Geometric structures on filtered manifolds

Tohru MORIMOTO

Introduction

Geometry deals with spaces and structures. In differential geometry, the spaces are usually differential manifolds and the structures are usually defined on them in terms of differential quantities and called geometric structures of order k if the defining quantities involve only derivatives of order up to k. The general equivalence problem is to find criteria to decide whether or not two geometric structures are (locally) equivalent. It is to this problem that the present work is devoted.

Let us first briefly mention the background. The general equivalence problem has been studied by many geometers since S. Lie. In particular, É. Cartan, in his study of infinite groups [1], invented a general method to treat the equivalence problem on the basis of the method of moving frames and the theory of Pfaff systems in involution, and found important applications in various domains of his work. However, his method was rather of the nature of a general heuristic principle not settled in precise mathematical concepts.

As was brought to light by C. Ehresmann and others, one of the fundamental concepts underlying his method is that of principal fibre bundle and G-structure. The extensive works which followed, in particular, I. M. Singer - S. Sternberg [21] and S. Sternberg [22], gave a rigorous foundation to deal with the general equivalence problem as that of G-structures and clarified important aspects of Cartan's ideas.

But the theory of G-structures as achieved there did not seem adequate to treat the equivalence problem in full generality: Even if one confines oneself to the equivalence problem of G-structures (the first order geometric structures), one has to deal with higher order geometric structures in a way suitable to find the higher order invariants of G-structures, and moreover it is necessary to develop a theory including the intransitive stuctures.

In answer to this, we developed in [14] a general scheme to treat the equivalence problem on the basis of the higher order "non-commutative" frame bundles, and gave a method to solve the general equivalence problem in a neighbourhood of every generic point in the analytic category.

On the other hand, in applications to various geometric problems the general method of G-structures is not always effective. For instance, the deep work of Cartan on *les systèmes de Pfaff à cinq variables* [2] is far from being well understood merely by the usual approach of G-structures. Here he elaborated a more refined method fitting in with the structures considered: a method of reduction by using Pfaff systems and of constructing what is now called a Cartan connection.

In his series of papers (in particular, [24], [26]), N. Tanaka developed this aspect extensively as the geometry of differential systems, and found various applications, especially in CR geometry [25] and in the geometric study of ordinary differential equations [27].

The present work, growing from the question what relation there is between the Tanaka theory and our general method developed in [14], aims to give a unified view on the equivalence problem by integrating the two methods.

The starting point of Tanaka's approach is to replace the usual underlying manifold, on which geometric structures are considered, by a manifold equipped with a regular differential system [24]. We take also this starting point, but a little generally we introduce the notion of a filtered manifold: It is a differentiable manifold M equipped with a tangential filtration $F=\{F^p\}_{p\in Z}$, where F^p (or denoted by $F^p TM$) is a sequence of subbundles of the tangent bundle TM of M satisfying: (i) $F^p \supset F^{p+1}$, (ii) $F^0=0$, $\cup F^p = TM$, and (iii) $[\underline{F}^p, \underline{F}^q] \subset \underline{F}^{p+q}$ for all $p, q \in Z$, $\underline{F}^{\cdot}$ denoting the sheaf of sectios of $F^{\cdot}$. Notice that a manifold equipped with a regular differential system gives rise to a filtered manifold (M, F) such that its filtration is generated by F^{-1}, that is, $\underline{F}^{p-1}=[\underline{F}^{-1}, \underline{F}^p]+\underline{F}^p$ for all $p<0$. Notice also that any manifold may be regarded as a filtered manifold endowed with the trivial filtration $F_{tr}^{-1} TM = TM$.

In this paper it is on filtered manifolds that we shall deal with geometric structures. The main goal is to construct a general scheme to treat the equivalence problem of geometric structures in a way well adapted to the underlying filtered manifolds.

The costruction is based on two ideas (or two reflections on the differentiation in differential geometry). The first basic idea, stemming from Tanaka [24], may be called in our terminology the utilization of weighted ordering in differentiation. For a differential operator on a

filtered manifold (M, F), it is often more natural and better to use a weighted ordering induced from the tangential filtration F rather than the usual ordering: The weighted order is defined for a vector field X to be $\leq k$ if X is a section of $F^{-k}TM$ and is extended to any differential operator in the obvious manner. We shall use this weighted ordering in a geometric setting. Thus for a filtered manifold (M, F), the usual rôle of the tangent bundle will be played by the bundle of nilpotent graded Lie algebras $grTM = \oplus F^p TM / F^{p+1}TM$. This point of view has an effect of changing the notions of frame bundle, G-structure and higher order geometric structure. All are considered with respect to the weighted ordering, which has an advantage in providing us with more refined tools than usual.

The second basic idea, stemming from our previous work [14], is the introduction of the notion of tower. This notion is an abstraction of certain properties that the homogeneous spaces possess. Recall that the local nature of a homogeneous space L/G is completely characterized as follows: First $L \to L/G$ is a principal G-bundle, secondly there is given an absolute parallelism on L defined by an $\mathfrak{l}$-valued 1-form θ (the Maurer-Cartan form) which satisfies: (i) $R_a^* \theta = Ad(a)^{-1}\theta$ for $a \in G$, (ii) $\theta(\tilde{A}) = A$ for $A \in \mathfrak{g}$, and (iii) the structure function γ, defined by $d\theta + \frac{1}{2}\gamma(\theta, \theta) = 0$, is constant (indeed the bracket operation of the Lie algebra $\mathfrak{l}$ of L).

