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Spectral orders and differences

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1. Introduction

The purpose of this paper is to investigate the relationship between differences of functions and majorization inequalities. More specifically, we shall extend the following theorem of Lorentz-Shimogaki-Day (see [12, Proposition 1, p. 34] and [5, Proposition (6. 1) (ii), p. 941]) to the case when (X, Λ, μ) is any totally σ-finite measure space:

THEOREM L-S-D. *Let (X, Λ, μ) be a finite measure space. If f, $g \in L^1(X)$, then $f^* - g^* \prec f - g$ and $|f^* - g^*| \ll |f - g|$.*

In the above theorem, $\prec$ and $\ll$ mean the Hardy, Littlewood and Pólya preorders (precisely defined in chapter 2).

Our Main Theorems are Theorems 1 and 2 in chapter 2. Proofs of them are easy; but they have many important applications in analytical fields. Theorems 1 and 2 extend recent results obtained by Chong [4, the left hand side inequality of (3. 7), p. 148] and by Chiti [1, Theorem, p. 24], and as a corollary to Theorem 2 (Corollary 8), we can show that, in any Orlicz spaces, convergence of a sequence $\{f_n\}$ to f implies convergences of $\{f_n^*\}$ to f^*, and $\{|f_n|^*\}$ to $|f|^*$, where, in general, h^* means the decreasing rearrangement of a measurable function h.

2. Preliminaries and statements of the Main Theorems

Let (X, Λ, μ) be a measure space. Throughout the paper, we assume that $\infty \geqq a = \mu(X) > 0$ and m is Lebesgue measure on $[0, a)$. Denote by $\mathfrak{M}(X)$ the set of all extended-real valued measurable functions on X, and let $L^1(X)$ and $L^\infty(X)$ stand for the set of all integrable functions and essentially bounded functions on X respectively. Any μ a. e. equal functions are identified. To each f in $\mathfrak{M}(X)$, assign its *decreasing rearrangement* f^* (see [13], [2], [9] and [15]): f^* is a uniquely determined, non-increasing and right continuous function on $[0, a)$ which is *equidistributed* with f, that is, $d_f(s) \equiv \mu([f > s]) = m([f^* > s])$ for all $s \in \boldsymbol{R} = (-\infty, \infty)$. In fact, the function f^* is defined by $f^*(t) = \sup\{s : d_f(s) > t\}$, provided that $\sup \phi = -\infty$, where ϕ denotes the empty set.

Define $\mathfrak{P}(X)=\{f\in\mathfrak{M}(X): \lim_{s\to-\infty} d_f(s)=\mu(X) \text{ or } f^+\in L^1(X)+L^\infty(X)\}$, where $L^1(X)+L^\infty(X)$ means the algebraic sum of $L^1(X)$ and $L^\infty(X)$ in $\mathfrak{M}(X)$. Further, let (X, Λ, μ) and (X', Λ', μ') be two measure spaces with $\mu(X)=\mu'(X')=a$. Then *Hardy-Littlewood-Pólya weak spectral order* $\ll$ for pairs of functions $f\in\mathfrak{P}(X)$ and $g\in\mathfrak{P}(X')$ is defined by the following (see [15, Definition 2]): If $f\in\mathfrak{P}(X)$ and $g\in\mathfrak{P}(X')$, we write $f\ll g$ whenever

$$\int_0^s f^*(t)\,dt \leqq \int_0^s g^*(t)\,dt \qquad \text{for all} \quad s\in(0, a);$$

we say that f is *majorized* by g whenever $f\ll g$. Further, if $f\ll g$ and both integrals $\int_0^a f^*(t)\,dt$ and $\int_0^a g^*(t)\,dt$ are definite and equal to each other, we write $f< g$.

