_u=\exp_{\tilde{\sigma}(u)}\cdot\sigma_*$ *for* $\sigma\in\mathfrak{P}(M)$,

(2) $\exp_u=\exp_{ua}$ *for* $a\in G_0$,

(3) $\sigma\cdot\exp_u=\exp_u\cdot\sigma_*$ *for* $\sigma\in\rho_u^{-1}(G_0)\subset\mathfrak{P}_x(M)$.

PROOF. (1) $\sigma\cdot\exp_u X=\sigma\cdot\pi\cdot\phi_1^\xi(u)=\pi\cdot\tilde{\sigma}\cdot\phi_1^\xi(u)$. Hence from $\tilde{\sigma}_*(B(\xi))=B(\xi)$, we have $\sigma\cdot\exp_u X=\pi\cdot\phi_1^\xi(\tilde{\sigma}(u))$. On the other hand $\sigma_*(X)=\sigma_*\cdot\bar{l}(u)\,\xi=\bar{l}(\tilde{\sigma}(u))\,\xi$. Therefore we get $\sigma\cdot\exp_u X=\exp_{\tilde{\sigma}(u)}\sigma_* X$.

(2) $\omega(R_{a_*}B(\xi))=R_a^*\,\omega(B(\xi))=\mathrm{Ad}\,(a^{-1})\,\omega(B(\xi))=\mathrm{Ad}\,(a^{-1})\,\xi$. Since $a\in G_0$ we get $\mathrm{Ad}\,(a^{-1})\,\xi\in\mathfrak{g}_{-1}$. Hence we have $R_{a_*}B(\xi)=B(a^{-1}\xi)$, i.e. $R_a\cdot\phi_t^\xi\cdot R_{a^{-1}}=\phi_t^{a^{-1}\xi}$. From $X=\bar{l}(u)\,\xi=\bar{l}(ua)\,a^{-1}\xi$, we have $\exp_{ua}X=\pi\cdot\phi_1^{a^{-1}\xi}(ua)=\pi\cdot R_a\cdot\phi_1^\xi(u)=\pi\cdot\phi_1^\xi(u)=\exp_u X$.

(3) follows from (1) and (2). q. e. d.

Now we will consider the orbital decomposition of M by G. The following Lemmas 7.3, 7.4 and 7.5 are due to S. Ishihara [1].

LEMMA 7.3. (cf. Remark 2 [1]). *If M has a fixed point x of G, then there exists a neighbourhood W of z such that $W\backslash\{x\}$ belongs to an open orbit of G of type (b). In particular x is an isolated fixed point of G.*

PROOF. We consider a normal coordinate $(U, \exp_u)$ around $x=\pi(u)$. We set $W=\exp_u(U)$. First we have $\rho_u(G)=B$. Hence setting $\tilde{G}=G\cap\rho_u^{-1}(G_0)$, we see that $\rho_u(\tilde{G})$ coincides with the identity component of G_0, which is identified with $GL^+(\mathfrak{g}_{-1})$ through l. From (3) of Lemma 7.2, it is seen that the action of $\tilde{G}$ on M is realized on U as the linear isotropy action of $\tilde{G}$. Moreover from $\sigma_*(X)=\bar{l}(\tilde{\sigma}(u))\,(\xi)=\bar{l}(u)\,(l\cdot\rho_u(\sigma)\,(\xi))$, we see that the linear isotropy action of $\tilde{G}$ on $T_x(M)$ is identified, through the frame $\bar{l}(u)$, with the action of $GL^+(\mathfrak{g}_{-1})$ on $\mathfrak{g}_{-1}$. Hence in order to see the action of $\tilde{G}$ around x, we have only to see the action of $GL^+(\mathfrak{g}_{-1})$ on U through $\bar{l}(u)$. Then it is easily seen that $W\backslash\{x\}$ belongs to an open orbit of $\tilde{G}$, hence of G.

Now we consider the isotropy subgroup G_y of G at $y\in W\backslash\{x\}$. Since $\tau\in G_y$ fixes the points x and y, τ carries a geodesic C joining x and y into

C. Hence τ_* leaves invariant the 1-dimensional subspace $\langle \dot{C}(y) \rangle$. On the other hand if G/G_y is an open orbit of type (a), the linear isotropy representation at y is irreducible, which is easily seen from $\mathfrak{b}_* = V + \mathfrak{gl}(V)$. Therefore G/G_y is an open orbit of type (b). q. e. d.