To treat in this spirit not only the homogeneous structures but also the inhomogeneous structures, we consider a principal fibre bundle with an absolute parallelism with less restrictive conditions. The first crucial generalization that we make is to liberate $\mathfrak{l}$ (the space in which θ takes values) from being a Lie algebra. We therefore consider, in stead of $(\mathfrak{l}, G, Ad)$, a triple (E, G, ρ), where G is a Lie group, E a vector space containing the Lie algebra $\mathfrak{g}$ of G, and ρ the representation of G on E such that $\rho(a)x = Ad(a)x$ for $a \in G$, $x \in \mathfrak{g}$. Then we have a filtration F_{tr} of G defined by the exact sequence:

$$1 \longrightarrow F_{tr}^{p+1}G \longrightarrow G \longrightarrow GL(E/F_{tr}^p\mathfrak{g})$$

with $F_{tr}^0 G = G$ (see § 2.2). In the case of a homogeneous space this is the filtration induced by the Tayler expansion of the actions of G at the origin of L/G. We say that the triple (E, G, ρ) is a skeleton on $V = E/\mathfrak{g}$ if it is formally effective, that is, $\cap F_{tr}^p G = \{e\}$. For reasons understood later on, we shall fix an identification: $E = V \oplus \mathfrak{g}$.

Now we say that a principal fibre bundle $P(M, G)$ provided with an absolute parallelism θ on P is a tower on M with skeleton (E, G, ρ) if θ takes values in E and satisfies: (i) $R_a^* \theta = \rho(a)^{-1}\theta$ for $a \in G$, (ii) $\theta(\tilde{A}) = A$

for $A \in \mathfrak{g}$. Here we make the second important generalization that G can be infinite-dimensional (a projective limit of finite-dimensional Lie groups).

Generalized in this way, the category of towers has the following remarkable properties: First every geometric structure can be represented as a tower P or as a truncation of tower $P/F_{tr}^{k+1}G$. Secondly, for every manifold M there is a universal tower $\mathscr{R}(M)$ in which every tower on M is canonically embedded. (See § 2.3 for further nice functiorial properties.)

We remark that $\mathscr{R}(M)$ has a system of local coordinates $(x_{j_1,\cdots,j_m}^{i})$ with $1 \leq i, j_1, \cdots, j_m \leq \dim M$, $m = 0, 1, 2, \cdots$, (the introduction of new variables which stand for the higher order derivatives, but without any commuting relation), while the usual infinite order frame bundle $\mathscr{F}(M)$ is embedded in $\mathscr{R}(M)$ by the equation $x_{j_{\sigma(1)},\cdots,j_{\sigma(m)}}^{i} = x_{j_1,\cdots,j_m}^{i}$ for all permutations σ. This is the reason why $\mathscr{R}(M)$ is called the non-commutative frame bundle of M.

Now let us turn our attention to a filtered manifold $M = (M, F)$. A tower (P, M, G, θ) with skeleton (E, G, ρ) is called a tower on the filtered manifold M if there is a filtration $\{F^p V\}$ of V invariant under the linear isotropy representation of G on V and if θ preserves the filtrations, namely $\pi_* \circ \theta^{-1}(F^p V) = F^p TM$, where π denotes the projection $P \to M$. There is also a universal tower $\mathscr{R}(M)$ on M in which every tower on M is canonically embedded. And any geometric structure on a filtered manifold M is represented as a tower on M or as its truncation.

If a skeleton (E, G, ρ) leaves invariant a filtration $\{F^p V\}$ we can introduce another filtratin $\{F^p G\}$ of G by the exact sequence (see § 2.2):

$$1 \longrightarrow F^{p+1}G \longrightarrow G \longrightarrow GL(E/F^p\mathfrak{g})/F^{p+1}GL(E/F^p\mathfrak{g}).$$

This is the filtration according to the weighted ordering mentioned above, and in studying the towers on filtered manifolds we let it play the rôle that the Taylor filtration F_{tr} does when the tangential filtration is trivial.

In this way we have made our foundation to study the geometric structures on filtered manifolds. Now to treat the equivalence problem, our general principle may be expressed as follows:

Given a geometric structure, first represent it as a truncation of a tower. Then reduce this tower to obtain a smaller one which has as nice structure as possible. The invariants of the latter will yield those of the original geometric structure.

Of course a geometric structure may, in general, much deviate from a homogeneous space. We shall make clear through our point of view what structures can be regarded as nice approximations of homogeneous spaces

and under what conditions the given geometric structure can be reduced to such nice ones.

The towers that have constant structure functions constitute an important class of towers that have the simplest structures. A tower P belonging to this class is an analogue of a homogeneous space (indeed, as explained earlier on, a local homogeneous space if $\dim P < \infty$). Of interest is thus the infinite-dimensional case. We shall clarify how such a tower can be determined from its truncated structure $P/F^k G$. In order this we introduce the important notion of involutivity for truncated towers on filtered manifolds by generalizing the notion of involutive C-bundle [14] and by using the generalized Spencer cohomology group. Since we are working through the weighted ordering, this notion of involutivity is, so to speak, the weighted version of the usual one.

It is nowadays a common philosophy which goes back to Cartan that the three objects on a manifold; Lie transformation groups, geometric structures, and differential equations, relate to each other intimately through the notion of involutivity. Our emphasis is that this trinity also holds if we replace the underlying manifold by a filtered manifold, and the notion of involutivity by the weighted version: In [15] we have determined the structure of a transitive filtered Lie algebra of depth μ (the infinitesimal object of a Lie pseudo-group acting on a filtered manifold) from involutive truncated filtered Lie algebras, which may be regarded as an algebraic aspect of the present geometric study. In [20] we study the formal integrability of differential equations on a filtered manifold by introducing the notion of weighted jet bundle, in which we see more explicitly the significance of the involutivity with respect to the weighted ordering.

