Hokkaido Mathematical Journal Vol. 15(1986), p. 113～147

Martin Boundaries of Plane Domains.

Dedicated to Professor Yukio Kusunoki on the occasions of his 60-th birthday.

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(Received February 14, 1985)

In the present paper we shall study Martin or N-Martin's[1),2)] boundary points of domains in the z-plane of some typical types.

§1. **Domain Ω.** Let Ω be the domain such that

$$\Omega=\{|z|<1\}-F,$$

where F is a closed set in $\{|z|\leqq 1,\ |\arg z|\geqq\frac{\pi}{2}\}$ and $F\ni\{z=0\}$. Let $G(z, p):p\in\Omega$ be a Green function. Put $K(z, p)=\frac{G(z,p)}{G(p_0,p)}$: $p_0=\{z=\frac{1}{2}\}$. Let $\{p_i\}$ be a divergent sequence in Ω such that $K(z, p_i)$ converges uniformly to an HP (a positive harmonic function) $U(z)$. Put $U(z)=K(z, p)$ and we say $\{p_i\}$ determines a boundary point p. Denote by Δ the set of boundary poists and put $\bar{\Omega}=\Omega+\Delta$. Let p_1 and p_2. Then the Martin's distance between p_1 and p_2 is given as

$$\overset{\mathrm{M}}{\mathrm{dist}}(p_1, p_2)=\sup_{z\in\Gamma}\left|\frac{K(z,p_1)}{1+K(z,p_1)}-\frac{K(z,p_2)}{1+K(z,p_2)}\right|,$$

where $\Gamma=\{|z-\frac{1}{2}|=\frac{1}{32}\}$.

The Martin's topology is introduced by this metric on $\bar{\Omega}$. We shall prove

THEOREM 1. *Let Ω be the domain. Let $\{p_i\}$ be a sequence in $\{|\arg z|<\frac{\pi}{2}-\delta\}$ with $p_i\to z=0$. Then $K(z, p_i)$ tends to a uniquely determined minimal function $K(z, q)$ for any $\{0<\delta<\frac{\pi}{2}\}$, i. e.*

$$\sup(\overset{\mathrm{M}}{\mathrm{dist}}(z, q)\ on\ \{|z|\leqq\frac{1}{n}, |\arg z|<\frac{\pi}{2}-\delta\}\to 0\ as\ n\to\infty.$$

We use often solutions of Dirichlet problems and H. M. (harmonic measure) of A. We denote by H_U^G the solution of the Dirichlet problem in

G with boundary value U on $\partial G+\Delta$. If $U=1$ on A and $=0$ elsewhere, we call H_U^G the H. M. of A denoted by $W(A, z, G)$.

LEMMA 1. 1). *Let $S_a=\{\mathrm{Re}\, z=0, |\mathrm{Im}\, z|\leqq a\}$ and let F be a continuum tending to $z=\infty$ in the z-plane such that $F\cap S_a=0$. Let $\mathfrak{M}$ be the module of the complementary set G of $F+S_a$. Then there exists an increasing function $\Psi(\mathfrak{M})$ of $\mathfrak{M}$ such that*

$$\mathrm{dist}(F, S_a)\geqq a\Psi(\mathfrak{M}),$$

where $\mathfrak{M}=\frac{1}{D(V(z))}$ and $V(z)$ is an HB (a positive bounded harmonic function) in G such that $V(z)=0$ on S_a, $=1$ on F and has an M. D. I. (minimal Dirichlet integral).

2). *Let T be an arc on $|z|=1$ containing $z=-1$, symmetric relative to $\mathrm{Im}\, z=0$ and T has length $2m$, where $m\leqq m_1<\pi$. Let F be a continuum in $|z|\leqq 1$ containing $z=1$. If there exists an HB in $\{|z|<1\}-F$ such that $V(z)=0$*

on T, $=1$ on F and $D(V(z))<M<\infty$, then there exist functions $\theta(m)$ and $L(M, m_1)$ such that

$$\mathrm{dist}(F. T)\geqq\frac{2\theta(m)\Psi\left(\frac{1}{2M}\right)}{L(M, m_1)},$$

where $\theta(m)=\frac{\sin m}{1+\cos m}$, $L(M, m_1)$ is a function of M and m_1 and $<\infty$ for $m_1<\pi$ and $0<M<\infty$.

PROOF of 1). We denote the module of the complementary set of $F+S_a$ by $\mathfrak{M}(F, S_a)$. By $\xi=\frac{z}{ia}$,

$$S_a\rightarrow S=(|\mathrm{Re}\,\xi|\leqq 1,\ \mathrm{Im}=0)\ \text{in the } \xi\text{-plane}.$$

Map the complementary set of S onto $\{|w|>1\}$ by

$$w=g(\xi)=\xi+\sqrt{\xi^2-1}:\ \xi=\frac{1}{2}\left(w+\frac{1}{w}\right).$$

Then $\mathfrak{M}(F, S_a)=\mathfrak{M}(F_\xi, S)=\mathfrak{M}(F_w, C)$, where $F_w=g(F_\xi)$, F_ξ is the image of F and $C=\{|w|<1\}$. Then by Grötsch's theorem there exists an increasing function $\Phi(\mathfrak{M})$ of $\mathfrak{M}$ such that

$$F_w\subset\{|w|\geqq 1+\Phi(\mathfrak{M})\}$$

and dist $(F_w, C)=\Phi(\mathfrak{M})$ is attained if and only if F_w is a ray:

$$F_w=\{1+\Phi(\mathfrak{M})\leqq|w|\leqq\infty,\ \arg w=\text{const.}\}.$$

Put $R=1+\Phi(\mathfrak{M})$ and let $\mathfrak{L}(R)$ be the image in the ξ-plane of $\{|w|=R\}$: Then F_ξ is outside of $\mathfrak{L}(R)$, i.e.

$$\operatorname{dist}(F_\xi, S)\geqq\operatorname{dist}(\mathfrak{L}(R), S).$$

Hence it is sufficient to investigate the behaviour of $\mathfrak{L}(R)$. Now $\mathfrak{L}(R)$ is given as

$$\frac{1}{2}(R+\frac{1}{R})\cos\theta+\frac{i}{2}(R-\frac{1}{R})\sin\theta : 0\leqq\theta\leqq 2\pi,\ w=Re^{i\theta}.$$

Put $\xi(R,\theta)=g^{-1}(w): w=Re^{i\theta}$ and $\beta(\theta)=\operatorname{dist}(\xi(R,\theta), S)$.

Then, Case 1. $|\operatorname{Re}(\xi(R,\theta))|\leqq 1$. In this case

$$\beta(\theta)=\frac{1}{2}(R-\frac{1}{R})|\sin\theta|.$$

Case 2. $|\operatorname{Re}(\xi(R,\theta))|>1$. Then

$$\beta(\theta)=[(\frac{1}{2}(R-\frac{1}{R})\sin\theta)^2+\{\frac{1}{2}(R+\frac{1}{R})\cos\theta-1\}^2]^{\frac{1}{2}}$$

Put $\alpha=\beta(0)$, i.e. $\alpha=\frac{1}{2}(R+\frac{1}{R})-1$. We have only to study for $0\leqq\theta\leqq\frac{\pi}{2}$.

Case 1. Let θ_0 be the θ such that $\operatorname{Re}(\xi(R,\theta_0))=1$.

Then $\cos\theta_0=\frac{2R}{1+R^2}$ and $\sin\theta_0=\frac{R^2-1}{R^2+1}$. Now

$$\beta(\theta_0)-\alpha=\frac{(R-1)^2}{1+R^2}\geqq 0.$$

Since $\beta(\theta)$ is increasing for $\theta_0\leqq\theta\leqq\frac{\pi}{2}$, $\beta(\theta)\geqq\alpha$ for $\theta_0\leqq\theta\leqq\frac{\pi}{2}$.

Case 2. In this case

$$\beta^2(\theta)-\alpha^2=(1-\cos\theta)\{(\frac{1}{R}+R-2)+(1-\cos\theta)\}\geqq 0.$$

Hence $\operatorname{dist}(\mathfrak{L}(R), S))=\alpha=\frac{1}{2}(R+\frac{1}{R})-1$. Let $\Psi(\mathfrak{M})=\alpha$. Then since $R=1+\Phi(\mathfrak{M})$,

$$\Psi(\mathfrak{M})=\frac{\Phi(\mathfrak{M})^2}{2(1+\Phi(\mathfrak{M}))}.$$

Consider F in the z-plane. Then $\Psi(\mathfrak{M})$ is the required function and we have Lemma 1. 1)

PROOF of 2). Map $\{|z|<1\}$ by $\xi=g(z)=\frac{1+z}{1+z}$ onto $|\arg\xi|<\frac{\pi}{2}$. Then

$$T\to S_a=\{\operatorname{Re}\xi=0,|\operatorname{Im}\xi|\leqq a\}.$$

By brief computation we have $a=\frac{\sin m}{1+\cos m}<\frac{\sin m_1}{1+\cos m_1}=N(m_1)<\infty$ by $m_1<\pi$. Put $\theta(m)=\frac{\sin m}{1+\cos m}$. Then $\theta(m)\nearrow$ as $m\nearrow$. Now $\theta(m)\leqq N(m_1)<\infty$. Let $\hat{F}_\xi$ be the symmetric image of $g(F)=F_\xi$ relative to $|\arg\xi|=\frac{\pi}{2}$. Let $V^*(z)$ be an HB in $\{|z|<1\}-F$ such that $V(z)=V^*(z)$ on $F+T$ and has M. D. I. Then $\frac{\partial}{\partial n}V^*(z)=0$ on $\{|z|=1\}-T-F$ and $D(V^*(z))\leqq M$. Put $V^*(\xi)=V^*(g^{-1}(\xi))$. We extend $V^*(\xi)$ into $|\arg\xi|\geqq\frac{\pi}{2}$ so that $V^*(\hat{\xi})=V^*(\xi)$, where $\hat{\xi}$ is the symmetric point of ξ relative to $|\arg\xi|=\frac{\pi}{2}$. Then the module of $\{|\xi|<\infty\}-F_\xi-\hat{F}_\xi-S_a\geqq\frac{1}{2M}$, where $\hat{F}_\xi$ is the symmetric image of F_ξ. Hence by 1)

$$\operatorname{dist}(S_a,F_\xi+\hat{F}_\xi)\geqq\left(\frac{\sin m}{1+\cos m}\right)\Psi\left(\frac{1}{2M}\right)=\lambda(m,M).$$

Let $J=\{\xi:\operatorname{dist}(S_a,\xi)<\lambda(m,M)\}$. Then $J\cap(F_\xi+\hat{F}_\xi)=0$.

Let J_z be the image of J and let $d=\operatorname{dist}(T,\partial J_z)$.

Then there exists a straight Λ of length d connecting a point $z_0\in T$ and a point $q\in\partial J_z$. Assume $\Lambda\not\subset J_z+\partial J_z$. Then there exists at least one inner point $q'(\neq q)$ of Λ such that $q'\in\partial J_z$. This means $\operatorname{dist}(T,\partial J_z)<d$. This is a contradiction. Hence $\Lambda\subset J_z+\partial J_z$. Let Λ_ξ be the image of Λ by $\xi=\frac{1+z}{1-z}$. Then $\Lambda_\xi\subset\bar{J}$ and Λ_ξ connects a point $\xi_0\in S_a$ and a point q_ξ on ∂J. By $z=\frac{\xi-1}{1+\xi}$

$$d=\int_{\Lambda_\xi}\left|\frac{dz}{d\xi}\right|d\xi=\int_{\Lambda_\xi}\frac{2}{|1+\xi|^2}d\xi.$$

$\Lambda_\xi\subset\bar{J}$ implies $|\xi|\leqq\theta(m)\Psi\left(\frac{1}{2M}\right)+\theta(m)$ in $\bar{J}$ and

$$|1+\xi|^2 \leqq \{1+\theta(m)(1+\Psi(\frac{1}{2M}))\}^2 \leqq \{1+\theta(m_1)(1+\Psi(\frac{1}{2M}))\}^2.$$

Let $L(m_1, M)=\{1+\theta(m_1)(1+\Psi(\frac{1}{2M}))\}^2$. Then

$$d \geqq \frac{2\theta(m)\Psi\left(\frac{1}{2M}\right)}{L(m_1, M)}.$$

We see at once $L(m_1, M)<\infty$ for $m_1<\pi$ and $0<M<\infty$. Hence

$$\mathrm{dist}(T, F) \geqq \mathrm{dist}(T, \partial J_z) \geqq d.$$

Thus $\theta(m)$ and $L(m_1, M)$ are the functions required and we have 2).