LEMMA 7. 4. (cf. Remark 1 [1]). *If M has a hyperorbit S of G, then for each point x of S there exists a neighbourhood W of z such that $W \backslash S$ belongs to one or two open orbits of G of type (a).*

PROOF. Let us fix $x \in S$. From Proposition 4. 4, there exists $u \in \pi^{-1}(x)$ such that $\iota_u^* \omega$ is a Lie algebra isomorphism of $\mathfrak{g}$ onto $\mathfrak{b}_{**}$. We consider a normal coordinate $(U, \exp_u)$ around x. We set $W = \exp_u(U)$. Let G_x be the isotropy subgroup of G at x. We denote by $\tilde{G}_x$ the identity component of $G_x \cap \rho_u^{-1}(G_0)$. Then from $\iota_u^* \omega(\mathfrak{g}) = \mathfrak{b}_{**} = H + \mathfrak{gl}(V, H) + V^*$, we get $l \cdot \rho_u(\tilde{G}_x) = \{a \in GL^+(V) | a(H) = H\}$. Obviously we have $\bar{l}(u)(H) = T_x(S) \subset T_x(M)$. The orbital decomposition of V by $l \cdot \rho_u(\tilde{G}_u)$ consists of the hyperplane H and two open orbits divided by H. Hence as in the proof of Lemma 7. 3, we conclude that $W \backslash S$ belongs to one or two open orbits of G.

Recall that H is spanned by the vectors $e_2, \cdots, e_n$ of V. Take a point $y = \exp_u \bar{l}(u)(\varepsilon e_1) \in W$. We consider the subgroup $K_y = \{\sigma \in \tilde{G}_x | \sigma(y) = y\}$ of $\tilde{G}_x$. Note that $l \cdot \rho_u(K_y)$ fixes each point on the line $\langle e_1 \rangle$ and carries each hyperplane parallel to H into itself. Now assume that y belongs to an open orbit of type (b). Then there exists a 1-dimensional subspace of $T_y(M)$ which is invariant by G_y. Since $K_y \subset G_y$, this subspace must coincide with $\langle \bar{l}(u)(e_1) \rangle$. We consider a geodesic C joining y and x defined by $C(t) = \exp_u \bar{l}(u)((1-t)\varepsilon e_1)$. Let G_y^0 be the identity component of G_y. Then $\sigma \in G_y^0$ preserves the direction $\dot{C}(0)$. Hence we have $\sigma(C(t)) = C(t)$. In particular $\sigma(x) = x$, i. e. $G_y^0 \subset G_x$. On the other hand we have $K_y = \tilde{G}_x \cap G_y$. Moreover, under the isomorphism $\iota_u^* \omega$ of $\mathfrak{g}$ onto $\mathfrak{b}_{**}$, $\mathfrak{gl}(V, H) + V^*$ (resp. $\mathfrak{gl}(V, H)$) corresponds to G_x (resp. $\tilde{G}_x$). Let $\mathfrak{g}'$ be the subalgebra of $\mathfrak{b}_{**}$ corresponding to $G_y^0 \subset G_x$. Then we have $\mathfrak{g}' \subset \mathfrak{gl}(V, H) + V^*$ and $\dim \mathfrak{gl}(V, H) \cap \mathfrak{g}' = \dim K_y = (n-1)^2$. Let p_1 be the projection of $\mathfrak{gl}(V, H) + V^*$ onto V^*. Since $\operatorname{Ker} p_1 = \mathfrak{gl}(V, H)$, we have

$$\begin{aligned} \dim p_1(\mathfrak{g}') &= \dim \mathfrak{g}' - \dim \operatorname{Ker} p_1 \cap \mathfrak{g}' \\ &= n^2 - (n-1)^2 = 2n - 1 > n = \dim V^* . \end{aligned}$$

This contradiction shows that y belongs to an open orbit of type (a).

q. e. d.

Summarizing the above discussion we obtain

LEMMA 7. 5. (cf. Remark 4 [1]). (1) *If M has a fixed point of G,* *then the orbital decomposition of M by G consists of isolated fixed points and a unique open orbit of type* (*b*).

(2) *If M has a hyperorbit of G, then the orbital decomposition of M by G consists of hyperorbits and open orbits of type* (*a*).