As another achievement of our method, we obtain a general criterion to construct a Cartan connection associated with a geometric structure.

If one recalls the definition of a Cartan connection (see § 3.10), one will notice immediately that a bundle with Cartan connection is nothing but a tower of which the size of the total space is *tight* (or more precisely, the space E in which θ takes values has a Lie algebra structure). Thus the problem of constructing a Cartan connection for a given geometric structure reduces to the problem of reducing a tower to a tight one. This point of view leads us to a general criterion and a unified method to construct Cartan connections, which generalizes all the results hitherto known on the existence of Cartan connections (Riemannian, conformal, and projective structures (cf. [9]), more generally the method of Tanaka constructing Cartan connections for certain geometric structures associated

with simple graded Lie algebras [26]).

A summary of contents now follows: In Chapter I we define a filtered manifold, and introduce the first order geometric structures on filtered manifold, a generalization of the G-structures.

Chapter II introduces the notion of tower which plays a central rôle in this paper, and constructs the universal tower $\mathscr{R}(M)$. In the case when the filtration is trivial, the truncation $\mathscr{R}^{(k)}(M)$ of $\mathscr{R}(M)$ coincides with what is called the non-commutative frame bundle of order $k+1$ in our previous paper [14]. Here we prefer to go to the infinite order since it much facilitates to understand the group theoretical natures of the structures and the fundamental properties of the structure functions.

Chapter III discusses the general equivalence problem. Of particular importance is the notion of involutivity. By virtue of this we are able to find all the invariants of a given geometric structure by a finite number of steps and able to solve the equivalence problem under the assumption of analyticity, provided that the structure is transitive. It gives a method to calculate the invariants of the structure in a way well-adapted to the underlying filtered structure, as illustrated by an example taken from Monge-Ampère equations. We also discuss how to treat intransitive structures, Finally a general criterion and method to construct Cartan connections is given. The Cartan connection of a conformal structure is explained in illustration of the general construction.

The main results of this paper were announced in [16] and [17].

Contents

Introduction

Chapter I. Filtered manifolds

1. 1. Definition of a filtered manifold.

A tangential filtration F on a differentiable manifold M is a sequence $\{F^p\}_{p \in Z}$ of subbundles of the tangent bundle TM of M such that the following conditions are satisfied:

i) $F^p \supset F^{p+1}$,

ii) $F^0 = 0$, $\cup_{p \in Z} F^p = TM$,

iii) $[\underline{F}^p, \underline{F}^q] \subset \underline{F}^{p+q}$, for all $p, q \in Z$,

where $\underline{F}^p$ denotes the sheaf of the germs of sections of F^p.

A filtered manifold is a differentiable manifold M equipped with a tangential filtration F. We shall often denote by the bold letter $\boldsymbol{M}$ the filtered manifold (M, F) and by $\{T^p\boldsymbol{M}\}$ or $\{F^p TM\}$ its tangential filtration.

An isomorphism of a filtered manifold $\boldsymbol{M}$ onto a filtered manifold $\boldsymbol{M}'$ is a diffeomorphism $\varphi : M \to M'$ such that $\varphi_* T^p\boldsymbol{M} = T^p\boldsymbol{M}'$ for all $p \in Z$, where φ_* denotes the differential of φ.

Let $\boldsymbol{M}$ be a filtered manifold. By definition there is an integer $\mu \geq 0$ such that $T^{-\mu}\boldsymbol{M} = TM$. The minimum of such integers is called the depth of $\boldsymbol{M}$.

Before proceeding, let us give some examples.

1) Trivial filtration. A differentiable manifold M itself may be regarded as a filtered manifold equipped with the trivial filtration defined by $F_{tr}^p TM = TM$ for $p < 0$ and $F_{tr}^q TM = 0$ for $q \geq 0$.

2) Tangential filtration derived from a regular differential system [24]. Let D be a differential system on a differentiable manifold M, that is, a

subbundle of the tangent bundle of M. Then there is associated a sequence of subsheaves $\{\mathscr{D}^p\}_{p<0}$ of $\underline{TM}$, called the derived systems of D, which is defined inductively by:

$$\begin{cases} \mathscr{D}^{-1}=\underline{D}, \\ \mathscr{D}^{p-1}=\mathscr{D}^p+[\mathscr{D}^p, \mathscr{D}^{-1}] \end{cases} \quad (p<0).$$

It then holds that:

$$[\mathscr{D}^p, \mathscr{D}^q] \subset \mathscr{D}^{p+q} \quad \text{for} \quad p, q<0.$$

Now suppose that the derived systems $\mathscr{D}^p$ are all vector bundles, that is, there are subbundles $D^p \subset TM$ such that $\underline{D}^p=\mathscr{D}^p$ for all $p<0$ (in this case the differential system D is called regular [24]). Then there exists a minimum integer $\mu \geq 1$ such that $D^p=D^{-\mu}$ for all $p \leq -\mu$. Setting

$$F^p TM = \begin{cases} 0 & (p \geq 0) \\ D^p & (-1 \geq p \geq -\mu) \\ TM & (p \leq \mu-1), \end{cases}$$

we have a filtered manifold $M=(M, F)$ derived from the regular differential system D.

There are two cases to distinguish. If $D^{-\mu} \subsetneqq TM$, then $D^{-\mu}$ is completely integrable and defines a foliation on M. In particular, if D is completely integrable the filtered manifold M is nothing but a foliated manifold. If $D^{-\mu}=TM$, we say that the tangential filtration F is generated by differential system D.