LEMMA 2. *Let D_r be a domain such that* $D_r=\{1<|z|<r, |\arg z|<\theta\}$: $r \geqq 4$, $\theta \leqq \pi$. *Let $U(z)$ be an HP in D_r such that $U(z)=0$ on $|z|=r$. Then there exists a const. l_1 depending on θ and δ but not r such that*

$$\frac{1}{l_1}<\frac{U(z_1)}{U(z_2)}<l_1,$$

where $2<|z_1|=|z_2|<r$, $|\arg z_1|$ *and* $|\arg z_2| \leqq \theta-\delta$: $0<\delta<\theta$.

$$\frac{1}{l_1}<\frac{\frac{\partial}{\partial n}U(z_1)}{\frac{\partial}{\partial n}U(z_2)}<l_1 : \ |z_i|=r, \ |\arg z_1|, \ |\arg z_2| \leqq \theta-\delta.$$

PROOF. Let z_1 and z_2 be points such that $|z_1|=|z_2|, |\arg z_1|$, $|\arg z_2|<\theta-\delta$. We can suppose without loss of generality

$$\arg z_1=\phi_1 \leqq \phi_2=\arg z_2 \text{ and } \phi_0=\frac{\phi_1+\phi_2}{2} \geqq 0.$$

Let $D'=\{1<|z|<r, \ \phi_1-\delta<\arg z<\phi_2+\delta\}$. Then $D' \subset D$. Let

$$F=\{\frac{3r}{4} \leqq |z| \leqq r, \ \phi_1 \leqq \arg z \leqq \phi_2\}.$$

By the mapping $\xi=\frac{z}{r}$, $D_r \to D_\xi$ and $D' \to D'_\xi$. Now

$$\partial D'_\xi = C_{\frac{1}{r}}+L_1+C_1+L_2,$$

where

$$C_{\frac{1}{r}}=\{|\xi|=\frac{1}{r},\ \phi_1-\delta\leqq\arg\xi\leqq\phi_2+\delta\}$$
$$L_1=\{\frac{1}{r}\leqq|\xi|<1,\ \arg\xi=\phi_1-\delta\},$$
$$C_1=\{|\xi|=1,\ \phi_1-\delta\leqq\arg\xi\leqq\phi_2+\delta\}$$
$$L_2=\{\frac{1}{r}\leqq|\xi|\leqq1,\ \arg\xi=\phi_2+\delta\}$$

Put $\xi_0=\frac{1}{2}e^{i\phi_0}$ and $T_\xi=L_1+C_{\frac{1}{r}}+L_2$. We map

$$D'_\xi\to\text{onto}\{|\eta|<1\}\quad\text{by } \eta=g(\xi),$$

so that $\xi_0\to\eta=0$, $\{\arg\ \xi=\phi_0\}\to\{\operatorname{Im}\ \eta=0\}$. Then $g(\frac{z_1}{r})$ and $g(\frac{z_2}{r})$ are symmetric relative to $\{\operatorname{Im}\eta=0\}$. Let $T=g(T_\xi)$. We shall estimate the length of T. Then the length of $T=2\pi W(T_\xi,\xi_0,D'_\xi)$. Clearly $W(T_\xi,\xi_0,D'_\xi)\downarrow$ as $r\uparrow$ and $\downarrow$ as $D'_\xi\uparrow$. Hence

$$\text{length of } T\geqq 2\pi W(T_\infty,\frac{1}{2},D^\infty_\xi)=2m_0, \tag{1}$$

where $T_\infty=\{0\leqq|\xi|\leqq1,\ \arg\xi=\theta\}+\{0\leqq|\xi|\leqq1,\ \arg\xi=-\theta\}$ and $D^\infty_\xi=\{0<|\xi|<1,\ |\arg\xi|<\theta\}$. Evidently m_0 depends on θ but not on r and δ. Also

$$\text{length of } T\leqq 2\pi W(T_{\frac{1}{4}},\frac{1}{2},D'')=2m_1, \tag{2}$$

where $T_{\frac{1}{4}}=\{\frac{1}{4}\leqq|\xi|\leqq1,\ \arg\xi=-\delta\}+\{|\xi|=\frac{1}{4},\ |\arg\xi|\leqq\delta\}+\{\frac{1}{4}\leqq|\xi|\leqq 1,\ \arg\xi=\delta\}$ and $D''=\{\frac{1}{4}<|\xi|<1,\ |\arg\xi|<\delta\}$, and m_1 depends on only δ. Let $G_\delta=\{\frac{1}{2}<|\xi|<1,\ 0<\arg\xi<\delta\}$. Then there exists an HB, $A(\xi)$ in G_δ such that $A(\xi)=1$ on $\{\arg\xi=0\}$, $A(\xi)=0$ on $\{\arg\xi=\delta\}$ and

$$D(A(\xi))=M_1<\infty.$$

Let $\bar{A}(\xi)=A(\bar{\xi})$, where $\bar{\xi}$ is the symmetric point of ξ relative to $\{\operatorname{Im}\xi=0\}$. Let $B(\xi)=\log\frac{|\xi|}{\frac{1}{2}}/\log\frac{6}{4}$ in $\{\frac{1}{2}<|\xi|<\frac{3}{4}\}$, $=0$ in $\{|\xi|<\frac{1}{2}\}$, $=1$ in $\{|\xi|\geqq\frac{3}{4}\}$. Then $D(B(\xi))=M_2<\infty$. We shall construct a Dirichlet function $V'(\xi)$ in D'_ξ as follows: $V'(\xi)=0$ in $\{\frac{1}{r}<|\xi|<\frac{1}{2}\}$, $V'(\xi)=\min(B(\xi),$

$A(\xi e^{-i\phi_2}))$ in $\{\frac{1}{2}<|\xi|<1,\ \phi_2\leqq\arg\xi\leqq\phi_2+\delta\}$, $V'(\xi)=1$ in $\{\frac{3}{4}\leqq|\xi|\leqq 1, \phi_1\leqq$ $arg\ \xi\leqq\phi_2\}=F_\xi(F_\xi$ is the image of F), $V'(\xi)=\min(B(\xi),\ \bar{A}(\xi e^{-i\phi_1})$ in $\{\frac{1}{2}\leqq|\xi|\leqq 1,\ \phi_1-\delta\leqq\arg\ \xi\leqq\phi_1\}$, $V'(\xi)=B(\xi)$ in $\{\frac{1}{2}\leqq|\xi|\leqq\frac{3}{4},\ \phi_1\leqq\arg\xi$ $\leqq\phi_2\}$. Then $V'(\xi)=0$ on T_ξ, $=1$ on F_ξ, continuous Dirichlet function and

$$D(V'(\xi))\leqq(2M_1+M_2)=M<\infty \quad \text{not depending on } r.$$

Hence there exists an HB, $V(\xi)$ in $D'_\xi-F_\xi$ such that $V(\xi)=0$ on T_ξ, $=1$ on F_ξ and

$$D(V(\xi))<M<\infty \quad \text{not depending on } r. \tag{3}$$

Hence by (1), (2), (3) and Lemma 1, 2,), there exists a const, ε_0 depending only on θ and δ such that

$$\mathrm{dist}(T, g(F_\xi))>\varepsilon_0.$$

Map D' by $\eta=f(z)$ onto $\{|\eta|<1\}$ so that $\{\arg\ z=\phi_0\}\to\{\mathrm{Im}\ \eta=0\}$ and $f(\frac{r}{2}e^{i\phi_0})=0$. Let $F=\{\frac{3r}{4}\leqq|z|\leqq r,\ \phi_1\leqq\arg\ z\leqq\phi_2\}$. Then

$$\mathrm{dist}(f(T),\ f(F))>\varepsilon_0>0,$$

where $T=\{|z|=1,\ \phi_1-\delta\leqq\arg\ \ z\leqq\phi_2+\delta\}+\{1\leqq|z|\leqq r,\ \arg\ \ z=\phi_1-\delta\}+\{1\leqq|z|\leqq r,\ \arg\ z=\phi_2+\delta\}$. Suppose $|z_1|=|z_2|\geqq\frac{3r}{4}$, $\arg\ z_1=\phi_1$, $\arg\ z_2=\phi_2$. Then z_1, $z_2\subset F$ and evidently $\eta_1=f(z_1)$ and $\eta_2=f(z_2)$ are symmetric relative to $\{\mathrm{Im}\ \eta=0\}$.

$$U(\eta_i)=\frac{1}{2\pi}\int U(e^{i\phi})\frac{1-\rho^2}{1-2\rho\cos(\phi-\Psi_i)+\rho^2}d\phi,$$
$$\eta_i=\rho e^{i\Psi_i},\quad i=1, 2.$$

Since $1-2\cos(\phi-\Psi_i)+\rho^2=(\mathrm{dist}(e^{i\phi},\ \eta_i))^2\geqq\varepsilon_0^2$ for $e^{i\phi}\in g(T)$, $\eta_i\in g(F)$,

$$\left(\frac{1-\rho^2}{4}\right)U(0)\leqq U(\eta_i)\leqq\left(\frac{1-\rho^2}{\varepsilon_0^2}\right)U(0);\ i=1, 2.$$

$$\left(\frac{\varepsilon_0}{2}\right)^2\leqq\frac{U(z_1)}{U(z_2)}\leqq\left(\frac{2}{\varepsilon_0}\right)^2.$$

Consider for $2\leqq|z_i|\leqq\frac{3r}{4}$. Let

$$D''=\{\frac{3}{4}|z_i|<|z|<|z_i|\frac{4}{3},\ \phi_1-\delta<\arg z<\phi_2+\delta\}.$$

Then $D''\subset D$ and $U(z)$ is an HP in D''. Map D'' by $\eta=\log\frac{z}{|z_i|}$. Then $D''\rightarrow\{|\mathrm{Re}\ \eta|<\log\frac{4}{3},\ \phi_1-\delta<\mathrm{Im}\ \eta<\phi_2+\delta\}$. Put $\rho=\min(\delta,\ \log\frac{4}{3})$ and $F=\{\mathrm{Re}\ \eta=0,\ \phi_1\leqq\mathrm{Im}\ \eta\leqq\phi_2\}$ *and* $C(\frac{\rho}{2},\ p_i)=\{|\eta-p_i|<\frac{\rho}{2}\}$. Then we can find at most n_δ number of circles $C(\frac{\rho}{2},p_i)$ such that $\rho_i\in F$, $\sum_i C(\frac{\rho}{2},p_i)\supset F$. We see at once the number n_δ attains its maximum in case $\{|\mathrm{Re}\ \eta|<\log\frac{4}{3},\ |\mathrm{Im}\ \eta\ |<\theta\}$ and $F=\{\mathrm{Re}\ \eta=0,|\ \mathrm{Im}\ \eta\ |\leqq\theta-\delta\}$. Then after Harnak's principle

$$(\frac{1}{3})^{2n_\delta}\leqq\frac{U(z_1)}{U(z_2)}\leqq 3^{2n_\delta},\qquad z_1,\ z_2\in F.$$

Put $l_1=\max(3^{2n_\delta},\ (\frac{2}{\varepsilon_0})^2)$. Then we have

$$\frac{1}{l_1}<\frac{U(z_1)}{U(z_2)}<l_1,\ \text{for}\ z_1=\rho e^{i\phi_1},\ z_2=\rho e^{i\phi^2}:\ 2\leqq\rho\leqq r.$$

Now z_1 and z_2 are arbitraly points in $\{2\leqq|z|<r,\ |\arg z|\leqq\theta-\delta\}$. Hence

$$\frac{1}{l_1}<\frac{U(z_1)}{U(z_2)}<l_1$$

for $2\leqq|z_1|=|z_2|<r$ and $|\arg z_i|\leqq\theta-\delta$. By $U(z)=0$ on $|z|=r$, we have at once

$$\frac{1}{l_1}<\frac{\partial}{\partial n}U(z_1)/\frac{\partial}{\partial n}U(z_2)<l_1:\ |z_1|=|z_2|=r,|\arg z_i|\leqq\theta-\delta.$$

Thus we have the Lemma.

Let $L_i=\{\frac{1}{r}\leqq|z|<\infty,\ \arg z=i\theta\}$, $r\geqq 8$, $0<\theta<\frac{\pi}{8}$, $i=0,\ \pm1,\ \pm2,\ \pm3,\ \pm4$. Let Λ_1 be a simple compact analytic curve in $\{1\leqq|z|<\infty\}$, $0\leqq\arg z\leqq\theta\}$ connecting two points $p_0=1+\delta_0$, $p_1=(1+\delta_1)e^{i\theta}$: $\delta_0,\ \delta_1\geqq 0$ such that Λ_1 has no common points with L_0+L_1 except p_0 and p_1 and $\mathrm{dist}(z=0,\Lambda_1)=1$. Then there exists a point $p_1^*\in\Lambda_1$ such that

$$p_1^*=e^{i\theta^*},\ 0\leqq\theta^*\leqq\theta.$$

Let $T_i(i=0, \pm1, \pm2, \pm3,)$ be a symmetric transformation with respect to L_i. Let $\Lambda_{i+1}=T_i(\Lambda_i): 1\leqq i\leqq 3$, $\Lambda_{-1}=T_0(\Lambda_1)$, $\Lambda_{i-1}=T_{-i}(\Lambda_i)$, $-1\geqq i\geqq -3$. Let D_r: $r\geqq 8$ be a simply connected domain such that

$$\partial D_r=\{|z|=\frac{2}{r},\ |\arg z|\leqq 4\theta\}+\sum_{i=-4}^{i=4}\Lambda_i+\{\frac{2}{r}\leqq|z|\leqq 1+\delta_0,\ \arg z=4\theta\}$$
$$+\{\frac{2}{r}\leqq|z|\leqq 1+\delta_0,\ \arg z=-4\theta\}.$$

Let $z\in\{0\leqq\arg z\leqq\theta\}\cap D_r$. We call $z+T_1(z)+T_2T_1(z)+T_3T_2T_1(z)+T_0(z)+T_{-1}T_0(z)+T_{-2}T_{-1}T_0(z)+T_{-3}T_{-2}T_{-1}T_0(z)$ the equivalent class of z. If z_1 and z_2 are contained in the same class, we denote by $z_1\approx z_2$.