The preorders $\ll$ and $<$ were originally introduced in $L_+^\infty(0, 1)$ by Hardy, Littlewood and Pólya [7], and have been studied by many authors (see, for examples, [8], [10], [11], [6] [13], [2], [3], and recently published books [14] and [16]). Among results for the preorder $\ll$, the most improtant one is the following (see [15, Theorem 2. 2] for the present general form):

THEOREM H-L-P. *Let (X, Λ, μ) and (X', Λ', μ') be two measure spaces with $\mu(X)=\mu'(X')$, and let $f\in\mathfrak{P}(X)$ and $g\in\mathfrak{P}(X')$. Then, $f\ll g$ if and only if*

$$\int_X (f-u)^+\,d\mu \leqq \int_{X'} (g-u)^+\,d\mu' \qquad \textit{for all} \quad u\in\boldsymbol{R}.$$

Now we state our Main Theorems.

THEOREM 1. *Let (X, Λ, μ) be a totally σ-finite measure space with $\mu(X)=a$, and let $f, g\in\mathfrak{M}(X)$. If $f-g\in\mathfrak{P}(X)$ and $f^*-g^*\in\mathfrak{P}([0, a))$, then $f^*-g^*\ll f-g$.*

THEOREM 2. *Let (X, Λ, μ) be a totally σ-finite measure spece, and let $f, g\in\mathfrak{M}(X)$. If $f-g$ and f^*-g^* are well-defined a. e., then $|f^*-g^*|\ll |f-g|$.*

3. Proof of Main Theorems

To prove main theorems, we need the following two lemmas. In the sequel, for each α, $\beta\in\boldsymbol{R}$, $\alpha\wedge\beta$ and $\alpha\vee\beta$ mean $\min(\alpha, \beta)$ and $\max(\alpha, \beta)$ respectively, while $\boldsymbol{N}$ stands for the set of natural numbers.

LEMMA 3. *Let (X, Λ, μ) be a measure space, and let $f\in\mathfrak{M}(X)$. If $f_n\in\mathfrak{M}(X)$ is defined by $f_n(x)=(f(x)\wedge n)\vee(-n)$ for each $x\in X$ and each*

$n\in N$, then $f^*=\lim_{n\to\infty} f_n^*$.

PROOF. Suppose first that $f^*(t)>s$. If $n>|s|$, then $d_{f_n}(s)=d_f(s)>t$; so that $f_n^*(t)>s$. Hence $\liminf_{n\to\infty} f_n^*\geqq f^*$. Suppose next that $\limsup_{n\to\infty} f_n^*(t)>s$. Then $f_n^*(t)>s$ for infinitely many n; so that $d_{f_n}(s)>t$ for infinitely many n. Therefore $d_f(s)>t$, that is $f^*(t)>s$. Hence $f^*\geqq\limsup_{n\to\infty} f_n^*$, and then $f^*=\limsup_{n\to\infty} f_n^*=\liminf_{n\to\infty} f_n^*=\lim_{n\to\infty} f_n^*$.

LEMMA 4. *Let (X, Λ, μ) be a totally σ-finite measure space, and let f, $g\in\mathfrak{M}(X)$. If $f-g$ and f^*-g^* are well-defined a. e., then*

$$\int_{[0,a)}(f^*-g^*)^+\,dm\leqq\int_X(f-g)^+\,d\mu\,. \tag{3. 1}$$

PROOF. We devide the proof in three steps.

Step 1°: (3. 1) holds whenever $0\leqq f$, $g\in L^\infty(X)$. Suppose that $0\leqq f$, $g\in L^\infty(X)$; so that f^*, $g^*\in L^\infty([0, a))$; hence f^*-g^* is well-defined. Since (X, Λ, μ) is totally σ-finite, there exists an increasing sequence $\{E_n\}$ of elements of Λ such that $\bigcup_{n=1}^{\infty} E_n=X$, with $\mu(E_n)<\infty$ for all $n\in N$. Define $f_n=f\chi_{E_n}$ and $g_n=g\chi_{E_n}$ for each $n\in N$. Then $\{f_n\}$ and $\{g_n\}$ are increasing sequences of measurable functions; so that $f_n^*\uparrow f^*$ and $g_n^*\uparrow g^*$. Hence $(f^*-g^*)^+=\lim_{n\to\infty}\chi_{[0,\mu(E_n))}(f_n^*-g_n^*)^+$. Further, since f, $g\in L^\infty(X)$, f_n, $g_n\in L^1(E_n)$ for all $n\in N$. Then, Theorem L-S-D and Theorem H-L-P yield