If a filtered manifold M (or M') is derived from a differential system D on M (resp. D' on M'), then M and M' are isomorphic if and only if (M, D) and (M', D') are isomorphic, that is, there is a diffeomorphism φ: $M \to M'$ such that $\varphi_* D=D'$.

3) Higher order contact manifold (cf. [29]). Let $\pi: M \to N$ be a fibred manifold. Let $J^k(M, N)$ be the bundle of k-jets of cross-sections of π. On this jet bundle we have a sequence of canonical differential systems $\{D^p\}$ called the higher order contact structure. In local coordinates it is expressed as follows: Let $(x^1, \cdots, x^n)$, $(x^1, \cdots, x^n, y^1, \cdots, y^m)$ be local coordinates systems of N and M respectively. Then $(x^1, \cdots, x^n, \cdots, p_a^i, \cdots)$, where

$$p_a^i = \frac{\partial^{|\alpha|} y^i}{\partial x^\alpha} \quad \text{with} \quad \alpha=(\alpha_1, \cdots, \alpha_n), \ |\alpha| \leq k,$$ gives a local coordinate system of

$J^k(M, N)$ called a canonical coordinates system. Put

$$\omega_a^i = dp_a^i - \sum_{j=1}^n p_{a+1_j}^i \, dx^j$$

for $|\alpha| \leq k-1$, with $\alpha+1_j = (\alpha_1, \cdots, \alpha_j+1, \cdots, \alpha_n)$, and define $D^p (p \leq -1)$ by the following Pfaff equations:

$$D^p : \omega_\alpha^i = 0 \qquad (i=1, \cdots, n \quad |\alpha| \leq k+p),$$

It is easy to see that D^p are well-defined subbundles of $TJ^k(M, N)$ and satisfy:

    i) $\underline{D^{p-1}} = \underline{D^p} + [\underline{D^p}, \underline{D^{-1}}]$,

    ii) $\overline{D^p} = TJ^k(M, N)$ for $p \leq -k-1$.

We thus obtain a canonical tangential filtration $\{D^p\}$ on $J^k(M, N)$ of depth $k+1$ generated by D^{-1}. It should be noted that if dim $M = n+1$, dim $N = n$ and $k=1$ then $J^1(M, N)$ is a contact manifold having D^{-1} as its contact structure.

    4) Standard filtered manifold. Let $\mathfrak{n}$ be a finite-dimensional Lie algebra endowed with a gradation $\mathfrak{n} = \bigoplus_{p \in Z} \mathfrak{n}_p$ such that

    i) $[\mathfrak{n}_p, \mathfrak{n}_q] \subset \mathfrak{n}_{p+q}$,

    ii) $\mathfrak{n}_p = 0$ $p \geq 0$.

Note that $\mathfrak{n}$ is therefore nilpotent. Let N be a Lie group whose Lie algebra is $\mathfrak{n}$. Set $\mathfrak{n}^p = \bigoplus_{i \geq p} \mathfrak{n}_i$ and identify $N \times \mathfrak{n}^p$ with a left invariant subbundle of TN, then $\{N \times \mathfrak{n}^p\}_{p \in Z}$ is a tangential filtration on N. The filtered manifold $N = (N, \{N \times \mathfrak{n}^p\})$ is called a standard filtered manifold of type $\mathfrak{n}$.

    1.2. The tangent space of a filtered manifold.

    Let M be a filtered manifold. The tangential filtration $\{T^p M\}$ defines on each tangent space $T_x M$, $x \in M$, the induced filtration $\{T_x^p M\}$. We denote by $T_x M$ this filtered vector space $(T_x M, \{T_x^p M\})$. Now by setting

$$gr_p T_x M = T_x^p M / T_x^{p+1} M,$$

we form a graded vector space:

$$gr T_x M = \bigoplus_{p \in Z} gr_p T_x M.$$

This vector space carries a natural bracket operation induced from the Lie bracket of vector fields: For $\xi \in gr_p T_x M$, $\eta \in gr_q T_x M$, take local cross-sections X, Y of $T^p M$, $T^q M$ respectively such that

$$\begin{cases} \xi \equiv X_x & \mod \quad T_x^{p+1} M, \\ \eta \equiv Y_x & \mod \quad T_x^{q+1} M, \end{cases}$$

and define

$$[\xi, \eta] \equiv [X, Y]_x \qquad \text{mod} \quad T_x^{p+q+1} M.$$

It is then easy to see that this bracket operation is well-defined and makes grT_xM a Lie algebra. Clearly we have:

i) $[gr_pT_xM, gr_qT_xM] \subset gr_{p+q}T_xM,$

ii) $gr_pT_xM = 0$ for $p \geq 0.$

This graded Lie algebra grT_xM is called the symbol algebra of M at x ([24]), and may be considered as the tangent space (algebra) at x of the filtered manifold M.

We say that a filtered manifold M is regular of type $\mathfrak{m}$ if the symbol algegras grT_xM are all isomorphic to a graded Lie algebra $\mathfrak{m}$.