7.3. In this paragraph we will prove the main theorem of this paper. First we have

PROPOSITION 7.6. *Let M be a connected manifold of dimension n ($n\geqq 3$) with a projectiove structure. Let G be a (connected) Lie subgroup of $\mathfrak{P}(M)$ with* $\dim G=n^2+n$. *If M has a fixed point of G, then M is projectively equivalent to $P^n(\boldsymbol{R})$, S^n or $S^n\setminus\{e\}$.*

PROOF. From Lemma 7.5, M has a unique open orbit O of type (*b*). This open orbit is projectively equivalent to Q or $\tilde{Q}$ according to Lemma 7.1. Then as in the proof of Theorem 3.4 [6] II, this equivalence induces a projective imbedding of M into $P^n(\boldsymbol{R})$ or S^n according as $O=Q$ or $\tilde{Q}$, which is compatible with the action of G and B_0. Since M has a fixed point of G, we conclude that M is projectively equivalent to $P^n(\boldsymbol{R})$, S^n or $S^n\setminus\{\text{one point}\}$. q.e.d.

PROPOSITION 7.7. *Let M be a connected manifold of dimension n ($n\geqq 3$) with a projective structure. Let G be a connected Lie subgroup of $\mathfrak{P}(M)$ with* $\dim G=n^2+n$. *If M has a hyperorbit of G, then M is projectively equivalent to $P^n(\boldsymbol{R})$ or S^n.*

PROOF. From Lemma 7.1, 7.4 and 7.5, there exists an open orbit O_1 of G, which is projectively equivalent to the affine space $\boldsymbol{R}^n$.

Now the proof is divided into several lemmas.

LEMMA 7.8. *Each hyperorbit H of G in M is diffeomorphic with $P^{n-1}(\boldsymbol{R})$ or S^{n-1}. In particular each hyperorbit is compact.*

PROOF. Let ∇ be a torsion free affine connection of M which induces the given projective structure on M. Then from the consideration of normal coordinates around H it is easily seen that H is a totally geodesic submanifold of M. Since ∇ is torsion free, H is an autoparallel submanifold of M (cf. Theorem 8.4 of Chapter VII [4]). Hence ∇ induces a torsion free affine connection ∇^H on H, which finally induces a projective structure on H. Moreover G acts on H as a group of projective transformations with respect to this projective structure on H. It is easily seen that the effective kernel of G is of dimension $n+1$, which is the radical of G. Hence H is a connected $(n-1)$-dimensional projectively connected manifold with $\dim\mathfrak{P}(H)=n^2-1=(n-1)^2+2(n-1)$. Therefore from Theorem 6.5, H is projectively equivalent to $P^{n-1}(\boldsymbol{R})$ or S^{n-1}. q.e.d.

From Lemma 7.4 it is obvious that $\bar{O}_1 \backslash O_1$ consists of hyperorbits of G. Take a hyperorbit H which is a member of $\bar{O}_1 \backslash O_1$. Then since H is connected we see that the following two cases can occur;

(1) $N = O_1 \cup H$ is an open submanifold of M,

(2) $N = O_1 \cup H$ is a manifold with a boundary H.

We will study the above two cases separately.

Case (1). First we have

LEMMA 7.9. $M = O_1 \cup H$.

PROOF. we have only to show that N is compact. Let $D(H)$ be the normal disk bundle of H in N and $\mathring{D}(H)$ be the interior of $D(H)$. Then if we identify $O_1 = N \backslash H$ with $\boldsymbol{R}^n$, $N \backslash \mathring{D}(H)$ is identified with a bounded closed subset of $\boldsymbol{R}^n$. Hence $N \backslash \mathring{D}(H)$ is compact. On the other hand since H is compact, $D(H)$ is compact. Therefore N is compact. q. e. d.

Let $\hat{p}(M)$ be the Lie algebra of all infinitesimal projective transformations of M. Since O_1 is projectively equivalent to $\boldsymbol{R}^n$, we have $\dim \hat{p}(O_1) = n^2 + 2n$. Moreover since M is flat, for each point x of H, there exists an open neighbourhood U_x of x such that $\dim \hat{p}(U_x) = n^2 + 2n$. On the other hand two (local) infinitesimal projective transformations coincide in the whole intersection of their domains if they coincide in an open subset. Hence from $\dim \hat{p}(O_1) = \dim \hat{p}(U_x) = n^2 + 2n$ for $x \in H$,, we get $\dim \hat{p}(M) = n^2 + 2n$. In other words each element of $\hat{p}(O_1)$ can be continued wholly on M. Since M is compact, we conclude that $\dim \mathfrak{P}(M) = n^2 + 2n$.