LEMMA 3. *Let D_r be the domain mentioned above. Let $U(z)$ be an HP in D_r vanishing on $\sum\Lambda_i$. Then there exists a const. l_2 not depending on r and the shape of Λ_1 such that*

$$\frac{1}{l_2}<\frac{U(z_1)}{U(z_2)}<l_2$$

where $z_2=T_j(z_1)$ in $\{\frac{4}{r}\leqq|z_1|<\infty,\ (j-1)\theta\leqq\arg z_1\leqq(j+1)\theta\}\cap D_r: j=0, \pm1$, and $\dfrac{1}{l_2}<\dfrac{\frac{\partial}{\partial n}U(z_1)}{\frac{\partial}{\partial n}U(z_2)}<l_2$ *on $\Lambda_{-2}+\Lambda_{-1}$: $j=-1$ on $\Lambda_{-1}+\Lambda_1$: $j=0$ and on $\Lambda_1+\Lambda_2$: $j=1$ respectively.*

PROOF. At first we consider only $D'_r=\{|\arg z|\leqq 3\theta\}\cap D_r$. Let $t_r=\{\frac{2}{r}\leqq|z|\leqq 1+\delta_1,\ \arg z=3\theta\}+\{|z|=\frac{2}{r},\ |\arg z|\leqq 3\theta\}+\{\frac{2}{r}\leqq|z|\leqq 1+\delta_1,\ \arg z=-3\theta\}$ and $F=\{\frac{3}{4}\leqq|z|\leqq 1,\ |\arg z|\leqq 2\theta\}$. Map D'_r by $\xi=g(z)$ onto $|\xi|<1$ so that $\{\arg z=0\}\to\{\mathrm{Im}\ \xi=0\}$, $\{z=\frac{1}{2}\}\to\{\xi=0\}$. We estimate the length of $g(t_r)$. $W(t_r, z, D'_r)\downarrow$ as $r\uparrow$ and $\downarrow D_r\uparrow$ for $r=$const. Hence

$$W(t_r, z, D'_r)\geqq W(t, z, D'_1),$$

where $t=\{0\leqq|z|\leqq 1,\ \arg z=3\theta\}+\{0\leqq|z|\leqq 1,\ \arg z=-3\theta\}$ and $D'_1=\{0<|z|<1, |\arg z|<3\theta\}$. Hence there exists a const. m_0 not depending on r and the shape of Λ_1 such that

$$\text{length of } g(t_r)\geqq m_0. \tag{4}$$

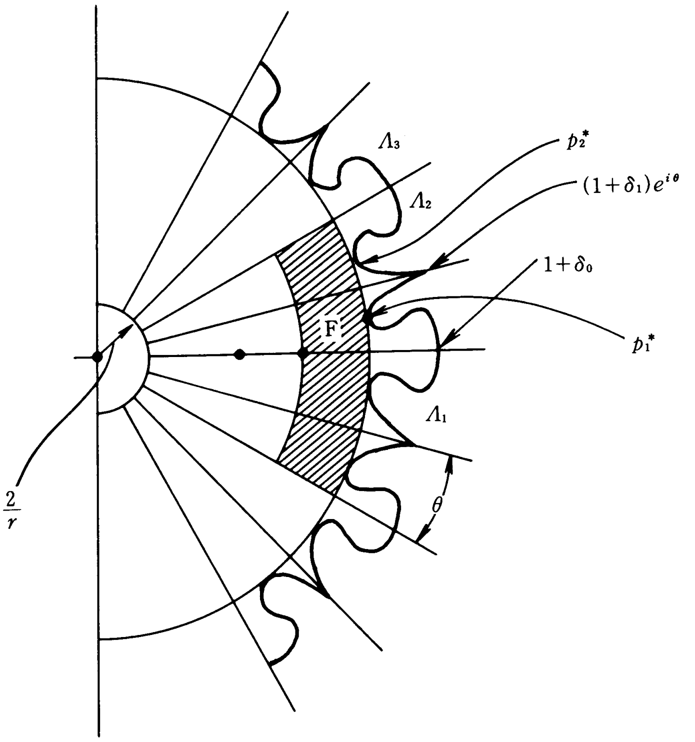

Fig. 1

Now $W(t_r, z, D'_r)=1-W(\sum_{i=-3}^{i=3}\Lambda_i, z, D'_r)$. Instead to estimate $W(t_r, z, D'_r)$ from the above, we estimate $W(\sum_{i=-3}^{i=3}\Lambda_i, z, D'_r)$ from the below. Λ_i separates $|z|=\frac{4}{r}$ from $z=\infty$ in $A_i=\{(i-1)\theta<\arg z<i\theta\}\cap D'_r$. We denote by G_i the component of A_i containing $z=\infty$ divided by Λ_i in A_i. Then G_i must contain a curve γ_i starting from p_i^*: $(p_i^*\approx p_1^*)$ and tending to $z=\infty$. Then $W(\sum_{i=-3}^{i=3}\Lambda_i, z, D'_r)\geqq W(\sum_{i=-3}^{i=3}\gamma_i, z, D_{\frac{1}{4}}-\sum_{i=-3}^{i=3}\gamma_i)\geqq W(\gamma_1, z, D_{\frac{1}{4}}-\gamma_1)$, where $D_{\frac{1}{4}}=\{\frac{1}{4}\leqq|z|<\infty, |\arg z|<3\theta\}$. Map $D_{\frac{1}{4}}$ by $\eta=f(z)$ onto $\{|\eta|<1\}$ so that $z=\frac{1}{2}\rightarrow\eta=0$ and $\{\mathrm{Im}\, z=0\}\rightarrow\{\mathrm{Im}\, y=0\}$. Let $J_1=\{|z|=1, 0\leqq\arg z\leqq\theta\}$. Then there exists a const $\rho_0<1$ such that

$$f(J_1)\subset\{|\eta|\leqq\rho_0\}.$$

Then $\gamma_\eta=f(\gamma_1)$ contains a curve γ'_η connecting $\rho_0 e^{i\phi}$ and $\{|\eta|=1\}$. Let $\hat{\gamma}'_\eta$ be the symmetric image of γ'_η relative to $\arg\eta=\phi$ and let $S=\{\rho_0\leqq|\eta|\leqq 1, \arg\eta=\phi\}$. Then $W(\gamma'_\eta, 0, C_\eta-\gamma'_\eta)=W(\hat{\gamma}'_\eta, 0, C_\eta-\hat{\gamma}'_\eta)$, where $C_\eta=\{|\eta|<1\}$. Since $\gamma'_\eta+\hat{\gamma}'_\eta$ encloses S,

$$W(\hat{\gamma}'_\eta, 0, C_\eta - \hat{\gamma}'_\eta) + W(\gamma'_\eta, 0, C_\eta - \gamma'_\eta) \geqq W(S, 0, C_\eta - S).$$

Hence $W(\gamma_\eta, 0, C_\eta - \gamma_\eta) \geqq W(\gamma'_\eta, 0, C_\eta - \gamma'_\eta) \geqq \frac{1}{2} W(S, 0, C_\eta - S)$.

Evidently $W(S, 0, C_\eta - S)$ does not depend on r and the shape of Λ_1. Hence there exists a const. α such that $W(t_r, z, D'_r) \leqq \alpha < 1$ at $z = \frac{1}{2}$, and there exists a const. m_1 not depending on r and Λ_1 such that

$$\text{length of } g(t_r) \leqq 2m_1 < 2\pi. \tag{5}$$

Since F is compact and since $\operatorname{dist}(F, \partial D_{\frac{1}{4}}) > 0$, there exists a Dirichlet function $V'(z)$ in $D_{\frac{1}{4}} - F$ such that $V'(z) = 0$ on $\partial D_{\frac{1}{4}}, = 1$ on F and $D(V'(z)) < M < \infty$. Hence there exists an HB $V(z)$ in $D'_r - F$ such that $V(z) = 0$ on $t_r, = 1$ on F and has M. D. I., i. e.

$$D(V(z)) \leqq M \text{ for any } r \geqq 8 \text{ and } \Lambda_1. \tag{6}$$

We shall study the behaviour of $\xi = g(z)$. We attend to the point p^*_1 on $\{|z|=1\}$: $p^*_1 = e^{i\theta^*}$.

Put $p^*_{-1} = T_0(p^*_1)$, $p^*_{-2} = T_{-1}(p^*_{-1})$, $p^*_2 = T_1(p^*_1)$. Then following 3 cases occur.

Case 1. $0 < \theta^* < \theta$. In this case $p^*_i \neq p^*_j$, $i \neq j$, $i, j, = \pm 1, \pm 2$.

Case 2. $\theta^* = 0$, in this case $p^*_1 = p^*_{-1}$, $p^*_2 \in L_2$, $p^*_{-2} \in L_{-2}$.

Case 3. $\theta^* = \theta$, in this case $p^*_1 \in L_1$, $p^*_{-1} \in L_{-1}$, $p^*_{\pm 1} = p^*_{\pm 2}$.

Put $F_\xi = g(F)$ and let Γ_ξ be the part of ∂F_ξ between $g(p^*_2)$ and $g(p^*_{-2})$ separating $\{\xi = 1\}$ from $g(t_r)$ and let Γ'_ξ be the arc on $\{|\xi| = 1\}$ between $g(p^*_2)$ and $g(p^*_{-2})$ and containing $\{\xi = 1\}$. Then $\Gamma_\xi + \Gamma'_\xi$ encloses a simply connected domain E_ξ : $\partial E_\xi \ni \{\xi = -1\}$ such that

$$E_\xi \supset F_\xi = g(F)$$

Since Γ_ξ separates $\{\xi = 1\}$ from $g(t_r)$, by (6) there exists an HB $V'(\xi)$ in $\{|\xi| < 1\} - E_\xi$ such that $V'(\xi) = 0$ on $g(t_r), = 1$ on E_ξ and

$$D(V'(\xi)) \leqq M < \infty. \tag{7}$$

Since $\bar{E}_\xi \ni \{\xi = 1\}$, by 4), 5), 7) and by Lemma 1. 2) there exists a const. ε_0 not depending on r and the shape of Λ_1 such that

$$\operatorname{dist}(g(t_r), E_\xi) > \varepsilon_0 > 0.$$

Let $J_1=\{\frac{1}{r}\leqq z\leqq 1,\ \arg z=\theta^*\}$ and $J_2=T_1(J_1)$, $J_{-2}=T_0(J_2)$. Let $J'_2=J_2\cap\{|z|\geqq\frac{3}{4}\}$. Then $J'_2+T_0(J'_2)\subset F$. Hence similarly as Lemma 2

$$(\frac{\varepsilon_0}{2})^2\leqq\frac{U(z')}{U(z'')}\leqq(\frac{2}{\varepsilon_0})^2, \tag{a}$$

for z' and z'' such that $|z'|=|z''|$. $z'+z''\in J'_2+T_0(J'_2)$, i.e. $z'=T_0(z'')$. For $\frac{4}{r}\leqq|z'|=|z''|\leqq\frac{3}{4}$. Consider $D''=\{\frac{3}{4}|z'|<|z|<\frac{4}{3}|z'|\,|\arg z|<3\theta\}$. Then $D''\subset D_r$. Hence also as Lemma 2 there exists a const. l'_2 such that

$$\frac{1}{l'_2}<\frac{U(z')}{U(z'')}<l'_2, \tag{b}$$

where $\frac{4}{r}<|z'|=|z''|\leqq\frac{3}{4}$, $|\arg z'|\leqq$and $|\arg z''|\leqq 2\theta$. Also there exists a const. l''_2 for z' and z'' on $C'_{\frac{2}{r}}=\{|z|=\frac{2}{r},\ |\arg z|\leqq 2\theta\}$ such that

$$\frac{1}{l''_2}<\frac{U(z')}{U(z'')}<l''_2. \tag{c}$$

Let Λ'_2 be the part of Λ_2 between $(1+\delta_1)e^{i\theta}$ and p^*_2. Let G be the domain such that $\partial G=C'_{\frac{2}{r}}+(J_2+\Lambda_1+\Lambda'_2)+T_0(J_2+\Lambda_1+\Lambda'_2)$. Then by (a), (b), (c)

$$U(z)<l_2U(T_0(z)) \text{ on } \partial G : l_2=\max((\frac{2}{\varepsilon_0})^2,\ l'_2,\ l''_2).$$

Hence by the maximum principle

$$U(z)\leqq l_2U(T_0(z)) \text{ in } D'_r\cap\{\frac{4}{r}\leqq|z|<\infty,\ |\arg z|\leqq\theta).$$

Next for $T_j: j=\pm1$, consider $\{\frac{2}{r}\leqq|z|\leqq\infty,\ (j-3)\theta\leqq\arg z\leqq(j+3)\theta\}\cap D_r$. Then similarly as before

$$U(z)\leqq l_2U(T_j(z)) \text{ in } \{\frac{4}{r}<|z|,\ (j-1)\theta<\arg z<(j+1)\theta\}\cap D.$$

Hence we have Lemma 3.

On Green functions. I. Let Ω be the domain in Theorem 1. Then $\Omega\cap\{|z|>r\}=\Omega_r: r<\frac{1}{16}$ consists of components. There exists a component

Ω_r' containing $z=\frac{1}{2}$. We can consider the Green function of Ω_r' the Green function $G_r(z, p)$ of Ω_r simply, where $G_r(z, p)=0$ in the other component. Denote by $\hat{z}$ the symmetric point of z with respect to Re $z=0$. Then

LEMMA 4. 1). $G_r(\hat{z}, \frac{1}{2}) \leqq G_r(z, \frac{1}{2}) : |arg\ z| \leqq \frac{\pi}{2}$.