1.3. First order frame bundle.

Let us fix our notation and terminology on the filtered vector spaces. We mean by a filtered vector space a vector space V endowed with a descending sequence $\{F^pV\}_{p \in Z}$ of its subspaces. If $V = (V, \{F^pV\})$, $W = (W, \{F^pW\})$ are filtered vector spaces, by an isomorphism $\varphi: V \to W$, we mean always an isomorphism of filtered vector spaces, that is, a linear isomorphism $\varphi: V \to W$ such that $\varphi(F^pV) = F^qW$ for all $p \in Z$. For a filtered vector space V we denote by $\mathrm{Aut}(V)$ the group of all isomorphisms of the filtered vector space V. Clearly $\mathrm{Aut}(V)$ is a closed Lie subgroup of $GL(V)$. We denote by $grV = \oplus gr_p V$ the graded vector space associated with V, that is, $gr_p V = F^pV/F^{p+1}V$. We denote by $\mathrm{Aut}(grV)$ the Lie group of all isomorphisms of grV as a graded vector space i.e., linear isomorphisms $\phi: grV \to grV$ such that $\phi(gr_pV) = gr_pV$. An isomorphism $\varphi: V \to W$ of filtered vector spaces induces in the obvious way a unique isomorphism of the associated graded vector spaces $gr\varphi: grV \to grW$. We have thus a surjective homomorphism

$$\mathrm{Aut}(V) \ni \varphi \longmapsto gr\varphi \in \mathrm{Aut}(grV).$$

The kernel of this homomorphism is clealy given by

$$F^1\mathrm{Aut}(V) = \{a \in \mathrm{Aut}(V) \mid (a - id_V)(F^pV) \subset F^{p+1}V \quad \forall p\}$$

Now let M be a filtered manifold. Choose a filtered vector space V isomorphic to the filtered vector space T_xM for some and hence all $x \in M$. (In order to assign V uniquely to M, we choose and fix once for all one filtered vector space in each isomorphic class of filtered vector spaces, say for examples, R^n with a suitable filtration.) To define what is to be called a 1-st order frame of M, there are two kinds of natural candidates: One is an isomorphism $\hat{z}: V \to T_xM$ of filtered vector spaces, another is

an isomorphism $z : grV \to grT_xM$ of the graded vector spaces. The former is first order in the ordinary sense and the latter in the weighted sence ([18]). Hence we call them respectively ordinary 1-st order frame and weighted 1-st order frame of M at x.

We denote by $\tilde{\mathscr{R}}^{(0)}(M)$ the set of all ordinary 1-st order frames of M and by $\mathscr{R}^{(0)}(M)$ the set of all weighted 1-st order frames of M.

It is immediate to see that $\mathscr{R}^{(0)}(M)$ (or $\tilde{\mathscr{R}}^{(0)}(M)$) is a principal fibre bundle with base space M and structure group $\mathrm{Aut}(grV)$ (resp. $\mathrm{Aut}(V)$). Note also that there is a natural projection

$$\tilde{\mathscr{R}}^{(0)}(M) \in \hat{z} \longmapsto gr\hat{z} \in \mathscr{R}^{(0)}(M).$$

Since for $\hat{z} \in \tilde{\mathscr{R}}^{(0)}(M)$ and $a \in \mathrm{Aut}(V)$, $gr(\hat{z}a) = gr(\hat{z})$ if and only if $a \in F^1\mathrm{Aut}(V)$, we can identify $\mathscr{R}^{(0)}(M)$ with the quotient bundle of $\tilde{\mathscr{R}}^{(0)}(M)$ by $F^1\mathrm{Aut}(V)$.

Let M and M' be filtered manifolds. If there is an isomorphisms $\varphi : M \to M'$ of filtered manifolds, then there is induced an isomorphism $\varphi^{(0)} : \mathscr{R}^{(0)}(M) \to \mathscr{R}^{(0)}(M')$ (or $\hat{\varphi}^{(0)} : \tilde{\mathscr{R}}^{(0)}(M) \to \tilde{\mathscr{R}}^{(0)}(M')$) of principal fibre bundles which sends $z \in \mathscr{R}^{(0)}(M)$ to $gr(\varphi_*) \circ z \in \mathscr{R}^{(0)}(M)$ (resp. $\hat{z} \in \tilde{\mathscr{R}}^{(0)}(M)$ to $\varphi_* \circ \hat{z} \in \tilde{\mathscr{R}}^{(0)}(M')$).

It is the weighted ordering that plays a fundamental rôle in the present study. We will therefore usually mean by the 1-st order frame bundle of M the weighted one $\mathscr{R}^{(0)}(M)$.

1. 4. First order geometric structures.

A weighted (or ordinary) first order geometric structure on a filtered manifold M is a principal subbundle P of $\mathscr{R}^{(0)}(M)$ (or resp. $\hat{P}$ of $\tilde{\mathscr{R}}^{(0)}(M)$). We write sometimes as $P = P(M, G)$ (resp. $\hat{P} = \hat{P}(M, \hat{G})$) to indicate the base space M and the structure group G (resp. $\hat{G}$), a Lie subgroup of $\mathrm{Aut}(grV)$ (resp. $\mathrm{Aut}(V)$).

The notion of isomorphism is defined in the natural manner : Two first order geometric strucures $P(M, G)$ and $P'(M', G')$ (or $\hat{P}(M, \hat{G})$ and $\hat{P}'(M', \hat{G}')$) on filtered manifolds M and M' are said to be isomorphic or equivalent if there exists an isomorphism $\varphi : M \to M'$ such that $\varphi^{(0)}(P) = P'$ (resp. $\hat{\varphi}^{(0)}(\hat{P}) = \hat{P}'$). It should be noted that if $P(M, G)$ and $P'(M', G')$ (resp. $\hat{P}(M, \hat{G})$ and $\hat{P}'(M', \hat{G}')$) are isomorphic then $G = G'$ (or $\hat{G} = \hat{G}'$).

