Hokkaido Mathematical Journal Vol. 22 (1993) p. 365-371

Higher dimensional semilinear parabolic problems

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(Received November 18, 1992, Revised May 6, 1993)

1. Introduction

There are two commonly used ways of approaching parabolic problems, i.e. "dynamic" semigroup technique (e.g. [AM], [WA]) and "static" a priori estimates method (e.g. [LA], DL]). In spite of the great power of "dynamical" approach (which leads to the general results for the possibly wide class of nonlinear problems), classical in the theory of partial differential equations "static" a priori technique can very often give exact in form and precise in assumptions existence-uniqueness theorems concerning regular solutions in the space of Hölder functions. Moreover dealing with the problem of local solvability one is also able to estimate (from below) the "life time" of the obtained solution in a way similar to the well known Peano theorem in the theory of ordinary differential equations.

However both in the "dynamic" and "static" studies of the classical solvability, growth of the space dimension n causes a feedback in the sense of increase (with respect to n) of the assumptions that should be put on the data of the considered problem. Since referring to needed assumptions as "sufficiently regular" makes the final result hardly applicable, we want in this note to deal with the case of higher space dimension coming back to the idea of our recent paper [CH], in which higher dimensional case was only mentioned in the Appendix.

We have presented in [CH] a classical approach to the $2m$-th ($m>1$) order initial-boundary value problem

$$(1)\qquad \begin{cases} u_t=-Pu+f(t,x,\ d^m u) & \text{in } D^T=(0,\ T)\times G \\ B_0 u=\ldots=B_{m-1}u=0 & \text{on } \partial G \\ u(0,\ x)=u_0(x) & \text{in } G \end{cases}$$

with $P=\sum_{|\alpha|,|\beta|\leq m}(-1)^{|\beta|}D^\beta(a_{\alpha,\beta}(x)D^\alpha)$, $d^m u=\left(u,\ \frac{\partial u}{\partial x_1},\cdots,\ \frac{\partial u}{\partial x_n},\ \frac{\partial^2 u}{\partial x_1^2},\cdots,\ \frac{\partial^m u}{\partial x_n^m}\right)$ and a bounded domain $G\subset R^n$ having smooth $C^{4m+\mu}$ boundary ∂G, where $\mu\in(0,1)$ is fixed from now on.

Partially supported by Silesian University under the grant "Dissipative Systems".

In this note we announce precise necessary assumptions and formulate the estimate-existence-uniqueness result in, outlined previously in the Appendix of [CH], higher dimensional case $n \geq 2m$ (especially we give the direct estimate of the "life time" of the solution). The exact stating of this result for higher dimensioal case is possible thanks to the more subtle use of the linear theory given in [LA, Chapt. VII, § 10 Th. 10.1, Th. 10.4], which plays an important role in the derivation of the estimate (4). In particular throughout this note we use both the notation and the general concept developed in [CH].

2 Assumptions.

Let for each pair of multi-indices $\alpha, \beta \in N^n$ with $|\alpha|, |\beta| \leq m$ the coefficient $a_{\alpha,\beta}$ of P belongs to the space $C^{2m+|\beta|+\mu}(clG)$, the coefficients of the boundary operators B_j $(j=0, \ldots m-1)$ are of the class (resp). $C^{4m-m_j+\mu}(\partial G)$ and nonlinear scalar function $f=f(t, x, p_1, \ldots, p_d)$ (d denotes the length of $d^m u$) is differentiable with respect to t and has all partial derivatives up to the order $2m$ with respect to $x, p_1, \ldots, p_d$. Moreover, let both f and all its partial derivatives which were just mentioned satisfy local Lipschitz condition with respect to t and functional arguments $p_1, \ldots, p_d$ and Hölder condition with respect to the space variable x (further, for simplicity, L will denote the common Lipschitz constant for the considered functions, relatively to a fixed compact subset of $[0, T] \times clG \times R^d$). We also assume the conditions E, F stated in [CH] (i.e. P is coercive, satisfies Green's Identity and the triple $(P, \{B_j\}, G)$ forms the regular elliptic boundary value problem) and claim that the initial function u_0 satisfies (necessary for the existence of Hölder solution) first order compatibility conditions according to the monograph [LA].