$$\begin{aligned}\int_{[0,a)}(f^*-g^*)^+\,dm&\leqq\liminf_{n\to\infty}\int_{[0,\mu(E_n))}(f_n^*-g_n^*)^+\,dm\\&\leqq\liminf_{n\to\infty}\int_{E_n}(f_n-g_n)^+\,d\mu\leqq\int_X(f-g)^+\,d\mu\,,\end{aligned}$$

on applying Fatou's Lemma to the sequence $\{(f_n^*-g_n^*)^+\}$.

Step 2°: (3. 1) holds whenever f, $g\in L^\infty(X)$. Suppose that f, $g\in L^\infty(X)$. Then the result of step 1° implies (3. 1), on using a general identity

$$(h-v)^*=h^*-v \quad \text{for any } h\in\mathfrak{M}(X) \text{ and any } v\in\boldsymbol{R}\,. \tag{3. 2}$$

Step 3°: (3. 1) holds whenever f, $g\in\mathfrak{M}(X)$ and both $f-g$ and f^*-g^* are well-defined a. e. To prove this, we may suppose, without loss of generality, that f, $g\in\mathfrak{M}(X)$ satisfy $(f-g)^+\in L^1(X)$ and that f^*-g^* is well-defined a. e. Here we note that

$$\big[(\alpha\wedge\gamma)\vee(-\gamma)-(\beta\wedge\gamma)\vee(-\gamma)\big]^+\leqq(\alpha-\beta)^+ \tag{3. 3}$$

for each α, β, $\gamma\in\boldsymbol{R}$: If $\alpha<\beta$, then the left hand side of (3. 3) is equal to

0; if $\alpha \geqq \beta$, then a general identity

$$|\alpha\wedge\gamma-\beta\wedge\gamma|+|\alpha\vee\gamma-\beta\vee\gamma|=|\alpha-\beta|$$

for any $\gamma\in\boldsymbol{R}$ implies

$$\begin{aligned}&\left[(\alpha\wedge\gamma)\vee(-\gamma)-(\beta\wedge\gamma)\vee(-\gamma)\right]^{+}\\ &\qquad\leqq\left|(\alpha\wedge\gamma)\vee(-\gamma)-(\beta\wedge\gamma)\vee(-\gamma)\right|\\ &\qquad\leqq|\alpha-\beta|=(\alpha-\beta)^{+}.\end{aligned}$$

Now, putting $\alpha=f(x)$, $\beta=g(x)$ for each $x\in X$ and $\gamma=n$, we have

$$(f_n-g_n)^{+}\leqq(f-g)^{+} \quad \text{a. e. for all } n\in\boldsymbol{N}, \tag{3. 4}$$

where $f_n\equiv(f\wedge n)\vee(-n)$ and $g_n\equiv(g\wedge n)\vee(-n)$ for each $n\in\boldsymbol{N}$. Since f_n, $g_n\in L^{\infty}(X)$, the result of step 2°, (3. 4) and Lemma 3 yield (3. 1), on applying the Fatou Lemma to the sequence $\{(f_n^{*}-g_n^{*})^{+}\}$.

PROOF OF THEOREM 1.

As a special case of the preceeding lemma, using (3. 2), we have

$$\int_{[0,a)}(f^{*}-g^{*}-u)^{+}\,dm\leqq\int_X(f-g-u)^{+}\,d\mu \text{ for any } u\in\boldsymbol{R}, \tag{3. 5}$$

whenever $f-g$ and $f^{*}-g^{*}$ are well-defined a. e. Then, by virtue of Theorem H-L-P, $f^{*}-g^{*}\ll f-g$ whenever f, $g\in\mathfrak{M}(X)$ satisfy both $f-g\in\mathfrak{P}(X)$ and $f^{*}-g^{*}\in\mathfrak{P}([0,a))$.

PROOF OF THEOREM 2.