2). *Let* $0<\delta<\frac{\pi}{4}$ *and* $r<\frac{1}{16}$. *Then there exist const.s* l_3 *and* $l_{3,\delta}$ *depending only* δ *but on* r *such that*

$$\frac{\partial}{\partial n} G_r(z, \frac{1}{2}) \leqq l_3 \frac{\partial}{\partial n} G_r(|z|, \frac{1}{2}),\ |z|=r.$$

$$\frac{\partial}{\partial n} G_r(z, \frac{1}{2}) \geqq \frac{1}{l_{3,\delta}} \frac{\partial}{\partial n} G_r(|z|, \frac{1}{2}) : |z|=r, |arg\ z| \leqq \frac{\pi}{2}-\delta.$$

PROOF. Put $V(z)=G_r(z, \frac{1}{2})-G_r(\hat{z}, \frac{1}{2}) : |\arg\ z| \leqq \frac{\pi}{2}$. Then $V(z)$ is an SPH. (a positive superharmonic function) in $\{|\arg\ z|<\frac{\pi}{2}\} \cap \{\Omega-F\}$ and $V(z)=0$ on $\{|z|=1\}+\{|z|=r\}+\{|\arg\ z|=\frac{\pi}{2}\}$, $\geqq 0$ on $\hat{F}$, where $\hat{F}$ is the symmetric image of F. By the minimum principle

$$G_r(\hat{z}, \frac{1}{2}) \leqq G_r(z, \frac{1}{2}) : |\arg\ z| \leqq \frac{\pi}{2}. \tag{8}$$

Let $D=\{\frac{2}{3}<|z|<1,\ -\frac{\pi}{4}-\frac{\pi}{12}<\arg\ z<0\} \supset D'=\{\frac{2}{3}<|z|<1,\ -\frac{\pi}{4}<\arg\ z<-\frac{\pi}{12}\}$. Map D by $\xi=(\frac{81}{16}e^{\frac{2\pi i}{3}}z^4$. Then $D \to D_\xi=\{1<|\xi|<\frac{81}{16}, -\frac{2\pi}{3}<\arg\xi<\frac{2\pi}{3}\} \supset D'_\xi=\{1<|\xi|<\frac{81}{16}, -\frac{\pi}{3}<\arg\ \xi<\frac{\pi}{3}\}$. Hence by Lemma 2, there exists a const. m_1 such that for any HP $U(z)$ vanishing on $|z|=1$,

$$U(\xi)<m_1 U(\xi') : 2<|\xi|=|\xi'|,\ \arg\ \xi=-\frac{\pi}{3},\ \arg\ \xi'=\frac{\pi}{3}.$$

Hence

$$G_r(z, \frac{1}{2}) \leqq m_1 G_r(z', \frac{1}{2}) : 1>|z|=|z'| \geqq \frac{2}{3} \cdot 2^{\frac{1}{4}},\ \arg\ z=-\frac{\pi}{4},\ arg\ z'=-\frac{\pi}{12}.$$

Apply this method twice to $\{-\frac{\pi}{6}<\arg\ z<\frac{\pi}{6}\}$ and $\{\frac{\pi}{6}-\frac{\pi}{12}<\arg\ z<\frac{5}{12}\pi\}$. Then since $\frac{5}{12}\pi>\frac{\pi}{4}$,

$$G_r(z, \frac{1}{2}) \leqq m_1^3 G_r(z', \frac{1}{2}),$$

$$1 > |z| = |z'| > \frac{2}{3} \cdot 2^{\frac{1}{4}},\ \arg z = -\frac{\pi}{4},\ \arg z' = \frac{\pi}{4}. \qquad (9)$$

$G_r(z, \frac{1}{2})$ is an HP in $\{\frac{1}{16} < |z| < 1, |\arg z| < \frac{\pi}{2}\} - \{z = \frac{1}{2}\}$. There exists a const. m_2 by Harnack's principle such that

$$G_r(z, \frac{1}{2}) < m_2 G_r(z', \frac{1}{2}) : \frac{1}{8} < |z| = |z'| \leqq \frac{2}{3} \cdot 2^{\frac{1}{4}},\ \arg z = -\frac{\pi}{4},\ \arg z' = \frac{\pi}{4} \qquad (10)$$

Since $G_r(z, \frac{1}{2}) = 0$ on $|z| = r$, (putting $\theta = \frac{\pi}{2}$, $\delta = \frac{\pi}{4}$), by Lemma 2 there exists a const. m_3 such that

$$G(z, \frac{1}{2}) \leqq m_3 G(z', \frac{1}{2}) : r < |z| = |z'| \leqq \frac{1}{8},\ \arg z = -\frac{\pi}{4},\ \arg z' = \frac{\pi}{4} \qquad (11)$$

Hence by 9), 10), 11)

$$G_r(z, \frac{1}{2}) \leqq m_4 G_r(z', \frac{1}{2}) : r < |z| = |z'| < 1,$$

$$\arg z = -\frac{\pi}{4},\ \arg z' = \frac{\pi}{4} \text{ and } m_4 = \max(m_1^3, m_2, m_3). \qquad (12)$$

Let $\hat{\hat{z}}$ be the symmetric point of $z : |\arg z| \leqq \frac{\pi}{4}$ relative to $\arg z = \frac{\pi}{4}$. Then by (8) and (12)

$$G_r(\hat{\hat{z}}, \frac{1}{2}) \leqq G_r(z', \frac{1}{2}) \leqq m_4 G_r(z, \frac{1}{2}) : |\hat{\hat{z}}| = |z'| = |z|,\ \arg \hat{\hat{z}} = \frac{3\pi}{4},$$

$$z' = \frac{\pi}{4},\ z = -\frac{\pi}{4}.$$

Hence by the minimum principle

$$G_r(\hat{\hat{z}}, \frac{1}{2}) \leqq m_4 G_r(z, \frac{1}{2}) : |\arg z| \leqq \frac{\pi}{4} \qquad (13)$$

Similarly

$$G_r(\check{\check{z}}, \frac{1}{2}) \leqq m_4 G_r(z, \frac{1}{2}) : |\arg z| \leqq \frac{\pi}{4}, \qquad (14)$$

where $\check{\check{z}}$ is the symmetric point of z relative to $\arg z = -\frac{\pi}{4}$. On the other hand, by Lemma 2, we can prove similarly as (12), there exists a const. m_5 such that

$$G_r(z, \frac{1}{2}) \leqq m_5 G_r(|z|, \frac{1}{2}) : |\arg z| \leqq \frac{\pi}{4}.$$

Put $l_3 = m_4 \cdot m_5$. Then we have by (7), (13), (14)

$$G_r(z, \frac{1}{2}) \leqq l_3 G_r(|z|, \frac{1}{2}). \tag{15}$$

Also by Lemma 2, we see by the methods as before, there exists a const. $l_{3,\delta}$ such that

$$G_r(z, \frac{1}{2}) \geqq \frac{1}{l_{3,\delta}} G_r(|z|, \frac{1}{2}) : r < |z| < \frac{1}{8}, |\arg z| \leqq \frac{\pi}{2} - \delta. \tag{16}$$

Since $G_r(z, \frac{1}{2}) = 0$ on $|z| = r$, we have 2) by (15) and (16) and we have Lemma 4.

Green functions, II. Let $\theta = \frac{2\pi}{8n}$ (n is a positive integer). Let $L_i = \{0 < |z| < \infty,\ \arg z = i\theta : i = 0, \pm 1, \pm 2, \dots, \pm 4n\}$.

$$A_i = \{0 < |z| < 1,\ (i-1)\theta \leqq \arg z \leqq i\theta\}.$$

T_i is a symmetric transformation with respect to L_i. Let $z_1 \in A_1$, $z_{i+1} = T_i(z_i) : 1 \leqq i \leqq 2n-1$, $z_{-1} = T_0(z_1)$, $z_{i-1} = T_i(z_i) : -4n+1 \leqq i \leqq -1$. We call $\sum_i z_i$ an equivalent class of $z_1 \in A_1$ and denote by $\{z_1\}$. If two points z_1 and z_2 are in the same $\{z\}$, we denote by $z_1 \approx z_2$. We remark for $z \in L_0 + L_1$, there exist only $2n$ equivalent points. Let Λ_1 be a simple compact analytic curve in A_1 separating $z=0$ from $|z|=1$ in A_1, $\Lambda_1 \cap (L_0 + L_1) = p_0 + p_1$ and $\max_{z \in \Lambda_1} |z| = r$. Let $\Lambda_{i+1} = T_i(\Lambda_i) : 1 \leqq i \leqq 4n-1$. $\Lambda_{-1} = T_0(\Lambda_1)$, $\Lambda_{i-1} = T_i(\Lambda_i)$ $: -4n+1 \leqq i \leqq -1$. Then

$$\Lambda_r = \sum_{i=-4n}^{i=4n} \Lambda_i$$

is a closed Jordan curve. Let D_r be a doubly connected domain bounded by $\Lambda_r + \{|z| = 1\}$. Put

$$\Omega^\Lambda = \Omega \cap D_r.$$

Let $G(z, \frac{1}{2})$ be a Green function of Ω^Λ. Then

LEMMA 5. 1) $\frac{\partial}{\partial n}G(\hat{z},\frac{1}{2})\leqq\frac{\partial}{\partial n}G(z,\frac{1}{2}):|arg\ z|\leqq\frac{\pi}{2},\ z\in\Lambda_r.$

2) *There exists a const.* l_4 *not depending* r *and the shape of* Λ_1 *such that*

$$\frac{\partial}{\partial n}G(z',\frac{1}{2})\leqq l_4\frac{\partial}{\partial n}G(z,\frac{1}{2}):z\approx z',\ z\in\Lambda_1.$$

3) *There exists a const.* l_4' such that

$$\frac{\partial}{\partial n}G(z',\frac{1}{2})\leqq\frac{1}{l_4'}\frac{\partial}{\partial n}G(z,\frac{1}{2}):z\approx z',\ z\in\Lambda_1,|arg\ z'|\leqq\frac{\pi}{2}-2\theta.$$

PROOF. Since D_r is symmetric with respect to $|\arg z|=\frac{\pi}{2}$, we have as Lemma 4, 1)

$$G(\hat{z},\frac{1}{2})\leqq G(z,\frac{1}{2}):|\arg z|\leqq\frac{\pi}{2}. \tag{a}$$

Let $D_j=D_r\cap\{|z|<\frac{1}{2},\ (j-4)\theta<\arg\ \ z\leq(j+4)\theta\}$. Consider $\xi=\frac{re^{-j(i\theta)}}{z}$. Then $D_j\rightarrow$onto a domain $\{2r<\xi<1,|\arg\xi|<4\theta\}$. Hence by Lemma 3 there exists a const.m_1 *depending only* n (or $\frac{2\pi}{8n}=\delta$) but on r and the shape of Λ_1 such that

$$\frac{1}{m_1}\leqq\frac{G(z,\frac{1}{2})}{G(T_j(z),\frac{1}{2})}\leqq m_1: \tag{17}$$

$$z\in\Omega^{\Lambda}\cap\{|z|\leqq\frac{1}{4},\ (j-1)\theta<\arg z<(j+1)\theta\},|j|\leqq 2n-3.$$

Hence

$$\frac{1}{m_1^{2n}}\leqq\frac{G(z',\frac{1}{2})}{G(z'',\frac{1}{2})}\leqq m_1^{2n}:|z'|=|z''|\leqq\frac{1}{4}:\ \arg z'=\frac{\pi}{4},\ \arg z''=-\frac{\pi}{4}. \tag{b}$$

For $\frac{1}{8}\leqq|z'|=|z''|<1$, $\arg z'=\frac{\pi}{4}$, $\arg z''=\frac{-\pi}{4}$ there exists a const.m_2 similarly as (10) and (11) such that

$$\frac{1}{m_2}\leqq\frac{G(z',\frac{1}{2})}{G(z'',\frac{1}{2})}\leqq m_2. \tag{c}$$

Hence by (b), (c)

$$\frac{1}{m_3} \leqq \frac{G(z', \frac{1}{2})}{G(z'', \frac{1}{2})} \leqq m_3 : \tag{d}$$

$m_3 = \max(m_1^{2n}, m_2) : |z'| = |z''|$, $\arg z' = \frac{\pi}{4}$, $\arg z'' = -\frac{\pi}{4}$.

By (a), (d)

$$G(\tilde{z}, \frac{1}{2}) \leqq m_3 G(z, \frac{1}{2}) : |\arg z| \leqq \frac{\pi}{4}, \tag{e}$$

where $\tilde{z}$ is the symmetric point of z relative to $\arg z = \pm\frac{\pi}{4}$.

Hence by (a)

$$G(z'', \frac{1}{2}) \leqq m_3 G(z', \frac{1}{2}),\ z'' \approx z' \text{ and } |\arg z'| \leqq \frac{\pi}{4}.$$

By (17)

$$G(z', \frac{1}{2}) \leqq m_1^n G(z, \frac{1}{2}),\ z' \approx z'\ z \in A_1, |\arg z'| \leqq \frac{\pi}{4}.$$

Hence

$$G(z', \frac{1}{2}) \leqq l_4 G(z, \frac{1}{2}) : z' \approx z,\ z \in A_1,\ l_4 = m_3 \cdot m_1^n. \tag{18}$$

By (17)

$$G(z', \frac{1}{2}) \leqq \frac{1}{l_4'} G(z, \frac{1}{2}) : \ell_4' = m_1^{2n},$$

$$z' \approx z, |z| \leqq \frac{1}{4}, |\arg z'| \leqq (2n-2)\theta,\ z \in A_1. \tag{19}$$

Since $G(z, \frac{1}{2}) = 0$ on Λ_r, we have by (18) and (19) the Lemma 5.

For the following we modify Lemma 2 as

LEMMA 2'. *Let $D = \{0 < |z| < 1, |\arg z| < \theta\}$ and let $U(z)$ be an HP in D such that $U(z) = 0$ on $|z| = 1$. Then there exists a const. l_1 depending on θ and δ such that*

$$\frac{1}{l_1} < \frac{U(z')}{U(z'')} < l_1 : |z'| = |z''|, |\arg z'| \text{ and } |\arg z''| < \theta - \delta.$$

In fact, let $D_r=\{r<|z|<1\},|\arg z|<\theta\}$. Consider $U(z)$ in D_r, using $\xi=\frac{z}{r}$, we have by Lemma 2 there exists a const. l_1 not depending on r such that $\frac{1}{l_1}<\frac{U(z')}{U(z'')}<l_1$: $|z'|=|z''|>2r$, $|\arg z'|$ and $|\arg z''|<\theta-\delta$. Let $r\downarrow 0$. Then we have Lemma 2′.

Class $\mathfrak{H}$. Let Ω be the domain. We denote by $\mathfrak{H}^\delta$ the class of functions $\{K(z, p)\}$: $p\in\Delta$ such that $p_i\overset{M}{\to}p$, $p_i\to\{z=0\}$ *and* $p_i\in\{|\arg z|\leqq\frac{\pi}{2}-\delta\}$. Put

$$\mathfrak{H}=\bigcup_{\delta>0}\mathfrak{H}^\delta.$$

LEMMA 6. 1). *Let* $U(z)\in\mathfrak{H}$. *Then* $U(\hat{z})\leqq U(z)$ *and there exist consts.* λ_1 *and* $\lambda_{1,\delta}$ *such that*

$$U(z)\leqq\lambda_1 U(|z|),$$
$$U(z)\geqq\frac{1}{\lambda_{1,\delta}}U(|z|):|\arg z|\leqq\frac{\pi}{2}-\delta,\ 0<\delta<\frac{\pi}{2},$$

where $\hat{z}$ *is the symmetric point of* z *relative to* $\{\arg z=\frac{\pi}{2}\}$.

