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A family of difference sets having -1 as an invariant

By Masahiko MIYAMOTO

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A construction is given for difference sets having -1 as an invariant, whose parameters are

$$v=\frac{1}{2}3^{s+1}(3^{s+1}-1),\ k=\frac{1}{2}3^{s}(3^{s+1}+1),\ \lambda=\frac{1}{2}3^{s}(3^{s}+1),\ n=3^{2s}\ (s \text{ even}).$$

Let G be a finite group of order v. A subset D of order k is called a difference set in G with parameters (v, k, λ, n) in case every non-identity element g in G can be expressed in exactly λ way as $g=d_1^{-1}d_2$ with $d_1, d_2 \in D$. The parameter n is defined by $n=k-\lambda$. For any integer t, let $D(t)$ denote the image of D under the mapping $g \to g^t$, $g \in G$. If the mapping is an automorphism of G and $D(t)$ is a translate of D, then t is called a multiplier of D. But even if it is not an automorphism, $t=-1$ has an important property, that is, it makes a non-direct graph which has a regular automorphism.

In this paper, we will show an infinite series of difference sets having -1 as an invariant. SPENCE [1] showed a family of difference set with parameters

$$v=\frac{1}{2}3^{s+1}(3^{s+1}-1),\ k=\frac{1}{2}3^{s}(3^{s+1}+1),\ \lambda=\frac{1}{2}3^{s}(3^{s}+1),\ n=3^{2s}.$$

By modification of his argument, we will prove the following theorem.

THEOREM. *There exists a difference set with parameter*

$$v=\frac{1}{2}3^{s+1}(3^{s+1}-1),\ k=\frac{1}{2}3^{s}(3^{s+1}+1),\ \lambda=\frac{1}{2}3^{s}(3^{s}+1),\ n=3^{2s}$$

which has -1 as an invariant for each even integer $s \geq 2$.

PROOF. Let E denote the additive group of $GF(3^{s+1})$ and K_1 be the multiplicative group of $GF(3^{s+1})$, where s is an even integer ≥ 2. Then since s is even, we have $K_1=Z/2Z \times K$ for a subgroup K of odd order. Set $G=E*K$ be the semi-direct product of E by K. Then we have the following;

a) $|G|=\frac{1}{2}3^{s+1}(3^{s+1}-1)$,

b) K is a cyclic subgroup of order $r=\frac{1}{2}(3^{s+1}-1)$,

c) K acts on $E^{\#}$ as fixed point free automorphisms, and

d) K permutes all hyperplanes of E transitively and no elements of $K^{\#}$ fix a hyperplane of E.

Let H be a hyperplane of E and $k_1=1, k_2, \cdots, k_r$ be the elements of K. Then we will show that

$$D=(E-H)^*k_1 \cup \bigcup_{i=1}^{r} (H^{\sqrt{k_i}^{-1}})^*k_i$$

is a difference set in G having -1 as an invariant, where $\sqrt{k}$ is a square of k in K, which is well defined since K has an odd order. Using the group ring notation for ZE, it is readily seen that (cf. [1])

$$H^{\sqrt{k_1}^{-1}}+H^{\sqrt{k_2}^{-1}}+\cdots+H^{\sqrt{k_r}^{-1}}=3^s.\quad 1_E+\frac{1}{2}(3^s-1)\,E\,,$$

$$H^{\sqrt{k_i}^{-1}}\,H^{\sqrt{k_i}^{-1}}=3^s\,H^{\sqrt{k_i}^{-1}}\,,\qquad H^{\sqrt{k_i}^{-1}}\,H^{\sqrt{k_j}^{-1}}=3^{s-1}E\ (i\neq j)\,,$$

$$(E-H)\,(E-H)=3^s(H+E)\,,\ \ \textit{and}\ \ (E-H)\,H^{\sqrt{k_i}^{-1}}=2\cdot 3^{s-1}E\ (i\neq 1)\,.$$

To verify that D is a difference set in G it is sufficient to show that $D(-1)\,D=n1_G+\lambda G$, where n, λ are as above. We can easily check $D(-1)=D$. Using the above results, we obtain

$$\begin{aligned}
D(-1)\,D=&(E-H)\,(E-H)^*k_1+\sum_{j=2}^{r}(H^{\sqrt{k_i}}{}^*k_i^{-1})\,(H^{\sqrt{k_i}^{-1}}{}^*k_i)\\
&+\sum_{2\leq i\neq j\leq r}(H^{\sqrt{k_i}}{}^*k_i^{-1})\,(H^{\sqrt{k_j}^{-1}}{}^*k_j)\\
&+(E-H)\sum_{j=2}^{r}H^{\sqrt{k_j}^{-1}}{}^*k_j+\left(\sum_{i=2}^{r}H^{\sqrt{k_i}}{}^*k_i^{-1}\right)(E-H)^*1_K\\
&=3^2(E+H)^*1_K+\sum_{i=2}^{r}H^{\sqrt{k_i}}H^{\sqrt{k_i}}{}^*1_K+\sum_{2\leq i\neq j\leq r}^{i}H^{\sqrt{k_i}}\,H^{\sqrt{k_j}^{-1}k_i}{}^*k_i^{-1}k_j\\
&+2\cdot 3^{s-1}E^*(K-1_K)+\sum_{i=2}^{r}H^{\sqrt{k_i}}(E-H)^{\sqrt{k_i}}{}^*k_i^{-1}\\
&=3^2(E+H)^*1_K+\sum_{i=2}^{r}3^s\,H^{\sqrt{k_i}}{}^*1_K+\sum_{2\leq i\neq j\leq r}3^{s-1}E^*k_i^{-1}k_j\\
&+2\cdot 3^{s-1}E^*(K-1_K)+2\cdot 3^{s-1}E^*(K-1_K)\\
&=3^s\,E^*1_K+3^s\left(3^s\cdot 1_E+\frac{1}{2}(3^s-1)\,E\right)^*1_K+3^{s-1}(r+2)\,E^*(K-1_K)\\
&=3^{2s}\cdot 1_G+\frac{1}{2}3^s(3_s+1)\,E^*1_K+3^{s-1}\left(\frac{1}{2}(3^{s+1}-1)+2\right)E^*(K-1_K)\\
&=3^{2s}\cdot 1_G+\frac{1}{2}3^s(3^s+1)\,G\,.
\end{aligned}$$

So we have proved that D is a difference set having -1 as an invariant. This completes the proof of Theorem.

Reference

[1] E. SPENCE: A family of difference sets, J. Combin. Theory ser. A 22 (1977), 103–106.

Department of Mathematics
Ehime University