Let us observe how weighted and ordinary first order geometric strucures are related to each other. Let $\hat{P}(M, \hat{G})$ be an ordinary 1-st order geometric structure. Let us denote by q both the natural projections $\tilde{\mathscr{R}}^{(0)}(M) \to \mathscr{R}^{(0)}(M)$ and $\mathrm{Aut}(V) \to \mathrm{Aut}(grV)$ i.e., the quotient maps by the right action of $F^1\mathrm{Aut}(V)$. Then $q\hat{P}(M, q\hat{G})$ is a weighted 1-st

order geometric structure. If there is an isomorphisms $\hat{\varphi}^{(0)}$; $\hat{P}(\boldsymbol{M}, \hat{G}) \to$ $\hat{P}'(\boldsymbol{M}', \hat{G}')$, then it clearly induces an isomorphism $\varphi^{(0)}: q\hat{P}(\boldsymbol{M}, q\hat{G}) \to$ $q\hat{P}'(\boldsymbol{M}', q\hat{G}')$ which makes the following diagram commutative:

$$
\begin{array}{ccc}
\hat{P} & \xrightarrow{\hat{\varphi}^{(0)}} & \hat{P}' \\
\downarrow & & \downarrow \\
q\hat{P} & \xrightarrow{\varphi^{(0)}} & q\hat{P}'
\end{array}
$$

Conversely given a weighted 1-st order geometric structure $P(\boldsymbol{M}, G)$, let $\natural P$, $\natural G$ be the inverse images by q of P and G respectively. Then we see that $\natural P(\boldsymbol{M}, \natural G)$ is a principal subbundle of $\mathscr{R}^{(0)}(\boldsymbol{M})$ i.e., an ordinary 1-st order geometric structure. We see also that weighted 1-st order geometric structures $P(\boldsymbol{M}, G)$ and $P'(\boldsymbol{M}', G')$ are isomorphic if and only if so are $\natural P$ and $\natural P'$. It should be noted that an ordinary 1-st order geometric structure $\hat{P}$ is a (in general proper) subbundle of $\natural q\hat{P}$. This means that an ordinary 1-st order geometric structure may be regarded as a higher order geometric structure with respect to the weighted ordering.

In this paper we shall be concerned with the equivalence problem primarily for the weighted first order geometric structures.

1.5.   Examples.

We give a few examples of first order geometric structures on filtered manifolds without alluding to the detail.

1) Filtered manifolds themselves. In general a filtered manifold $\boldsymbol{M}$ is, even locally, highly non-trivial, so that the equivalence problem of $\mathscr{R}^{(0)}(\boldsymbol{M})$ already has various aspects and presents interesting problems. The work of Cartan on the Pfaff systems in five variables [2] provides us with a good example, in which he developed a deep and detailed study on the equivalence problem of differential systems on a five-dimensional manifold. In particular, a generic differential system D of rank 2 on a 5-dimensional manifold generates a filtration $\{D^p\}$ such that rank $D^{-2}=3$ and rank $D^{-3}=5$ ([24]). The invariants are calculated by constructing a so-called Cartan connection. (For the interpretation of [2] I have benefited from a lot of conversation with N. Tanaka and K. Yamaguchi.)

2) G-structures. If the filtration is trivial, then $\mathscr{R}^{(0)}(\boldsymbol{M})$ is just the usual linear frame bundle and a subbundle of $\mathscr{R}^{(0)}(\boldsymbol{M})$ just a so-called G-structure.

3) CR-structures. Let $\boldsymbol{M}$ be a filtered manifold of depth 2. An almost CR-structure on $\boldsymbol{M}$ is defined by $I \in \Gamma \operatorname{End}(T^{-1}\boldsymbol{M})$ such that $I^2 =$

$-id..$ Clearly to this structure corresponds a subbundle $P^{(0)}$ of $\mathscr{R}^{(0)}(M)$. The integrability conditions or Levi forms etc. can be expressed in terms of the structure function. For more detail we refer to [25].

4) Monge-Ampère equations. Let M be a 5-dimensional contact manifold, in other words a filtered manifold of depth 2 whose symbol algebras are isomorphic to the 5-dimensional Heisenberg Lie algebra. According to the geometric formulation of the Monge-Ampère equation [13], a hyperbolic M-A equation is defined by a direct sum decomposition: $T^{-1}M = \eta_1 \oplus \eta_2$ such that η_1 and η_2 are perpendicular. It therefore defines a subbundle of $\mathscr{R}^{(0)}(M)$, and the equivalence problem of the M-A equations may be treated as that of certain subbundles of $\mathscr{R}^{(0)}(M)$ (see § 3. 9).

5) Foliations with transversal structures (cf. [12]). Let M be a filtered manifold of depth 2 with $T^{-1}M$ completely integrable. Let G_{-2} be a Lie subgroup of $GL(gr_{-2}V)$ and let $G_0 = G_{-2} \times GL(gr_{-1}V)$. Then a reduction of $\mathscr{R}^{(0)}(M)$ to the group G_0 defines a foliation with a transversal structure, e. g., riemannian if $G_{-2} = O(gr_{-2}V)$.

6) Differential equations. A system of differential equations of order k is in general expressed as a submanifold R of a jet bundle $J^k(M, N)$. The higher order contact structure $\{D^p\}$ of the jet bundle induces on R a tangential filtration $\{F^p TR\}$ by $F^p TR = D^p \cap TR$, provided that the rank of $F^p TR$ is independent of the base points. This filtration plays a fundamental rôle in the geometric study of differential equations (cf. [27], [29]).

Chapter II. Towers on filtered manifolds

2. 1. Infinite dimensiolal manifolds and Lie groups.

In our study of geometric structures, though we essentially concern ourselves with finite-dimensional geometric structures, we will often deal with infinite dimensional objects such as manifolds, Lie groups and principal fibre bundles etc. The introduction of such infinite-dimensional objects has the advantage to give us clear perspectives to treat geometric structures.