2) *Let* $U(z)\in\mathfrak{H}$. *Then* $U(z)=0$ *on* $F+\{|z|=1\}$ *except at most a set of capacity zero, i. e.* $U(z)$ *is singular and* $\varlimsup_{z\to\xi}U(z)<\infty$: $\xi\in F$, $\xi\neq 0$.

3) *Let* $\Omega_r=\Omega\cap\{|z|>r\}:r<\frac{1}{16}$ *be the domain in Lemma 4. Let* γ *be an arc on* $\{|z|=r,\ |\arg z|<\frac{\pi}{2}-\delta\}$. *Let* $U(\gamma, z, \Omega_r)=H_\phi^{\Omega_r}$: $\phi=U$ *on* γ *and* $=0$ *elsewhere. Then there exists a const.* ε_1 *not depending on* r *such that*

$$U(\gamma, \tfrac{1}{2}, \Omega_r)\geqq\varepsilon_1(\textit{angular mes of }\gamma).$$

4) Ω^Λ *be the domain in Lemma* 5: $\theta=\frac{2\pi}{8n}$. *Let* Λ_i: $|i|\leqq 2n-2$ *and* $U(\Lambda_i, z,\Omega^\Lambda)=H_\phi^\Omega$: $\phi=U$ *on* Λ_i, $=0$ *elsewhere, then there exists a const.* ε_2 *depending on* θ *but not on* r *and the shape* Λ_1 *such that*

$$U(\Lambda_i, \tfrac{1}{2}, \Omega^\Lambda)>\varepsilon_2.$$

PROOF of 1) and 2). We can suppose $p_i\in\{|z|<\frac{1}{32}, 0\leqq\arg z<\frac{\pi}{2}-\delta\}$: $\delta<\frac{\pi}{8}$. Let $\hat{z}$ be the symmetric point of z relative to $|\arg z|=\frac{\pi}{2}$. Then

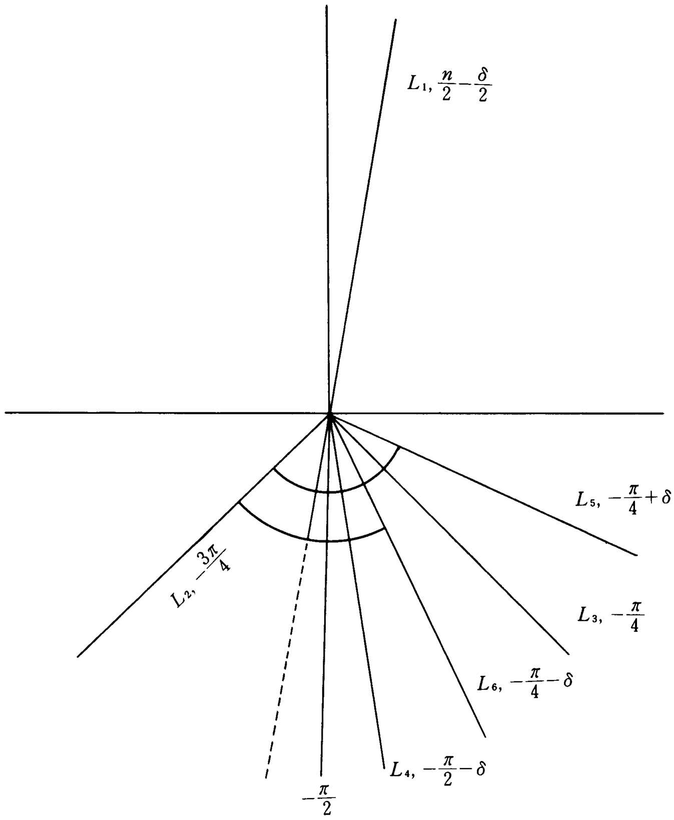

Fig. 2

as Lemma 4

$$K(\hat{z}, p_i) \leqq K(z, p_i): \ |\arg z| \leqq \frac{\pi}{2}. \tag{20}$$

Hence

$$K(z_2, p_i) \leqq K(z_3, p_i): \ |z_2| = |z_3|, \ z_2 \in L_2, \ z_3 \in L_3, \ L_2 = \{\arg z = -\frac{3\pi}{4}\}, \ L_3 = \{\arg z = -\frac{\pi}{4}\}.$$

$K(z, p)$ is an HP in $\{-\frac{\pi}{2} < \arg z < 0\}$, $=0$ on $|z|=1$. By Lemma 2′, there exists a const. l_1 such that

$$K(z_3, p_i) \leqq l_1 K(z_5, p_i),$$

$|z_3| = |z_5|$, $z_3 \in L_3$, $z_5 \in L_5 = \{\arg z = -\frac{\pi}{4} + \delta\}$. Now L_2 is symmetric to L_5 relative to $L_4 = \{\arg z = -\frac{\pi}{2} + \frac{\delta}{2}\}$. By the maximum principle

$$K(\hat{\hat{z}}, p_i) \leqq l_1 K(z, p_i): \ -\frac{\pi}{2} + \frac{\delta}{2} \leqq \arg z \leqq -\frac{\pi}{4}, \tag{21}$$

where $\hat{\hat{z}}$ is the symmetric point of z relative to L_4. Let $L_6=\{\arg z=-\frac{\pi}{4}-\delta\}$. Then similarly as before, there exists a const. l_2 such that

$$K(z', p_i)\leqq l_2K(z, p_1): |z|=|z'|, z'\in L_2, z\in L_6.$$

L_6 is symmetric to L_2 relative to $L_1=\{\arg z=\frac{\pi}{2}-\frac{\delta}{2}\}$. $K(z, p_i)-l_2K(z', p_i)$ is an SPH in $\{-\frac{\pi}{4}-\delta<\arg z<\frac{\pi}{2}-\frac{\delta}{2}\}$. Hence by the maximum principle

$$K(z', p_i)\leqq l_2K(z, p_i): -\frac{\pi}{4}-\delta<\arg z<\frac{\pi}{2}-\frac{\delta}{2}, \tag{22}$$

where z' is symmetric to z relative to L_1.
Put $l_3=\max(l_1, l_2)$. Then by (20), (21), (22)

$$K(z, p_i)\leqq l_3 \max_{\xi}(K(\xi, p_i)) \text{ for } (\xi=|z|, |\arg\xi|\leqq\frac{\pi}{2}-\delta) \text{ for any } z.$$

Suppose $p_i\in\{|z|<\varepsilon\}$ and consider $K(z, p_i)$ in $|\arg z|<\frac{\pi}{2}$. Then by Lemma 2′, there exists a const. l_δ depending on $\frac{\pi}{2}$, δ such that

$$\frac{1}{l_\delta}K(|z|, p_i)\leqq K(z, p_i)\leqq l_\delta K(|z|, p_i): 2\varepsilon<|z|<1, |\arg z|\leqq\frac{\pi}{2}-\frac{\delta}{2}.$$

Put $\lambda_1=l_3l_\delta$ and $\lambda_{1,\delta}=l_\delta$. Then

$$K(z, p_i)\leqq\lambda_1K(|z|, p_i)$$

and

$$K(z, p_i)\geqq\frac{1}{\lambda_{1,\delta}}K(|z|, p_i): |\arg z|\leqq\frac{\pi}{2}-\frac{\delta}{2}. \tag{23}$$

Let $p_i\to z=0$ and $\varepsilon\to 0$. Then

$$K(z, p)\leqq\lambda_1K(|z|, p),$$
$$K(z, p)\geqq\frac{1}{\lambda_{1,\delta}}K(|z|, p): |\arg z|\leqq\frac{\pi}{2}-\frac{\delta}{2}.$$

For case $p_i\in\{\arg z\geqq 0\}$, we used only angular domains whose boundary does not touch p_i. For $p_i\in\{\arg z<0\}$, we have the same result. Hence we have 1).

Let $p_i\in\{|z|<\varepsilon\}$ and $r>2\varepsilon$. Since $K(\frac{1}{2}, p_i)=1$, there exists a const.

$C(r)$ depending only on r such that $K(r, p_i) \leqq C(r)$. Hence by (23)

$$K(z, p_i) \leqq C(r)\lambda_1 W(\Gamma_r, z, \Omega_r): \ r<|z|<1,$$

where $\Gamma_r=\{|z|=r\}$. Hence by letting $p_i \to z=0$ and $\varepsilon \to 0$

$$K(z, p) \leqq \lambda_1 C(r) W(\Gamma_r, z, \Omega_r): \ |z| \geqq r.$$

Hence $K(z, p)$ is singular and $K(z, p)<\infty$ for $|z|>0$. Thus we have 2).

Proof of 3) and 4). Let $G_r(z, \frac{1}{2})$ be a Green function Ω_r. Then since $U(z) \in \mathfrak{H}$ is singular,

$$1=U(\tfrac{1}{2})=\frac{1}{2\pi}\int_{|z|=r} U(\xi)\frac{\partial}{\partial n}G_r(\xi, \tfrac{1}{2})ds.$$

By Lemma 4) and 6)

$$\frac{\partial}{\partial n}G_r(\hat{\xi}, \tfrac{1}{2}) \leqq l_3\frac{\partial}{\partial n}G_r(|\xi|, \tfrac{1}{2}) \text{ and } U(\hat{\xi}) \leqq U(\xi) \leqq \lambda_1 U(|\xi|),$$

whence

$$\frac{1}{2} \leqq \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} U(\xi)\frac{\partial}{\partial n}G_r(\xi, \tfrac{1}{2})d\theta \leqq \frac{1}{2}l_3\lambda_1 U(|\xi|)\frac{\partial}{\partial n}G_r(|\xi|, \tfrac{1}{2}),$$

and

$$U(|\xi|)\frac{\partial}{\partial n}G_r(|\xi|, \tfrac{1}{2}) \geqq \frac{1}{l_3\lambda_1}.$$

On the other hand γ is in $\{|z|=r, \ |\arg z| \leqq \frac{\pi}{2}-\delta\}$ and

$$\frac{\partial}{\partial n}G_r(\xi, \tfrac{1}{2}) \geqq \frac{1}{l_{3,\delta}}\frac{\partial}{\partial n}G_r(|\xi|, \tfrac{1}{2}), \ U(\xi) \geqq \frac{1}{\lambda_{1,\delta}}U(|\xi|).$$

Hence

$$U(\gamma, \tfrac{1}{2}, \Omega_r)=\frac{1}{2\pi}\int_\gamma U(\xi)\frac{\partial}{\partial n}G_r(\xi, \tfrac{1}{2})ds \geqq \frac{\text{ang}\cdot\text{mes}\,\gamma}{2\pi l_3\lambda_1 l_{3,\delta}\lambda_{1,\delta}}.$$

Put $\varepsilon_1=1/2\pi l_3\lambda_1 l_{3,\delta}\lambda_{1,\delta}$. Then we have 3).

Let $G(z, \frac{1}{2})$ be Green function of Ω^A. Then similarly as 3), by Lemma 5) and 6), 1), by putting $\delta=\frac{\pi}{4n}$,

$$\frac{1}{2}\leqq\frac{1}{2\pi}\int_{\Sigma\Lambda_i:\,|i|\leqq 2n}U(\zeta)\frac{\partial}{\partial n}G(\zeta,\frac{1}{2})ds\leqq\frac{4n\lambda_1 l_4}{2\pi}\int_{\Lambda_1}U(\zeta)\frac{\partial}{\partial n}G(\zeta,\frac{1}{2})ds.$$

$$U(\Lambda_i,\frac{1}{2},\Omega^{\Lambda})=\frac{1}{2\pi}\int_{\Lambda_i}U(\zeta)\frac{\partial}{\partial n}G(\zeta,\frac{1}{2})ds\geqq\frac{1}{4nl_4\lambda_1 l_4'\lambda_{1,\delta}}:\ |i|\leqq 2n-2.$$

Put $\varepsilon_2=1/4nl_4\lambda_1 l_4'\lambda_{1,\delta}$. Then we have 4).

On fine neighbourhoods of $p\in\Delta_1$ and canonical representations of SPH, where Δ_1 is the set of minimal boundary points.

Let G be an open set in Ω. If $K(z,p)-K_{CG}(z,p)>0$, we call G a fine neighbourhood of p and denote by

$$G\overset{K}{\ni}p,$$

where $K_{CG}(z,p)$ is the least positive SPH not smaller than $K(z,p)$ on CG. Then $V_n(p)\overset{K}{\ni}p$ and $V_M(p)\overset{K}{\ni}p$ are well known[1)2)], where $V_n(p)=\{z:\overset{M}{\text{dist}}(z,p)<\frac{1}{n}\}$ and $V_M(p)=\{z:K(z,p)>M\}:M<\sup_z K(z,p)$. Let E be a closed set in Ω. Let $U(z)$ be an SPH. Then

$$U_{E\cap\Delta}(z)=\lim_n U_{E\cap\Delta_n}(z)\ :\ \Delta_n=\{z:\overset{M}{\text{dist}}(z,\Delta)<\frac{1}{n}\}.$$

Then if $U_{E\cap\Delta}(z)>0$, then $U_{E\cap\Delta}(z)$ is represented by a canonical mass[1)] on $\bar{E}^M\cap\Delta_1$[1)]: $\bar{E}^M$ is the closure of E with respect to Martin's topology (in the following we denote it by $\bar{E}$ simply). Hence if $U_{E\cap\Delta}(z)>0$, $\bar{E}\cap\Delta_1\neq 0$.

Let $E=\{z:|\arg z|\leqq\frac{\pi}{4}\}$. Then $\gamma_m=E\cap\{|z|=r_m\}$ has ang. mes $=\frac{\pi}{2}$. Let $U(z)\in\mathfrak{H}$. Then $U(z)\in\mathfrak{H}^{\delta}$ for a positive number δ. Hence by Lemma 6. 3)

$$U_{E\cap\Delta}\ (\frac{1}{2})\geqq\lim_m U_{E\cap D_m}\ (\frac{1}{2})\geqq\varliminf_m U(\gamma_m,\frac{1}{2},\Omega_{r_m})\geqq\frac{\pi\varepsilon_0}{2}>0,$$

where $D_m=\{z:|z|\leqq\frac{1}{m}\}$. Hence $\bar{E}\cap\Delta_1\neq 0$ and there exist at least a sequence $\{p_i\}$ such that $p_i\in\{|\arg z|\leqq\frac{\pi}{4}\}$ and $p_i\overset{M}{\rightarrow}q\in\Delta_1$.