Then in the case of higher space dimension $n \geq 2m$ we have the following result:

3 The main result.

THEOREM. *For arbitrarily fixed $K>0$ there exist positive constants $\bar{C}_1, \bar{C}_2, \bar{C}_3, \bar{C}_4, \bar{C}_5$ such that for any hypothetical, classical solution u of the problem* (1), *as long as*

(2) $$\sum_{|\alpha| \leq m} \|D^\alpha (u-u_0)(t, \cdot)\|_{L^\infty(G)} \leq K$$

(*considering $t \in [0, T]$, $x \in G$*) *the following a priori estimates hold*:

(3) $$\|(u-u_0)(t, \cdot)\|_{0,2}^2 \leq \bar{C}_1 t e^{\bar{C}_2 t},$$

$$\|u_t(t,\cdot)\|_{0,\frac{n}{m-1}} \leq \bar{C}_3 t^{1-\frac{m-1}{n}} + \|f(0,\cdot,d^m u_0) - Pu_0\|_{0,\frac{n}{m-1}}. \tag{4}$$

Moreover inequalities (3), (4) *guarantee validity of our main a priori estimate*

$$\|u-u_0\|_{L^\infty(0,t;W^{m,\infty}(G))} \leq \bar{C}_4(t) \qquad \left(\bar{C}_4(0)=\frac{K}{2} \text{ and } \bar{C}_4(t)\nearrow\infty\right), \tag{5}$$

which, in turn, implies that (2) *holds until the time* T_0 *given by*

$$T_0=\min\{T_1, T\}. \tag{6}$$

Here $T_1>0$ *is given as the unique solution of the equation*

$$\begin{aligned}\nu_1 \bar{C}_3 T_1^{1-\frac{m-1}{n}} &+ \nu_1 \|f(0,\cdot,d^m u_0) - Pu_0\|_{0,\frac{n}{m-1}} \\ &+ \nu_1 |G|^{\frac{m-1}{n}} \sup_{clD^T}\{|f(t,x,d^m u_0) - Pu_0|\} + C_{\nu_1} \bar{C}_1^{\frac{1}{2}} T_1^{\frac{1}{2}} e^{\frac{\bar{C}_2}{2}T_1} = K\end{aligned} \tag{7}$$

with ν_1 *and* C_{ν_1} *specified in* (*resp.*) (24) (18). *In consequence, the preceding week norms estimates* (3)-(5) *lead to the condition*

$$\langle u-u_0\rangle_{C^{2m+\mu,1+\frac{\mu}{2m}}(clD^{T_0})} \leq \bar{C}_5, \tag{8}$$

being sufficient to establish solvability of the problem (1) *in the Hölder space* $C^{2m+\mu,1+\frac{\mu}{2m}}(clD^{T_0})$. *Furthermore, the* $C^{2m+\mu,1+\frac{\mu}{2m}}(clD^{T_0})$ *solution is unique.*

PROOF OF THE THEOREM. According to the notation of [CH] we put $v:=u-u_0$ and $g(t,x,d^m v):=f(t,x,d^m v+d^m u_0)-Pu_0$. Thus v satisfies:

$$\begin{cases} v_t=-Pv+g(t,x,d^m v) & \text{in } D^T \\ B_0 v=\ldots=B_{m-1}v=0 & \text{on } \partial G \\ v(0,x)=0 & \text{in } G \end{cases} \tag{9}$$

At the beginning we shall prove that as long as (2) holds, for all $\nu\in(0,\nu_0]$ (ν_0 given by (17)) the following (compare [DL], [CH]) "flexible" estimate is valid:

$$\begin{aligned}\|v(t,\cdot)\|_{m,\infty} \leq \nu\Big(\|v_t(t,\cdot)\|_{0,\frac{n}{m-1}} &+ |G|^{\frac{m-1}{n}} \sup_{clD^T}\{|g(t,x,d^m 0)|\}\Big) \\ &+ C_\nu \|v(t,\cdot)\|_{0,2},\end{aligned} \tag{10}$$

where $C_\nu\nearrow\infty$ as $\nu\searrow 0$. First we use Sobolev Embeddings and Nirenberg-Gagliardo inequalities to find that (α fixed with $|\alpha|\leq m$)

$$(11)\qquad \|D^\alpha v\|_{0,\infty}\le C_1\|D^\alpha v\|_{m-1,\frac{n+\frac{1}{2}}{m-1}}\le C_1C_4\|D^\alpha v\|_{m,\frac{n}{m-1}}^{\frac{n}{n+\frac{1}{2}}}\|D^\alpha v\|_{0,2}^{\frac{\frac{1}{2}}{n+\frac{1}{2}}}.$$