From Proposition 5.7, we see easily that a (n^2+n)-dimensional connected Lie subgroup of L (resp. $SL(n+1, \boldsymbol{R})$) is conjugate to B_* or B_o $\left(\text{resp. } \tilde{B}_o \text{ or } \tilde{B}_* = \left\{\begin{pmatrix} A & \xi \\ 0 & a \end{pmatrix} \in SL(n+1, \boldsymbol{R})\right\}^+\right)$. Then since M has a unique open orbit, M is projectively equivalent to $P^n(\boldsymbol{R})$.

Case (2). First we have

LEMMA 7.10. (1) *N is compact.*

(2) *There exists another open orbit O_2 of G such that $M = O_1 \cup H \cup O_2$.*

PROOF. (1) By considering a collar neighbourhood of $\partial N = H$ in N, we easily see that N is compact as in Lemma 7.9.

(2) Considering a normal coordinate around $x \in H$, we see that there exists another open orbit O_2 of G such that $H \subset \bar{O}_2$. Then $O_2 \cup H$ is a manifold with boundary, since otherwise we have $M = O_2 \cup H$. $O_1 \cup H \cup O_2$ is an open submanifold of M which is compact. Hence we get $M = O_1 \cup H \cup O_2$. q. e. d.

Similarly as in *Case* (1), we have $\dim \mathfrak{P}(M)=n^2+2n$. Since M has two open orbits we see that M is projectively equivalent to S^n. q. e. d.

Summarizing the above propositions and Theorem 6. 6, we obtain the main theorem of this paper.

THEOREM 7. 11. *Let M be a connected manifold of dimension n $(n\geqq 3)$ with a projective structure. If M admits a group of projective transformations of dimension n^2+n, then M is projectively equivalent to one of the following spaces;*

(1) $P^n(\boldsymbol{R})$; *the real projective space,*
(2) S^n; *the universal covering space of* (1),
(3) $S^n\backslash\{$*one point*$\}$,
(4) R^n; *the affine space,*
(5) $Q=P^n(\boldsymbol{R})\backslash\{$*one point*$\}$,
(6) $\tilde{Q}$; *the universal covering space of* (5).

§ 8. Remarks on the conformal case

In this section we will obseve that we can determine Riemannian manifolds of dimension n admitting groups of conformal transformations of the second largest dimension $\frac{1}{2}n(n+1)+1$ by the same method as above. In particular we note that this case has a close resemblance to the case of strongly pseudo-convex hypersurfaces (cf. [6]).

Let $$S^n=\left\{(x_0,\cdots,x_{n+1})\in P^{n+1}(\boldsymbol{R})\,\middle|\,2x_0\cdot x_{n+1}=x_1^2+\cdots+x_n^2\right\}$$

be the Möbius space of dimension n, where $(x_0,\cdots,x_{n+1})$ is the homogenous coordinate of $P^{n+1}(\boldsymbol{R})$. Then $S^n=L/L_0$, where

$L=O(n+1,1)$,

L_0; the isotropy subgroup of L at $o=(0,\cdots,0,1)\in S^n$.

The Lie algebra $\mathfrak{l}$ of L has a gradation given by

$$\mathfrak{l}=\left\{X\in\mathfrak{gl}(n+2,\boldsymbol{R})\,\middle|\,X=\begin{pmatrix}-a & {}^tv & 0\\ \xi & A & v\\ 0 & {}^t\xi & a\end{pmatrix} A\in\mathfrak{o}(n),\ \xi,\ v\in\boldsymbol{R}^n,\ a\in\boldsymbol{R}\right\},$$

$$\mathfrak{g}_{-1}=\left\{\begin{pmatrix}0 & {}^tv & 0\\ 0 & 0 & v\\ 0 & 0 & 0\end{pmatrix}\right\},\qquad \mathfrak{g}_0=\left\{\begin{pmatrix}-a & 0 & 0\\ 0 & A & 0\\ 0 & 0 & a\end{pmatrix}\right\},\qquad \mathfrak{g}_1=\left\{\begin{pmatrix}0 & 0 & 0\\ \xi & 0 & 0\\ 0 & {}^t\xi & 0\end{pmatrix}\right\},$$

$\mathfrak{l}_0=\mathfrak{g}_0+\mathfrak{g}_1$.