For our purpose it is enough to consider only those infinite-dimensional objects which are obtained as projective limits of finite-dimensinal ones, to which hold most of the basic notions and facts well-known in the finite-dimensional case.

In this section, for the sake of fixing the notation, we introduce a "p-category" and define the class of infinite-dimensional objects that we shall deal with.

2.1.1.   Let X be a set.   By a fibring of X we mean a family $\{X^{(i)}, \pi^i_j, \pi^i\}_{i,j \in Z}$ consisting of sets $X^{(i)}$, maps $\pi^i_j : X^{(j)} \to X^{(i)}$ $(i < j)$, $\pi^i :$ $X \to X^{(i)}$ which satisfies the following conditions:

    i) $\pi^i_j \circ \pi^j_k = \pi^i_k$ $(i < j < k)$

    ii) $\pi^i_j \cdot \pi^j = \pi^i$ $(i < j)$

    iii) $\pi : X \to \varprojlim X^{(i)}$ is bijective,

where $\varprojlim X^{(i)}$ denotes the projective limit of the system $(X^{(i)}, \pi^i_j)$ and π denotes the canonical map induced from π^i.

We say that two fibrings of X, $(X^{(i)}, \pi^i_j, \pi^i)$ and $(\bar{X}^{(i)}, \bar{\pi}^i_j, \bar{\pi}^i)$ are equivalent if for any i there exist j, k and maps $\alpha^i_j : X^j \to \bar{X}^i$, $\beta^i_k : \bar{X}^k \to X^i$ such that $\bar{\pi}^i = \alpha^i_j \circ \pi^j$, $\pi^i = \beta^i_k \circ \bar{\pi}^k$.   An equivalence class of fibrings of X is called a $\mathfrak{p}$-strucure on X and a set endowed with a $\mathfrak{p}$-structure will be called a $\mathfrak{p}$-space.   A fibring belonging to the $\mathfrak{p}$-structure is called an admissible fibring of the $\mathfrak{p}$-space.

Let X, Y be $\mathfrak{p}$-spaces.   A map $f : X \to Y$ is called a $\mathfrak{p}$-morphism if for admissible fibrings $(X^{(i)}, \pi^i_j, \pi^i)$, $(Y^{(i)}, w^i_j, w^i)$ respectively of X and Y and for any i there exist j and a map $f^i_j : X^{(j)} \to Y^{(j)}$ such that $f^i_j \circ \pi^j = w^i \circ f$.   The map f is called a $\mathfrak{p}$-isomorphism if f is bijective and if both f and f^{-1} are $\mathfrak{p}$-morphisms.

2.1.2.   Now let us define a $\mathfrak{p}$-manifold ($\mathfrak{p}$-vector space, $\mathfrak{p}$-Lie group, or $\mathfrak{p}$-Lie algebra) by specializing the sets $X^{(i)}$ and the maps π^i_j, α^i_j, β^i_k (which appeared in the definition above of a $\mathfrak{p}$-space) as follows: The sets $X^{(i)}$ are required to be differential manifolds (vector spaces, Lie groups, or Lie algebras resp.) of finite dimensions and the maps π^i_j, α^i_j, β^i_k to be surjective submersions (linear maps, Lie group homomorphisms, or Lie algebra homomorphisms resp.).

A morphism of $\mathfrak{p}$-manifolds, $\mathfrak{p}$-vector spaces $\mathfrak{p}$-Lie groups, or $\mathfrak{p}$-Lie algebras is defined by specializing the maps f^i_j above to be a differentiable map, linear map, or homomorphism respectively, and the morphism is rather referred to as differentiable map, ($\mathfrak{p}$-)linear map, or (Lie group or Lie algebra) homomorphism respectively.

Similarly we have the notions of $\mathfrak{p}$-principal fibre bundle and $\mathfrak{p}$-vector bundle: In particular a $\mathfrak{p}$-principal fibre bundle $P(M, G)$ is a projective limit of finite-dimensional principal fibre bundles $\{P^{(i)}(M^{(i)}, G^{(i)})\}$.

A few remaks are now in order.

REMARK 2.1.1.   Let X be a $\mathfrak{p}$-manifold ($\mathfrak{p}$-vector space, or $\mathfrak{p}$-Lie group etc.).   If one (and hence all) admissible fibring $(X^{(i)}, \pi^i_j, \pi^i)$ is

bounded to the above, that is, there exists i_0 such that π_i^j are all bijective for $i_0 \leq i < j$, X is identified with a usual finite-dimensional manifold (or vector space etc.). Conversely a finite-dimensional manifold may be regarded as a $\mathfrak{p}$-manifold with the trivial fibring.

REMARK 2.1.2. We shall use freely the algebraic structures that $\mathfrak{p}$-objects inherit in the natural way. For instance a $\mathfrak{p}$-Lie group obviously has a group structure. However as to topology, we should pay some attention: If X is a $\mathfrak{p}$-manifold, it has the natural topology, the projective limit topology of the manifold topology $X^{(i)}$. If X is a $\mathfrak{p}$-vector space (or $\mathfrak{p}$-Lie algebra), it is rather convenient to give it a topology by assigning $\{Ker\ \pi^i\}$ as a fundamental system of neighbourhoods of the origin, i.e., the projective limit topology of $X^{(i)}$, with the discrete topology on $X^{(i)}$. Since a $\mathfrak{p}$-vector space over $\boldsymbol{R}$ or $\boldsymbol{C}$ may be also viewed as a $\mathfrak{p}$-manifold, we have two different natural topologies on it. This may be somewhat confusing, but when topologies of $\mathfrak{p}$-objects are concerned, it will be clear by context which topology is referred to.