We continue further with the Young's inequality and interpolation inequality for the intermediate derivatives ([AD, Th. 4. 14] with $\varepsilon=\frac{\delta}{C'_\delta}\le\varepsilon_0$) until we get

$$(12)\qquad \|D^\alpha v\|_{0,\infty}\le\delta\|D^\alpha v\|_{m,\frac{n}{m-1}}+C'_\delta\|D^\alpha v\|_{0,2}\le\delta\|v\|_{2m,\frac{n}{m-1}}+C'_\delta\|D^\alpha v\|_{0,2}$$
$$\le\delta\|v\|_{2m,\frac{n}{m-1}}+C'_\delta\left(\frac{\delta}{C'_\delta}\|v\|_{2m,2}+C_0^{\frac{2m}{2m-|\alpha|}}\left(\frac{C'_\delta}{\delta}\right)^{\frac{|\alpha|}{2m-|\alpha|}}\|v\|_{0,2}\right),$$

where $0<\delta\le\delta_0=\frac{1}{2n+1}C_1C_4\varepsilon_0^{\frac{1}{2n+1}}(2n)^{\frac{2n}{2n+1}}$, $C'_\delta=\frac{1}{2n+1}(C_1C_4)^{2n+1}\left(\frac{(2n+1)\delta}{2n}\right)^{-2n}$. Applying evident inequality $\|v\|_{2m,2}\le d_1^{\frac{n-2m+2}{2n}}|G|^{\frac{n-2m+2}{2n}}\|v\|_{2m,\frac{n}{m-1}}$, $\left(\text{here } d_1=\frac{(2m+n)!}{(2m)!n!}\right)$ and Calderon-Zygmund estimate, we increase the right side of (12) coming to the condition

$$(13)\qquad \|D^\alpha v\|_{0,\infty}\le C_5\delta\left(1+d_1^{\frac{n-2m+2}{2n}}|G|^{\frac{n-2m+2}{2n}}\right)(\|Pv\|_{0,\frac{n}{m-1}}+\|v\|_{0,\frac{n}{m-1}})+C_{\delta,\alpha}\|v\|_{0,2},$$

where $C_{\delta,\alpha}=C'_\delta C_0^{\frac{2m}{2m-|\alpha|}}\left(\frac{C'_\delta}{\delta}\right)^{\frac{|\alpha|}{2m-|\alpha|}}$. Now, similarly as it was done in [CH, Lem. 1, conditions (18)-(21)], we find required estimate of the $L^{\frac{n}{m-1}}$ norm of Pv:

$$(14)\qquad \|Pv\|_{0,\frac{n}{m-1}}\le\|v_t\|_{0,\frac{n}{m-1}}+\|g(t,\cdot,d^m0)\|_{0,\frac{n}{m-1}}+Ld^{1-\frac{m-1}{n}}\|v\|_{m,\frac{n}{m-1}}.$$

Since $\|v\|_{0,\frac{n}{m-1}}\le\|v\|_{m,\frac{n}{m-1}}\le|G|^{\frac{m-1}{n}}\|v\|_{m,\infty}$, thus collecting the inequalities (13), (14) and defining constants $\overline{C}_6=|G|^{\frac{m-1}{n}}\sup_{(t,x)\in clD^T}\{|g(t,x,d^m0)|\}$, $\overline{C}_7=(d_1|G|)^{\frac{n-2m+2}{2n}}$ we obtain that

$$(15)\qquad \|D^\alpha v\|_{0,\infty}\le C_5(Ld^{1-\frac{m-1}{n}}+1)|G|^{\frac{m-1}{n}}\delta(1+\overline{C}_7)\|v\|_{m,\infty}$$
$$+C_5\delta(1+\overline{C}_7)(\|v_t\|_{0,\frac{n}{m-1}}+\overline{C}_6)+C_{\delta,\alpha}\|v\|_{0,2}.$$