LEMMA 7. *Let $p\in\Delta_1$ and $K(z,p)$ be singular. Then sup $K(z,p)=\infty$ and then,*

1) $K_{CV_M(p)\cap\Delta}(z,p)=0\ :\ M<\infty$.

2) $G\overset{K}{\ni}p$ *if and only if* ${}_{\Delta}(K_{CG}(z,p))=0=\lim_{M=\infty}{}_{V_M(p)}(K_{CG}(z,p))$.

Also $G \overset{K}{\ni} p$ *implies* $K_{CG\cap\Delta}(z, p)=0$.

3) *Let* Ω *be the domain. Let* $q \in \Delta_1$ *corresponding to an HP in* $\mathfrak{H}$, *i. e.* $K(z, q) \in \mathfrak{H}$, $q \in \Delta_1$, q *lies on* $z=0$ *and* $K(\hat{z}, q) \leqq K(z, q)$. *Let* γ *be a closed set consisting analytic curves clustering nowhere in* $\Omega \cap \{|\arg z| \leqq \frac{\pi}{2}\}$ *such that* $C\gamma$ *(complementary set of* γ*)* $\overset{K}{\ni} q$. *Let* $\hat{\gamma}$ *be the symmetric image of* γ *relative to* $\{|\arg z|=\frac{\pi}{2}\}$. *Then*

$$_{\Delta}(K_{\gamma+\hat{\gamma}}(z, q))=0, \text{ i.e. } C(\gamma+\hat{\gamma}) \overset{K}{\ni} q.$$

PROOF of 1). Since $K(z, p)$ is singular, we see at once $\sup_z K(z, p)=\infty$. Assume $_{CV_M(p)\cap\Delta}K(z, p)>0$. Then $_{CV_M(p)\cap\Delta}K(z, p)$ is an HP. By the minimality of $K(z, p)$, $_{CV_M(p)\cap\Delta}K(z, p)=aK(z, p)$: $a>0$. Evidently $_{CV_M(p)\cap\Delta}K(z, p) \leqq M$. This is a contradiction. Hence we have 1).

2) Let $G \overset{K}{\ni} p$. Assume $_{\Delta}(K_{CG}(z, p))>0$ (or $K_{CG\cap\Delta}(z, p)>0$). Then this is an HP and by the minimality, $_{\Delta}(K_{CG}(z, p))=aK(z, p)$(or $K_{CG\cap\Delta}(z, p)=a'K(z, p)$) : $a, a'>0$. Hence

$$K_{CG}(z, p)=\beta K(z, p)+V(z) : \beta=a \text{ or } a',$$

where $V(z)$ is an SPH$\geqq 0$ and $V(z)=(1-\beta)K(z, p)$ on ∂G. By the definition of $K_{CG}(z, p)$, $V(z) \geqq (1-\beta)K_{CG}(z, p)$.

$$K_{CG}(z, p) \geqq \beta K(z, p)+(1-\beta)K_{CG}(z, p).$$

Hence for $0<\beta\leqq 1$, we have $K_{CG}(z, p) \geqq K(z, p)$. This contradicts $G \overset{K}{\ni} p$.

Hence $G \overset{K}{\ni} p$ implies

$$_{\Delta}(K_{CG}(z, p))=K_{\Delta\cap CG}(z, p)=0.$$

$G \overset{K}{\not\ni} p$ implies $K_{CG}(z, p)=K(z, p)$, whence $_{\Delta}(K_{CG}(z, p))=K(z, p)>0$.

We show $_{\Delta}(K_{CG}(z, p))=0$ if and only if $\lim_{M=\infty} {}_{V_M(p)}(K_{CG}(z, p))=0$.

$$_{\Delta\cap V_M(p)}(K_{CG}(z, p)) \leqq {}_{\Delta}(K_{CG}(z, p)) \leqq {}_{\Delta\cap CV_M(p)}(K_{CG}(z, p))+ {}_{\Delta\cap V_M(p)}(K_{CG}(z, p)).$$

But ${}_{\Delta\cap CV_M(p)}(K_{CG}(z, p))=0$ by 1). We have

$$ {}_{V_M(p)}(K_{CG}(z, p))\geqq{}_{\Delta\cap V_M(p)}(K_{CG}(z, p))={}_{\Delta}(K_{CG}(z, p)). $$

Let $M\nearrow\infty$. Then $\lim\limits_{M} V_M(p)\subset\Delta$ by the maximum principle. Hence $\lim\limits_{M=\infty} {}_{V_M(p)}(K_{CG}(z, p))\leqq{}_{\Delta}(K_{CG}(z, p))$. Hence

$$ \lim_{M=\infty} {}_{V_M(p)}(K_{CG}(z, p))={}_{\Delta}(K_{CG}(z, p)). $$

3) Let $U_1(z)=H_{\phi}^{\Omega-\gamma-\hat{\gamma}}: \phi=K(z, q)$ on γ, $=0$ elsewhere, $U_2(z)=H_{\psi}^{\Omega-\gamma-\hat{\gamma}}$: $\psi=K(z, q)$ on $\hat{\gamma}$, $=0$ elsewhere.

Then
$$ K_{\gamma+\hat{\gamma}}(z, q)=U_1(z)+U_2(z) \tag{23} $$
and
$$ U_1(z)\leqq K_{\gamma}(z, q). \tag{24} $$

Let $G(z, z_0): z_0\in\{|\arg z|\leqq\frac{\pi}{2}\}$ be Green function of $\Omega-\gamma-\hat{\gamma}$. Then as Lemma 5 $G(\hat{z}, z_0)\leqq G(z, z_0)$ and $\frac{\partial}{\partial n}G(\hat{z}, z_0)\leqq\frac{\partial}{\partial n}G(z, z_0)$, $z\in\gamma$, where $\hat{z}$ is the symmetric point of z. Put $V(z)=K_{\gamma+\hat{\gamma}}(z, q)$. Then since $K(z, q)\in\mathfrak{H}$,

$$ K(\hat{\xi}, q)=V(\hat{\xi})\leqq V(\xi)=K(\xi, q): \xi\in\gamma,\ \hat{\xi}\in\hat{\gamma}. \tag{25} $$

$U_2(z_0)=\frac{1}{2\pi}\int_{\hat{\gamma}}V(\hat{\xi})\frac{\partial}{\partial n}G(\hat{\xi}, z_0)ds\leqq\frac{1}{2\pi}\int_{\gamma}V(\xi)\frac{\partial}{\partial n}G(\xi, z_0)ds=U_1(z_0)$. Now z_0 is an arbitrarly point in $\{|\arg z|\leqq\frac{\pi}{2}\}$ and

$$ U_2(z)\leqq U_1(z): |\arg z|\leqq\frac{\pi}{2}. $$

Hence
$$ V(z)=U_1(z)+U_2(z)\leqq 2U_1(z): |\arg z|\leqq\frac{\pi}{2}. \tag{26} $$

On the other hand, $V(z)-V(\hat{z})=0$ on $|\arg z|=\frac{\pi}{2}$, $\geqq 0$ on $\gamma+\hat{F}$: $\hat{F}$ is the symmetric image of F. Hence

$$ V(\hat{z})\leqq V(z): |\arg z|\leqq\frac{\pi}{2}. \tag{27} $$

Hence by 24) and 26)

$$V(\hat{z}) \leqq V(z) \leqq 2K_\gamma(z, q) : |\arg z| \leqq \frac{\pi}{2}. \tag{28}$$

Let $D_n = \{|z| < \frac{1}{n}\}$. Then since $K(z, q) < \infty$ for $|z| > 0$ by Lemma 6.2, for any given D_n, there exists an M_n such that $D_n \supset V_{M_n}(q) = \{z \in \Omega : K(z, q) > M_n\}$. Hence

$$\Delta \supset \lim D_n \supset \lim_M V_M(q). \tag{29}$$

By the assumption $C(\gamma) \overset{K}{\ni} q$, $0 = \lim_\Delta(K_\gamma(z, q)$, i.e.

$$0 = \lim_n {}_{D_n}(K_\gamma(z, q))_{z=\frac{1}{2}} = \lim_n \frac{1}{2\pi} \int_{\partial D_n \cap \Omega} K_\gamma(\zeta, q) \frac{\partial}{\partial n} G^n(\zeta, \frac{1}{2}) ds \geqq$$
$$\lim_n \frac{1}{2\pi} \int_{\partial D_n \cap \{|\arg \xi| \leqq \frac{\pi}{2}\}} K_\gamma(\zeta, q) \frac{\partial}{\partial n} G^n(\zeta, \frac{1}{2}) ds : \tag{30}$$

$G^n(z, \frac{1}{2})$ is a Green fumction of $\Omega - D_n$.

Clearly $\frac{\partial}{\partial n} G^n(\hat{\xi}, \frac{1}{2}) \leqq \frac{\partial}{\partial n} G^n(\zeta, \frac{1}{2})$: $\zeta \in \partial D_n \cap \{|\arg z| \leqq \frac{\pi}{2}\}$.

By (28)

$$_{D_n}(K_{\gamma+\hat{\gamma}}(\frac{1}{2}, q)) = \frac{1}{2\pi} \int_{\partial D_n \cap \Omega} V(\xi) \frac{\partial}{\partial n} G^n(\zeta, \frac{1}{2}) ds$$
$$\leqq \frac{1}{\pi} \int_{\partial D_n \cap \{|\arg z| \leqq \frac{\pi}{2}\}} V(\xi) \frac{\partial}{\partial n} G^n(\zeta, \frac{1}{2}) ds$$
$$\leqq \frac{2}{\pi} \int_{\partial D_n \cap \{|\arg z| \leqq \frac{\pi}{2}\}} K_\gamma(\zeta, q) \frac{\partial}{\partial n} G^n(\zeta, \frac{1}{2}) ds.$$

Hence by (30)

$$\lim_n {}_{D_n}(K_{\gamma+\hat{\gamma}}(z, q)) = 0.$$

At last by (29) and (2) we have

$$_\Delta(K_{\gamma+\hat{\gamma}}(z, q)) = 0.$$

Thus we have 3).

On the behaviour of Martin's topology.

$$\overset{\mathrm{M}}{\operatorname{dist}}(p_1, p_2) = \sup_{z \in \Gamma} \left| \frac{K(z, p_1)}{1 + K(z, p_1)} - \frac{K(z, p_2)}{1 + K(z, p_2)} \right| : \Gamma = \{|z - \frac{1}{2}| = \frac{1}{32}\}.$$

Let $\Omega'=\Omega\cap\{|z-\frac{1}{2}|>\frac{1}{8}\}$. Then $K(z, p)$ are uniformly bounded in $\{|z-\frac{1}{2}|\leqq\frac{1}{16}\}$ for $p\in\Omega'$. We define a new distance $\delta^*(p_1, p_2): p_i\in\Omega'$ as

$$\sup_{z\in\Gamma}|K(z, p_1)-K(z, p_2)|.$$

We denote by M^*—the topology induced by this metric. Then we see at once M^*—top. and Martin's topology are isomorphic in $\bar{\Omega}'$. Let $q\in\Delta_1$ over $\{z=0\}$ corresponding to a function in $\mathfrak{H}$. Put $V_n^*(q)=\{z:\delta^*(z, q)<\frac{1}{n}\}$. Then $\{V_n(q)\}$ and $\{V_n^*(q)\}$ are equivalent. To study the behaviour of $V_n(q)$ we investigate $V_n^*(q)$ instead of $V_n(q)$.

LEMMA 8. *Let $q\in\Delta_1$ corresponding to a function in $\mathfrak{H}$. Let $g(z)=\delta^*(z, q)$: $z\in\Omega'$. Then clearly $g(z)$ is continuous in Ω'. Let G be a compact domain in Ω'. Then*

$$g(z)\lneqq \max_{z\in\partial G} g(z): z\in G$$

and $\{z: g(z)=const.\}$ does not contain an open set. Therefore $\{z: g(z)\geqq\delta>0\}$ is not compact in Ω'.

PROOF. Put $M=\max_{z\in\partial G} g(z)$ and $M^*=\max_{z\in G} g(z)$. Suppose $M^*>M$. Put $E=\{z\in G: g(z)=M^*\}$. Then dist $(E, \partial G)>0$. There exists a point $z_0\in\partial E\cap G$ and r such that $C(r, z_0)\subset G$ and $\partial C(r, z_0)\cap CE\neq 0$, where $C(r, z_0)=\{|z-z_0|<r\}$. Then mes $\partial C(r, z_0)\cap CE)>0$. $g(z_0)=M^*$ means

$$M^*=\sup_{z\in\Gamma}|K(z, z_0)-K(z, q)|,\ \sup_{z\in\Gamma}|K(z, z')-K(z', q)|$$
$$<M^* \text{ for } z'\notin E.$$

Since $C(r, z_0)\ni z_0$ and $\partial C(r, z_0)$ is compact,

$$K(z, z_0)=K_{C(r, z_0)}(z, z_0)=H_{\Psi}^{\Omega-C(r, z_0)}: z\notin C(r, z_0),$$

where $\Psi=K(z, z_0)$ on $\partial C(r, z_0)$ and $=0$ elsewhere. Hence there exists a positive, continuous unit mass distribution[1] $\mu(z_\tau)$ exists on $\partial C(r, z_0)$ such that

$$K(z, z_0)=\int_{\partial C(r, z_0)} K(z, z_\tau)d\mu(z_\tau): z\notin C(r, z_0).$$

There exists a point t_0 on Γ not depending on r such that

$$M^* = |K(t_0, z_0) - K(t_0, q)| \leqq \int |K(t_0, z_\tau) - K(t_0, q)|\, d\mu(z_\tau). \quad (31)$$

Now $\int_{\partial C(r, z_0) \cap CE} d\mu(z_\tau) > 0$ and $|K(t_0, z_\tau) - K(t_0, q)| < M^* : z \in (\partial C(r, z_0) \cap CE)$. The term on the right hand of (31) $< M^*$. This is a contradiction. Hence $M^* = M$. If $0 \neq E \subsetneqq \bar{G}$, we have the same contradiction. Hence $E = G$ or $E \cap G = 0$. Suppose $E = G$. Then there exist a point $z_0 \in G$ and $C(r, z_0) \subset G$. Then by (31), there exist a point $t_0 \in \Gamma$ not depending on r such that

$$M = |K(t_0, z) - K(t_0, q)| = \int_{\partial C(r, z_0)} |K(t_0, z_\tau) - K(t_0, q)|\, d\mu(z_\tau).$$

Hence $K(t_0, z_\tau) = K(t_0, q) + C : C = M$ or $-M$ for $z_\tau \in C(r, z)$ for any $r <$ dist $(z, \partial G)$, whence $K(t_0, z_\tau) = \text{const} : z_\tau \in C(r, z_0)$, i.e.

$$K(z_0, z_\tau) = \frac{G(t_0, z_\tau)}{G\left(\frac{1}{2}, z_\tau\right)} = \frac{G(z_\tau, t_0)}{G\left(z_\tau, \frac{1}{2}\right)} = \text{const}.$$

Whence $G(z, t_0) = \text{const.} \times G(z, \frac{1}{2})$. This is a contradiction. Hence $E \cap G = 0$. i.e. $g(z) < \max_{z \in \partial G} g(z) : z \in G$. The other assertions are contained in the above discussion.