On changing the role of f and g in Lemma 4, and again using (3. 2), we have

$$\int_{[0,a)}(f^{*}-g^{*}+u)^{-}\,dm\leqq\int_X(f-g+u)^{-}\,d\mu \text{ for any } u\in\boldsymbol{R}, \tag{3. 6}$$

whenever $f-g$ and $f^{*}-g^{*}$ are well-defined a. e. Besides,

$$(|h|-v)^{+}=(h-v)^{+}+(h+v)^{-} \text{ for any } h\in\mathfrak{M}(X) \text{ and } v\in\boldsymbol{R}_{+}=[0,\infty).$$

Then (3. 5) and (3. 6) yield

$$\int_{[0,a)}(|f^{*}-g^{*}|-u)^{+}\,dm\leqq\int_X(|f-g|-u)^{+}\,d\mu$$

for any $u\in\boldsymbol{R}_{+}$ whenever $f-g$ and $f^{*}-g^{*}$ are well-defined a. e.; hence $|f^{*}-g^{*}|\ll|f-g|$, on using Theorem H-L-P.

REMARK. If (X,Λ,μ) and (X',Λ',μ') are finite measure spaces with $\mu(X)=\mu'(X')$, and if $f\in L^{1}(X)$ and $g\in L^{1}(X')$ satisfy that $f<g$, then $|f|\ll|g|$,

by a Theorem of Luxemburg [13, Theorem 9. 5, p. 107]. Therefore Theorem 2 is an immediate consequence of Theorem 1 when f, $g \in L^1(X)$ and $\mu(X) < \infty$. But the theorem of Luxemburg is not true when $\mu(X) = \mu'(X') = \infty$.

4. Some Consequences

As an immediate corollary to Theorem 1, we obtain the following Theorem, which is a part of Theorem 3. 8 of Chong [4, p. 148] (for the another part of the theorem, see [15, Theorem 3. 5]):

THEOREM 5. *Let (X, Λ, μ) be a finite measure space. If f^+, $g^- \in L^1(X)$ or f^-, $g^+ \in L^1(X)$, then $f^* - g^* \prec f - g$.*

Theorems 1 and 2, combined with a result of the preceding paper [15, Theorem 3. 1], yield the following:

THEOREM 6. *Let (X, Λ, μ) be a totally σ-finite measure space with $\mu(X) = a$, and let f, $g \in \mathfrak{M}(X)$. Further, let Φ: $\bar{\boldsymbol{R}} \to \bar{\boldsymbol{R}} = [-\infty, \infty]$ be an increasing, left continuous and convex function.*

(i) *If $f-g$, $\Phi(f-g) \in \mathfrak{P}(X)$ and $f^* - g^*$, $\Phi(f^* - g^*) \in \mathfrak{P}([0, a))$, then $\Phi(f^* - g^*) \ll \Phi(f-g)$.*

(ii) *If $f-g$ and $f^* - g^*$ are well-defined a. e., then $\Phi(|f^* - g^*|) \ll \Phi(|f-g|)$, provided that $\Phi(|f-g|) \in \mathfrak{P}(X)$ and $\Phi(|f^* - g^*|) \in \mathfrak{P}([0, a))$.*

The following corollary of Theorem 6 extends the Main Theorem of Chiti [1, Theorem, p. 24].

COROLLARY 7. *Let (X, Λ, μ) be a totally σ-finite measure space with $\mu(X) = a$, and let Φ: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ be a convex and increasing function. If f, g: $X \to \boldsymbol{R}$ are measurable and $f^* - g^*$ is well-defined a. e., then*

$$\int_0^a \Phi\left(|f^*(t) - g^*(t)|\right) dt \leqq \int_X \Phi(|f-g|)\, d\mu .$$

REMARK. The notation f^* used by Chiti in [1] means $|f|^*$.

COROLLARY 8. *In any Orlicz spaces, convergence of a sequence $\{f_n\}$ to f implies convergences of $\{f_n^*\}$ to f^*, and $\{|f_n|^*\}$ to $|f|^*$.*

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