Summing both sides of (15) with respect to α with $|\alpha|\le m$, for $C_\delta=\sum_{|\alpha|\le m}C_{\delta,\alpha}$ and any positive δ satisfying

$$\delta \leq \min\left\{\frac{1}{2n+1} C_1 C_4 \varepsilon_0^{\frac{1}{2n+1}} (2n)^{\frac{2n}{2n+1}}, \right.$$
$$\left. \frac{1}{2dC_5(Ld^{1-\frac{m-1}{n}}+1)|G|^{\frac{m-1}{n}}(1+\bar{C}_7)}\right\},$$

we get the estimate

$$\|v\|_{m,\infty} \leq \frac{1}{2}\|v\|_{m,\infty} + dC_5\delta(1+\bar{C}_7)(\|v_t\|_{0,\frac{n}{m-1}} + \bar{C}_6) + C_\delta \|v\|_{0,2}. \tag{16}$$

Substituting $\nu := 2dC_5\delta(1+\bar{C}_7)$ in (16) we come immediately to (10) with

$$\nu_0 = \min\left\{2dC_5\delta_0(1+\bar{C}_7), \frac{1}{(Ld^{1-\frac{m-1}{n}}+1)|G|^{\frac{m-1}{n}}}\right\} \tag{17}$$

and

$$C_\nu = 2\sum_{|\alpha|\leq m} C_0^{\frac{2m}{2m-|\alpha|}} \left(\frac{1}{2n+1}\left(\frac{2n+1}{2n}\right)^{-2n} (C_1C_4)^{2n+1}\right)^{\frac{2m}{2m-|\alpha|}} \times$$
$$\times (2dC_5(1+\bar{C}_7))^{\frac{4mn+|\alpha|}{2m-|\alpha|}} \nu^{-\frac{4mn+|\alpha|}{2m-|\alpha|}}. \tag{18}$$

Condition (10) is thus proved. For the proof of a priori estimate (3) we refer to [CH, Lem. 2]. We proceed now to justification of (4). First, let us note that v_t is a classical solution of

$$\begin{cases} z_t = (-P + \sum_{|\alpha|\leq m} g_\alpha D^\alpha) z + g_t \\ B_0 z = \ldots = B_{m-1} z = 0 \\ z(0, x) = g_0(x) \end{cases} \tag{19}$$

were $g_t(t, x) = \frac{\partial g}{\partial t}(t, x, d^m v(t, x))$, $g_\alpha(t, x) = \frac{\partial g}{\partial (D^\alpha{}_u)}(t, x, d^m v(t, x))$ and $g_0(x) = g(0, x, d^m 0)$. Furthermore, as long as (2) holds, as a result of continuity we have

$$\|g_t(t, \cdot)\|_{0,\infty} \leq M, \qquad \|g_\alpha(t, \cdot)\|_{0,\infty} \leq M \quad |\alpha| \leq m.$$

Let us denote by $T_{\max}$ ($T_{\max} \in (0, T]$) hypothetical, maximal time until which (2) holds. Then from the linear theory stated in [LA, Chapt. VII, §10, Th. 10.4 with $t=2m, 1=0, s=0$] (using also notation of [LA]) we obtain that

$$\|v_t\|_{W^{2m,1}_{\frac{n}{m-1}}(D^{T_{\max}})} \leq c\left(MT^{\frac{m-1}{n}}|G|^{\frac{m-1}{n}} + \|g_0\|_{B^{2m-\frac{2m(m-1)}{n}}_{\frac{n}{m-1}}(G)}\right) =: \bar{C}_3. \tag{20}$$

Thus because of (20), as long as inequality (2) is valid, we obtain in particular:

$$(21)\qquad \left(\int_0^t\int_G |v_{tt}(\tau, x)|^{\frac{n}{m-1}}dx\,d\tau\right)^{\frac{m-1}{n}} \leq \bar{C}_3.$$