Proof of Theorem 1. We shall show.

1) There exists only one minimal function in $\mathfrak{H}$. We denote such function $U(z) = K(z, q) : q \in \Delta_1$.

2) Let $A^\delta = \{|\arg z| \leqq \frac{\pi}{2} - \delta\}$. Then $\bar{A}^\delta \cap \Delta_1$ consists of only one point q.

3) Let $A(\theta_1, \theta_2) = \{\theta_1 < \arg z < \theta_2\}$: $-\frac{\pi}{2} < \theta_1 < \theta_2 < \frac{\pi}{2}$. Then $CV_n^*(q) \cap A(\theta_1, \theta_2)$ does not contain a continuum tending to $z = 0$ for any n.

4) For any n and $\delta > 0$ there exists r depending on n and δ such that

$$(\{|z| < r\} \cap A^\delta) \subset V_n^*(q).$$

Thus we have the Theorem 1.

Proof. Let $U(z) \in \mathfrak{H}$. Then there exists $\delta > 0$ such that $U(z) \in \mathfrak{H}^\delta$. Then there exist const. s, $\lambda_1, \lambda_\delta$ depending on δ by Lemma 6 such that

$$U(z) \leqq \lambda_1 U(|z|) \text{ and } U(z) \geqq \frac{1}{\lambda_\delta} U(|z|) : |\arg z| < \frac{\pi}{2} - \delta.$$

Let $\lambda_r = \{|z| = r, |\arg z| \leqq \frac{\pi}{4}\}$. Put $M(r) = \max U(z)$ on λ_r and $N(r) =$

min $U(z)$ on λ_r and $W(\lambda_r, z, \Omega_r)$ be H.M of λ_r, where $\Omega_r=\Omega\cap\{|z|>r\}$. Then by Lemma 6

$$\frac{M(r)}{N(r)}\leqq\lambda_5=\lambda_1\cdot\lambda_\delta \qquad \text{and } 1=U(\frac{1}{2})\geqq U(\lambda_r, \frac{1}{2}, \Omega_r)\geqq\frac{\pi\varepsilon_1}{4}>0.$$

$$N(r)W(\lambda_r, z, \Omega_r)\leqq U(\lambda_r, z, \Omega_r)\leqq M(r)W(\lambda_r, z, \Omega_r). \tag{32}$$

Putting $z=\frac{1}{2}$,

$$N(r)\leqq\frac{1}{w(\lambda_r, \frac{1}{2}, \Omega_r)}, \quad M(r)\geqq\frac{\pi\varepsilon_1}{4w(\lambda_r, \frac{1}{2}, \Omega_r)}. \tag{32'}$$

We can find a sequence $\{U(\gamma_{\frac{1}{n}}, z, \Omega_{\frac{1}{n}})\}$ tending to an HP $V(z)$ $(\leqq U(z))>0$ by Lemma 6. Let $U(z)\in\mathfrak{H}$. Then $U(z)\in\mathfrak{H}^\delta$ for a number δ. Let $A=A(\theta_1, \theta_2)=\{\theta_1\leqq\arg z\leqq\theta_2\}:|\theta_i|<\frac{\pi}{2}-\delta$. Then $U_{A\cap\{|z|\geqq r\}}(z)\geqq U(\gamma_r, z, \Omega_r)>0$ by lemma 6 not depending on r. Let $r\to 0$. Then $U_{A\cap\Delta}(z)>0$ and $U_{A\cap\Delta}(z)$ is represented by a canonical mass on $\bar{A}\cap\Delta_1$. This implies there exists a sequence $\{p_i\}$ in A such that $p_i\to q\in\Delta_1\cap\bar{A}$. Hence $\mathfrak{H}$ has at least one minimal function. We shall show $\mathfrak{H}$ has only one minimal function. Let $U_i(z): i=1, 2$ be minimal in $\mathfrak{H}^\delta$. We can find $V_i(z)$ from $\{U_i(\gamma_r, z, \Omega_r)\}$ *as* $r\to 0$. By the minimality of $U(z)$

$$V_i(z)=a_iU_i(z): i=1, 2.$$

By 32) and 32') we have $V_1(z)\leqq\lambda_5 V_2(z)$.

By the minimality of $U_i(z)$, $U_1(z)=a'\,U_2(z): a'>0$.

Now $U_i(\frac{1}{2})=1$. Whence $U_1(z)=U_2(z)$. Thus $\mathfrak{H}$ has only one minimal function.

2) Let $\gamma_r=\{|z|=r,\ \theta_1\leqq\arg z\leqq\theta_2\}$, $A=\{\theta_1\leqq\arg z\leqq\theta_2\}:-\frac{\pi}{2}<\theta_1<\theta_2<\frac{\pi}{2}$.

Then $U_{A\cap\Delta}(z)\geqq\lim_{r\to 0} U(\gamma_r, z, \Omega_r)>0$. $\bar{A}\cap\Delta_1=$one point q by 1). Hence we have 2).

PROOF OF 3. Since $\{V_n^*(q)\}$ and $\{V_n(q)\}$ are equivalent, we use $V^*(q)$ instead of $V_n(q)$. Assume $CV_n^*(q)$ has a continuum in $A(\theta_1, \theta_2)$ connecting a point $z_0:|z_0|>r_0$ and $z=0$. Then we can find a continuum γ in it satisfying the condition of Lemma 7). 3) Then $\gamma\subset CV_n^*(q)\subset CV_m(q)$ $(m$

depends on n). Hence by Lemma 7. 2)

$$0={}_\Delta(K_\gamma(z,q))={}_\Delta(K_{\gamma+\hat\gamma}(z,q)),$$

where $\hat\gamma$ is the symmetric image of γ. Now $\gamma+\hat\gamma$ dividedes $\{|z|<r_0\}$. Evidently $V_M(q)\overset{K}{\ni}q: M<\infty$ and $C(r_0)=\{z\in\Omega:|z|<r_0\}\supset V_M(q)\overset{K}{\ni}q$ for large M, because $U(z)<\infty$ for $|z|>0$ by Lemma 6. Hence

$$(C(r_0)-\gamma-\hat\gamma)\overset{K}{\ni}q.$$

Hence there exists exactly one component G of the above such that $G\overset{K}{\ni}q$. G must be contained in A_1 or A_2:

$$A_1=\{|z|<r_0,\ \theta_1<\arg z<\pi-\theta_1\},$$
$$A_2=\{|z|<r_0,\ -\pi-\theta_2<\arg z<\theta_2\}.$$

We can suppose without loss of generality $G\subset A_1$. Then by Lemma 7. 2)

$$K_{\Delta\cap CG}(z,q)=0.$$

On the other hand by Lemma 6

$$K_{\Delta\cap CG}(z,q)\geqq K_{A'\cap\Delta}(z,q)>0,$$

by $CG\supset A'=\{-\frac{\pi}{2}+\delta<\arg z<\theta_1\}:-\frac{\pi}{2}+\delta<\theta_1$. This is a contradiction. Hence we have 3).

PROOF OF 4. Assume 4) is false. Then there exist $\delta>0$ and a const. $\varepsilon_0=\varepsilon_0(\delta)$ such that there exists a sequence $\{z_i\}$ such that

$$z_n\to z=0 \text{ in } A^\delta=\{|\arg z|<\frac{\pi}{2}-\delta\}:\delta>0, \text{ and } \delta^*(z_n,q)>\varepsilon_0>0.$$

Then we can find Ψ_0 such that $|\Psi_0|\leqq\frac{\pi}{2}-\delta$, $A(\Psi_0)=\{|\arg z-\Psi_0|<\frac{\delta}{12}\}$ has a subsequence $\{z'_n\}$ of $\{z_n\}$. For simplicity we denote by $\{z_n\}$ also. We can suppose $\Psi_0\geqq0$. Put $\theta=\frac{2\pi}{8n}$. Then we find an integer n such that

$$\theta\geqq\frac{\delta}{6},\ 5\theta+\frac{\delta}{12}<\delta. \tag{33}$$

Because $\frac{5}{11}(\frac{\pi}{\delta})<\text{n}<3(\frac{\pi}{6})$ and $(3-\frac{5}{11})(\frac{\delta}{6})>2$, by $0<\delta<\frac{\pi}{2}$. Let $A_i=$

$\{i\theta<\arg\ z<(i+1)\theta\}: i=0,\pm 1,\dots,\pm 2n$. Then by (33) there exists a number i_0 such that

Case A). $A_{i_0}\supset A(\Psi_0)$, $A_j\cap A(\Psi_0)=0: j\neq i_0$ and $-2n+5<i_0<2n-5$ or
Case B). $A_{i_0}+A_{i_0+1}+\{\arg\ z=i_0\theta\}\supset A(\Psi_0): -2n+4\leqq i_0\leqq 2n-4$.
$A_j\cap A_{\Psi_0}=0: j\neq i_0$ and $\neq i_0+1$.

At present we consider the case A). Let

$$A^+=\{(i_0+1)\theta<\arg\ z(i_0+2)\theta\},$$
$$A^-=\{(i_0-1)\theta<\arg\ z<i_0\theta\}.$$

Put $A=A^++A_{i_0}+A^-+\{\arg\ z=(i_0+1)\theta\}+\{\arg\ z=i_0\theta\}\subset\{|\arg\ z|<\frac{\pi}{2}-3\theta\}$. Let $E=C^\circ V^*(q)\cap A: C^\circ V^*(q)=\{z: \delta^*(z,q)>\varepsilon_0\}$. Let $\{z_n\}$ be a subsequence of $\{z_n\}$ in A_{i_0} and let E_n be a component of E) containing z_n. Then

Case 1. Any E_n does not tend to $z=0$ by 3).
Case 2. If the number of $\{E_n\}$ is finite, there exists at least a component E_0 containing a subsequence $\{z_m\}$. This implies E_0 tends to $z=0$. This also contradicts 3). Hence it is sufficient to consider the case
Case 3. There exist a sequence $\{E_m\}$.

Let $a_m=\sup|z|$ for $z\in E_m$. Then $\lim_m a_m=0$.

Assume $\overline{\lim_m}\ a_m>d>0$. Then there exists a sequence $\{E_m\}$ such that $E_m\ni z_m$: $E_m\cap\{|z|\geqq\frac{d}{2}, z\subset\bar{A}\}\neq 0$ and $z_m\to 0$. Put $\Gamma=\sum E_m+\{|z|>\frac{d}{2}, z\in\bar{A}\}$. Then Γ is a continum in $\bar{A}$ and tends to $z=0$. Let $\varepsilon_1=\min(\delta^*(z,q)): z\in\{|z|\geqq\frac{d}{2}, z\in\bar{A}\}$. Then $\varepsilon_1>0$. Hence

$$\Gamma\subset CV^*_{\varepsilon_2}(q): \varepsilon_2=\min(\varepsilon_0,\varepsilon_1).$$

This contradicts also 3). Hence $\lim_m a_m=0$. By Lemma 8, E_m is not compact in A. Hence E_m must intersect $\{\arg\ z=(i_0+2)\theta\}$ or $\{\arg\ z=(i_0-1)\theta\}$. Now $z_n\in A_{i_0}$, E_m intersects
1) $\{\arg\ z=(i_0\oplus 1)\theta\}$ and $\{\arg\ z=(i_0+2)\theta\}$ or
2) $\{\arg\ z=i_0\theta\}$ and $\{\arg=(i_0-1)\theta\}$. It is sufficient to consider the case 1.

In this case by $\lim_m a_m=0$, since E_m is a closed domain we can find a compact analytic curve Λ_m in E_m separating $z=0$ in A^+ from $|z|=1$ and satisfying the conditions of Lemma 3 and $\Lambda_m \to \{z=0\}$ as $m\to\infty$. By Lemma 6, there exists a positive const.ε_1 depending only n but Λ_m such that

$$U(\Lambda_m, \frac{1}{2}\Omega^{\Lambda}) > \varepsilon_1 \quad m=1, 2, \ldots$$

where Ω^{Λ} is a domain constructed from Λ_m in the manner of Lemma 6. 4). Now $\{V_n(q)\}$ and $\{V_n^*(q)\}$ are equivalent. There exist $V_m(q)$ such that $V_m(q) \subset V_n^*(q)$. $V_m(q) \overset{K}{\ni} q$ implies

$$0 = K_{CV_m(q)\cap\Delta}(z, q) \geqq K_{CV_n^*(q)\cap\Delta}(z, q). \tag{34}$$

Now $\lim_m a_m = 0$ means, for any $\Delta_l = \{z : \overset{M}{\mathrm{dist}}(z, \Delta) < \frac{1}{l}\}$, there exists m_0 such that $\Lambda_m \subset \Delta_l : m > m_0$. Hence

$$K_{CV_n^*(q)\cap\Delta_l}(z, q) \geqq K_{\Lambda_m\cap\Delta_l}(z, q) = K_{\Lambda_m}(z, q) \geqq U(\Lambda_m, \frac{1}{2}, \Omega^{\Lambda}) > \varepsilon_1 : \text{for}$$
$$z = \frac{1}{2}.$$

Let $m\to\infty$ and then $l\to\infty$. Then $K_{CV_n^*(q)\cap\Delta}(z, q) > 0$. This contradicts (34). Case B. Put $A^+ = \{(i_0+2)\theta < \arg z < (i_0+3)\theta\}$, $A^- = ((i_0-1)\theta < \arg z < i_0\theta\}$, $A' = \{i_0\theta \leqq \arg z \leqq (i_0+2)\theta\}$. Then we have the same contradiction. Thus we have Theorem 1.