Moreover the graded Lie algebra $\mathfrak{l}$ can be described as follows. Let $V(\cong \boldsymbol{R}^n)$ be the n-dimensional euclidean vector space and V^* be the dual space of V. We denote by ξ_* the image of $\xi\in V$ under the isomorphism of V onto V^* induced from the innerproduct of V, i. e. $\langle \xi^*, v\rangle=(\xi, v)$ for $v\in V$, where $(\ ,\ )$ is the innersproduct of V. Then

$$\mathfrak{l}=V+\mathfrak{co}(V)+V^*,$$

under the identification (p 134 [2]);

$$\begin{pmatrix} 0 & {}^t v & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix}\in\mathfrak{g}_{-1}\longmapsto v\in V, \qquad \begin{pmatrix} 0 & 0 & 0 \\ \xi & 0 & 0 \\ 0 & {}^t\xi & 0 \end{pmatrix}\in\mathfrak{g}_1\longmapsto \xi_*\in V^*,$$

$$\begin{pmatrix} -a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix}\in\mathfrak{g}_0\longmapsto A-aI_n\in\mathfrak{co}(V).$$

Then we have

$$[v, v']=0,\ [\xi_*, \xi'_*]=0,\ [U, v]=Uv,\ [\xi_*, U]=({}^tU\xi)_*,$$
$$[U, U']=UU'-U'U,\ [v, \xi_*]=v\xi_*-{}^t(v\xi_*)+(v, \xi)\, I_n,$$

where v, v', ξ, $\xi'\in V$ and U, $U'\in\mathfrak{co}(V)$. Hence we have

(8. 1) $\qquad [\mathfrak{g}_{-1}, \mathfrak{g}_1]=\mathfrak{g}_0,$

(8. 2) $\qquad \big[[v, \xi_*], \xi'_*\big]=(\xi, \xi')v_*-(v, \xi')\,\xi_*-(v, \xi)\,\xi'_*,$

(8. 3) $\qquad \big[v, [v', \xi_*]\big]=(v, \xi)\,v'-(v, v')\,\xi+(v', \xi)\,v.$

Let $\mathfrak{k}$ be a graded subalgebra of $\mathfrak{l}$. Then we get easily

LEMMA 8. 1. *Assume that* $\mathfrak{k}_{-1}\neq\{0\}$ *and* $\mathfrak{k}_1\neq\{0\}$, *then we have*

(1) $(\mathfrak{k}_{-1})_*=\mathfrak{k}_1$. *In particular if* $\mathfrak{k}_{-1}=\mathfrak{g}_{-1}$ *or* $\mathfrak{k}_1=\mathfrak{g}_1$, *then* $\mathfrak{k}=\mathfrak{l}$.

(2) *Set* $\mathfrak{co}(V, \mathfrak{k}_{-1})=\{U\in\mathfrak{co}(V)|\,U(\mathfrak{k}_{-1})\subset\mathfrak{k}_{-1}\}$, *then* $\tilde{k}=\mathfrak{k}_{-1}+\mathfrak{co}(V, \mathfrak{k}_{-1})+\mathfrak{k}_1$ *is a graded subalgebra of* $\mathfrak{l}$ *containing* $\mathfrak{k}$ *such that* $\dim\tilde{\mathfrak{k}}=\dim\mathfrak{l}-(n-s)\,(s+2)$, *where* $s=\dim\mathfrak{k}_{-1}$.

From this lemma we get

PROPOSITION 8.2. *Let* $\mathfrak{k}$ *be a proper graded subalgebra of* $\mathfrak{l}$. *Then* $\dim\mathfrak{k}\leqq\dfrac{1}{2}\,n(n+1)+1(=\dim\mathfrak{l}-n)$. *The equality holds if and only if* $\mathfrak{k}=\mathfrak{l}_0$ *or* $\mathfrak{b}=\mathfrak{g}_{-1}+\mathfrak{g}_0$.

Let B be the analytic subgroup of L corresponding to $\mathfrak{b}\subset\mathfrak{l}$. Then we have

PROPOSITION 8.3. (1) *B is the identity component of the isotropy subgroup of L at $o' = (1, 0, \cdots, 0) \in S^n$.*

(2) *The orbital decomposition of S^n by B consists of a unique open orbit Q and a fixed point o'. Q is conformally equivalent to the equclidean space $\boldsymbol{R}^n$.*

(3) *There exists $\sigma \in B$ such that o is the only fixed point of σ in Q.*

Now using the above propositions in the proofs of Proposition 7.1 [6] I and Theorem 3.4 [6] II and form the unique existence theorem of the normal conformal connection (Theorem 4.2 [2]), we obtain

THEOREM 8.4. *Let M be a connected manifold of dimension n $(n \geqq 3)$ with a conformal structure. If M admits a group of conformal transformations of the second largest dimension $\frac{1}{2}n(n+1)+1$, then M is conformally equivalent to the Möbius space S^n or the euclidean space $\boldsymbol{R}^n$.*

The above theorem is first obtained by T. Nagano [5] by a different method.

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