Applying next Newton Integral Formula, together with Hölder and (generalized) Minkowski's inequalities we find the estimate

$$(22)\qquad \|v_t(t,\cdot)-v_t(0,\cdot)\|_{0,\frac{n}{m-1}}=\left(\int_G |v_t(t,x)-v_t(0,x)|^{\frac{n}{m-1}}dx\right)^{\frac{m-1}{n}}$$
$$=\left(\int_G\left|\int_0^t v_{tt}(\tau,x)d\tau\right|^{\frac{n}{m-1}}dx\right)^{\frac{m-1}{n}} \leq \int_0^t\left(\int_G |v_{tt}(\tau,x)|^{\frac{n}{m-1}}dx\right)^{\frac{m-1}{n}}d\tau$$
$$\leq t^{1-\frac{m-1}{n}}\left(\int_0^t\int_G |v_{tt}(\tau,x)|^{\frac{n}{m-1}}dx\,d\tau\right)^{\frac{m-1}{n}},$$

which because of (21) leads directly to (4). Collecting now estimates (10), (3) and (4) we come to the condition

$$(23)\qquad \|v(t,\cdot)\|_{m,\infty}\leq \nu(\bar{C}_3 t^{1-\frac{m-1}{n}}+\|g_0\|_{0,\frac{n}{m-1}}+\bar{C}_6)+C_\nu \bar{C}_1^{\frac{1}{2}} t^{\frac{1}{2}} e^{\frac{\bar{C}_2}{2}t}.$$

From (23), choosing $\nu=\nu_1$ given by

$$(24)\qquad \nu_1:=\min\left\{\nu_0, \frac{1}{2}K(\|g_0\|_{0,\frac{n}{m-1}}+\bar{C}_6)^{-1}\right\}$$

and using the same argumentation as in [CH, Lem. 4 formulas (33)-(34)] we obtain immediately (5), verifying also (2) together with the "life time" T_0 (6) and the condition (7) determining auxiliary time value T_1.

Having justified precisely that v belongs to $L^\infty(0, T_0; W^{m,\infty}(G))$, we refer to [CH] for the rest of the proof. Using the argumentation of [CH, Appendix, conditions (65)-(66)] we get

$$(25)\qquad D^\alpha v\in C^{\mu,\mu}(clD^{T_0})$$

and next, exactly as it was done in [CH, Lem. 4, formulas (41)-(42)], we get the $C^{2m+\mu,1+\frac{\mu}{2m}}(clD^{T_0})$ estimate claimed in (8). Existence of the solution for (1) can be derived now by standard "method of continuity" with the use of the Leray-Schauder Principle; then also, as there has been shown in [CH, Sec, 2. 1], the solution is unique.

Because $C^{2m,1}(clD^{T_0})$ somoothness of v_t was used in the proof, we add for the completeness verification of the assumed regularity (based on linear theory). Since $g\in C^{m+\mu,\frac{m+\mu}{2m}}(clD^{T_0})$ then from [LA, Chapt. VII, §10, Th. 10.1] we have that $v\in C^{3m+\mu,1+\frac{m+\mu}{2m}}(clD^{T_0})$. Next we find that $g\in C^{2m+\mu,\frac{2m+\mu}{2m}}(clD^{T_0})$ and hence $v\in C^{4m+\mu,2+\frac{\mu}{2m}}(clD^{T_0})$. Thus we conclude finally that solution u of the problem (1) belongs to the Hölder space $C^{4m+\mu,2+\frac{\mu}{2m}}(clD^{T_0})$.

Our considerations are completed.

THE FINAL REMARK. Let us note that given in (10) stronger version of proved previously in [CH, Lem. 1] inequality (15) allows to estimate in a flexible manner $W^{m,\infty}$ norm of the solution, and finally establish (2) for certain positive time $T_0 \leq T$. Moreover, thanks to "flexibility" of (10), although the assumptions needed in higher dimensional case $n \geq 2m$ are stronger than stated in [CH] (sufficient for $n < 2m$) conditions (A)-(F), they do not growth any more for larger value of n. Whereas for the existence of the classical solution in [FR] or [AM] the conditions imposed on f must have been getting stronger and stronger, relatively to the growth of the space dimension n and sometimes they also have not been given explicitly (as in [FR, Th.19, Sec. 7, Chapt. 10]).

Acknowledgement. The author is very grateful to the Referee for the remarks improving the manuscript of the paper.

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