§ 2. **Domain $\Omega^{\frac{1}{3}}$.** Let D be a domain $\{|z|<1, |\arg z| < \frac{3\Psi}{2}\}$. $\Psi \leqq \frac{2\pi}{3}$. Let F be a closed set in $\{|z|<1, |\arg z| \geqq \frac{\Psi}{2}\}$ such that $\Omega^{\frac{1}{3}} = D - F$ is a domain. We suppose Martin's topology introduced on $\bar{\Omega}^{\frac{1}{3}}$. We define the classes $\mathfrak{H}$ and $\mathfrak{H}^{\delta}$ as § 1. i. e. $\mathfrak{H}^{\delta} = K(z, p) : p \in \Delta$, $p_i \overset{M}{\to} p$, $p_i \in \{|\arg z| < \frac{\Psi}{2} - \delta\} : 0 < \delta < \frac{\Psi}{2}$. Then for $U(z) \in \mathfrak{H}$, $U(\hat{z}) \leqq U(z)$ and $G(\hat{z}, \xi) \leqq G(z, \xi)$ or $G^{\Lambda}(\hat{z}, \xi) \leqq G^{\Lambda}(z, \xi)$, where $G(z, \xi)$ or $G^{\Lambda}(z, \xi)$ are Green function of $\Omega_r = \{|z| > r\} \cap \Omega^{\frac{1}{3}}$ or Ω^{Λ} respectively and $\hat{z}$ is the symmetric point of z : $|\arg z| < \frac{\Psi}{2}$ with respect to $\arg z = \frac{\Psi}{2}$ or $= -\frac{\Psi}{2}$. The proofs of Theorem 1 depend on the observation of symmetric image of functions relative to some rays chosen suitably. The methods can be applied on this domain without any essential alteration but some trivial modefication. Hence we have

THEOREM 2. *There exists only one minimal function* $K(z, q)$ *in* $\mathfrak{H}$ *and* $\lim_n z_n=0$ *in* $\{|\arg z|<\frac{\Psi}{2}-\delta\}:\delta>0$ *implies* $z_n \overset{M}{\rightarrow} q$.

§ 3. Let F be a closed set in $|z|<1$ such that $\Omega=\{|z|<1\}-F$ is a domain and $\partial F\ni\{z=0\}$. We suppose an N-Martin topology[2] is defined over $\Omega+\Delta$, where Δ is the set of boundary points with respect to the N-Martin topology. We proved

THEOREM[3] 3. *If* F *is irregular at* $\{z=0\}$, *then* $\Delta(on\ z=0)$ *consists of only one point which is clearly* N*-minimal.*

Let A be a set of enumerably infinite number of analytic curves A_n clustering nowhere in $0<|z|<1$. *Put* $F=\sum(A_n+A_n^0)$, where A_n^0 is a domain bounded by A_n and A_n^0 may be empty. We suppose $\Omega=\{|z|<1\}-F$ is a domain. Let $\hat{\Omega}$ be the same leaf as Ω. Identity ∂F with $\partial\hat{F}$. Then we have a Riemann surface $\tilde{R}$ called the double of Ω. $\tilde{R}$ has a compact boundary $\Gamma+\hat{\Gamma}$ on $\{|z|=1\}$ and has one boundary component $\mathfrak{p}$. For a set E in Ω we denote by $\hat{E}$ the symmetric image with respect to ∂F. It is well known $\tilde{R}$ is an end of a Riemann surface $\tilde{R}^*\in O_g$.[4] Let $N(z, p):p\in\Omega$ be an N -Green function.
Then

$$N(z, p)=G(z, p)+G(z, \hat{p}),$$

where $G(z, p)$ is a Green function of $\tilde{R}$.

Rings and modified rings. Let J be a ring domain in Ω with two boundary components γ_1 and γ_2, where γ_2 separates $\{z=0\}$ from γ_1 and γ_1 separates γ_2 from $|z|=1$. Usually module $\mathfrak{M}(J)$ of J is given as

$$\mathfrak{M}(J)=D(U(z))=2\pi M,$$

where $U(z)=H_\phi^J:\phi=0$ on γ_1 and $=M$ on γ_2 and $\int_{\gamma_1}\frac{\partial}{\partial n}Uds=2\pi$.

Let J' be a ring in $\{0<|z|<1\}$ with two boundary components γ_1 and γ_2 as above. If $J=J'-F$ is a simply connected, we call J a modified ring and J' the primitive of J.
$\partial J=(\gamma_1+\gamma_2-F)+(\partial F\cap J')$. Module of J is given as

$$\mathfrak{M}(J)=D(V(z))=2\pi M,$$

where $V(z)$ is an HB in J such that $V(z)=0$ on $\gamma_1-F,=M$ on γ_2-F, $\int_{\gamma_1-F}\frac{\partial}{\partial n}V(z)ds=2\pi$ and $V(z)$ has M. D. I.(minimal Dirichlet integral). Let J

be a modified ring and J' be its primitive. Suppose $\mathfrak{M}(J')=2\pi M'$. Let $U'(z)$ be an HB in J' such that $U'(z)=0$ on $\gamma_1-F, =M'$ on γ_2-F and $U'(z)$ has M. D. I. Then $D_J(U')\leqq 2\pi M'$ and $\int_{\gamma_1-F}\frac{\partial}{\partial n}U'\,ds\leqq 2\pi$. This implies

$$\mathfrak{M}(J)\geqq\mathfrak{M}(J').$$

For the convenience, we define the primitive of a ring J in Ω by J in itself, i.e. $J=J'$.

Let J_1 and J_2 be rings or modefied ring and let J'_i be the primitives of J_i. If $J'_1\cap J'_2=0$ and J'_2 separates $z=0$ from J'_1, we denote $J_2<J_1$. Then

THEOREM 4. *Let Ω be a domain such that $\Omega=\{|z|<1\}-F$ as before. If there exists a sequence of rings or modified rings J_1, J_2, ... such that*

$$J_1>J_2>J_3\dots J_n\to\{z=0\}\ \text{as}\ n\to\infty,\ \text{and}\ \sum\mathfrak{M}(J'_n)=\infty.$$

Then $\Delta_1=\Delta=$ one point, where Δ and Δ_1 are set of boundary points and of N-minimal poists over $\{z=0\}$.

PROOF. Case 1. $\sum\mathfrak{M}(J'_n)=\infty$, *where the summation is over only modified rings. Then* $\tilde{\gamma}_n=(\gamma_{n,i}-F)+(\hat{\gamma}_n-\hat{F})$ is connected and $\tilde{J}_n=(J'_n-F)+(\hat{J}'_n-\hat{F})$ is an ordinaly ring with module $\frac{\mathfrak{M}(J_n)}{2}\left(\geqq\frac{\mathfrak{M}(J_n)}{2}\right)$ in $\tilde{R}$, where $\gamma_{n,i}(i=1,2,)$ are boundary of J'_n: J'_n is the primitive of J_n. Denote by $\tilde{R}_{n_i}$ the compact part of $\tilde{R}$ divided by $(\gamma_n-F)+(\hat{\gamma}_n-\hat{F})$. Then $\tilde{R}_n$ is an exhaustion of $\tilde{R}$. Then there exists a number n_0 such that $\tilde{R}_{n_0}\ni p$. Consider $G(z,p)$ in $\tilde{R}-\tilde{R}_{n_0}$. Then $G(z,p)$ is an HBD. Hence by M. Heins's theorem[5)]

$$G(z,p)\ \text{has limit as}\ z\to\mathfrak{p}\ \text{in}\ \tilde{R}. \tag{35}$$

Case 2. $\sum\mathfrak{M}(J_n)=\infty$, where the summation is only rings. We can suppose $\sum_{n=1}^{\infty}\mathfrak{M}(J_n)=\infty$, where J_n is a ring: $n=1,2,\dots$. In this case both J_n and $\hat{J}_n$ are rings in $\tilde{R}$. Let $\tilde{R}_n$ be the compact part of $\tilde{R}$ divided by $\gamma_n+\hat{\gamma}_n$(of J'_n and $\hat{J}'_n$). Then $\tilde{R}-\tilde{R}_n$ is an end and has a boundary component. Since $p\in\Omega$, there exists a number n_0 such that $R_{n_0}\ni p$ and $G(z,p)$ is an HBD in $\tilde{R}-\tilde{R}_{n_0}$. Attend to the closed set F in $|z|<1$. If F is irregular, our assertion is Theorem 3. In the following we suppose F is regular at $z=0$. Let Λ be a curve in Ω tending to $z=0$. Then $S(z)=G(z,p)-G(\hat{z},p)=0$ on ∂F by $\hat{z}=z$. $S(z)\to 0$ as $z\to\{z=0\}$ in Λ. Hence

$$\max_{z\in\Lambda\cap J_n}|S(z)|=\varepsilon_n:\ \varepsilon_n\downarrow 0\ \text{as}\ n\to\infty.$$

Let J be one of J_n and let $U(z)$ be the HB function defining $\mathfrak{M}(J)$. Let $A_\rho = \{z \in J : e^U = \rho\}$ and $\hat{A}_\rho$ be the symmetric image of A_ρ in $\hat{J}$. Put $\tilde{A}_\rho = A_\rho + \hat{A}_\rho$, $\tilde{L}_\rho = \int_{\tilde{A}\rho} |\frac{\partial}{\partial s} G(z, p)|\, ds.$ and

$L(\tilde{J}) = \min_{1 \leqq \rho \leqq e^{M(J)}} \tilde{L}_\rho$. Then by Schwarz's inequality

$$D_{J+\hat{J}}(G(z, p)) \geqq 2\tilde{L}(J)^2 \mathfrak{M}(J).$$

$$\infty > D_{\tilde{R}-\tilde{R}_{n_0}}(G(z, p)) \geqq 2\sum \tilde{L}^2(J_n)\mathfrak{M}(J_n)$$

implies there exists a sequence of $\tilde{A}_{\rho(n')}$ of $\tilde{J}_{n'}$ such that

$$\text{Os of } G(z, p) \text{ on } A_{\rho(n')} + \text{Os of } G(z, p) \text{ on } \hat{A}_{\rho(n')} = \delta_{n'} \downarrow 0 \text{ as } n' \to \infty,$$

where Os means the oscillation.
Since Λ and $\hat{\Lambda}$ intersects $A_{\rho(n')}$ and $\hat{A}_{\rho(n')}$ respectively,

$$\text{Os of } G(z, p) \text{ on } \tilde{A}_{\rho(n')} < \varepsilon_{n'} + \delta_{n'}.$$

Let $\tilde{R} - \tilde{R}_{n'}$, be the part of $\tilde{R} - \tilde{R}_{n'}$ divided by $\tilde{A}_{\rho(n')}$ which contains $\mathfrak{p}$. Since $\tilde{R}$ is an end of a Riemann surface $\in 0_g$,

$$\sup_{\tilde{R}-\tilde{R}_{n'}} G(z, p) - \inf_{\tilde{R}-\tilde{R}_{n'}} G(z, p) < \varepsilon_{n'} + \delta_{n'}.$$

Let $z \to \mathfrak{p}$ and $n' \to \infty$. Then

$$G(z, p) \text{ has limit as } z \to \mathfrak{p}. \tag{36}$$

By (35) and (36) $G(z, p)$ has limit as $z \to \mathfrak{p}$. Since p is arbitrary, $G(z, p)$. tends uniquely determined function $G(z, \mathfrak{p})$ as $p \to \mathfrak{p}$. i.e. $\mathfrak{p}$ is of harmonic dimension 1. Evidently $N(z, p) = G(z, p) + G(z, \hat{p}) \to 2G(z, \mathfrak{p})$ as $p \to z = 0$ and $G(z, \mathfrak{p})$ is N-minimal. Hence Δ over $\{z=0\} = \Delta_1$ over $\{z=0\}$ consists of only one point.

Applications. Map Ω conformally onto a domain in $|w|<1$ with radial slits such that $z=0 \to w=0$. Then the mapping function $w=f(z)$ must be-$\exp(U(z)+iV(z))e^{i\theta}$, where

$$U(z) = \int N(z, p)\, d\mu(p),$$

μ is a positive unit mass on Δ_1 (over $z=0$) and V is the conjugate of U.

COROLLARLY. *If there exists a sequence of rings or modified rings in* Ω *of Theorem* 4, *the mapping function is uniquely determined except rotations. As a special case.* $\Omega=\{|z|<1\}-\sum S_n$, *where* Ω *is a domain and* S_n *is a circular slit and* $S_n \to z=0$ *as* $n\to\infty$. *Then the mapping function* $\Omega\to a$ *domain with radial slits is uniquely determined except rotations.*

References

[1] R. S. MARTIN, Minimal positive harmonic functions. Trans. Amer. Math. Soc., 49 (1941).

[2] Z. KURAMOCHI, Potentials on Riemann surfaces. J. Fac. Sci. Hokkaido Univ., Ser. I., 16 (1962).

[3] Z. KURAMOCHI, Applications of quasi-Dirichlet bounded harmonic functions. Lecture note in Math., 1014, Springer (1981).

[4] Z. KURAMOCHI, On covering surfaces, Osaka Math. J. V (1953).

[5] M. HEINS, Riemann surfaces of infinite genus. Ann. Math. 55 (1952).

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