REMARK 2.1.3. We can also introduce $\mathfrak{F}$-categories of Lie groups and Lie algebras by defining a $\mathfrak{F}$-Lie group to be a $\mathfrak{p}$-manifold G endowed with a group structure such that the group operations of G are differentiable , and $\mathfrak{F}$-Lie algebra to be a $\mathfrak{p}$-vector space L endowed with a Lie algebra structure such that the bracket operation is continuous. In our following discussion we sometimes deal with $\mathfrak{F}$-Lie algebras but not $\mathfrak{F}$-Lie groups.

2.1.3 We briefly mention the differential calculus on $\mathfrak{p}$-manifolds. Let M be a $\mathfrak{p}$-manifold with an admissible fibring $\{M^{(i)}, \pi_i^j, \pi^i\}$. Put

$$TM = \text{proj lim}\,(TM^{(i)}, (\pi_i^j)_*),$$

where $TM^{(i)}$ denotes the tangent bundle of $M^{(i)}$ and $(\pi_i^j)_*$ the differential of π_i^j. Denote by $(\pi^i)_*$ the canonical projection: $TM \to TM^{(i)}$. Then we have a $\mathfrak{p}$-manifold TM with $(TM^{(i)}, (\pi_i^j)_*, (\pi^i)_*)$ as an admissible fibring and with a canonical projection $\pi: TM \to M$. Clearly TM is a $\mathfrak{p}$-vector bundle over M and is uniquely determined by M up to isomorphisms. So we call it the tangent bundle of M.

We can also consider differential forms on a $\mathfrak{p}$-manifold M. Let $\wedge^p TM$ denote the p-th exterior product of TM, which is again a $\mathfrak{p}$-vector bunble over M. Note that when we form the tensor product $X \otimes Y$ of $\mathfrak{p}$-vector spaces X, Y it is to be understood as the complete tensor product with respect to the $\mathfrak{p}$-vector space topologies. Now given a $\mathfrak{p}$-vector

space E, an E-valued differential p-form on M is a differential map ω: $\wedge^p TM \to E$ such that the restriction of ω to each fibre: $\omega_x: \wedge^p T_x M \to E$, is $\mathfrak{p}$-linear. We have also the exterior differentiation d and the usual formulas for differential forms.

If G is a $\mathfrak{p}$-Lie group the tangent space $T_e G$ to G at the neutral element e becomes in the natural manner a $\mathfrak{p}$-Lie algebra, which is called the Lie algebra of G and usually denoted by the corresponding German letter $\mathfrak{g}$. We have also the Maurer-Cartan form ω of G, a left invariant $\mathfrak{g}$-valued 1-form on G assigning to each $x \in G$ an isomorphism of $\mathfrak{p}$-vector spaces $\omega_x: T_x G \to \mathfrak{g}$, which satisfies:

$$d\omega + \frac{1}{2}[\omega, \omega] = 0.$$

2.1.4. At the end of this section let us make some notational convention on filtered objects.

If X is a $\mathfrak{p}$-vector space, $\mathfrak{p}$-Lie algebra or $\mathfrak{p}$-Lie group, and if $\{(X^{(i)}, \pi_j^i, \pi^i)\}$ is an admissible fibring of X, we call $\{Ker\ \pi^i\}$ an admissible filtration of X. If L is a $\mathfrak{p}$-Lie algebra, we mean by an admissible filtration of L a descending sequence of subspaces $\{L^p\}$ such that it constitutes a fundamental system of the neighbourhoods of the origin and satisfies:

$$[L^p, L^q] \subset L^{p+q}.$$

We shall mean by a filtered Lie group (Lie algebra, or vector space) a $\mathfrak{p}$-Lie group ($\mathfrak{p}$-Lie algebra, or $\mathfrak{p}$-vector space resp.) endowed with an admissible fildtration.

For filtered objects (vector spaces V, W, Lie groups G etc.) we denote by $\{F^p\}$ not only their filtrations but also the induced filtrations defined naturally on various associated spaces, for instance:

$$F^p(V \oplus W) = F^p V \oplus F^p W$$
$$F^p(V \oplus W) = \sum_{r+s=p} F^r V \oplus F^s W$$
$$F^p(G/F^h G) = F^p G/F^p G \cap F^h G$$
$$F^p \mathrm{Hom}(V, W) = \{\alpha \in \mathrm{Hom}(V, W) \mid \alpha(F^i V) \subset F^{i+p} W \quad \forall i\}$$
$$F^p GL(V) = \{\alpha \in GL(V) \mid \alpha - 1_V \in F^p \mathrm{Hom}(V, V)\}.$$

With this convention, if V is a finite-dimensional filtered vector space we have $\mathrm{Aut}(V) = F^0 GL(V)$. Note that $\mathrm{Aut}(V)$ becomes a filtered Lie group with the natural filtration defined by

$$F^p \mathrm{Aut}(V) = \mathrm{Aut}(V) \cap F^p GL(V).$$

When we take a quotient space, for instance $F^0 \mathrm{Aut}(V)/F^p \mathrm{Aut}(V)$, we

often write it simply as $F^0\mathrm{Aut}(V)/F^p$.

2. 2. Algebraic skeletons.

In this section we introduce the notion of skeleton and study the basic properties, to pave the way for introducing the notion of tower. Hereafter V always denotes a finite dimensional filtered vector space (V, F) such that

$$V = F^{-\mu}V \supset \cdots \supset F^p V \supset F^{p+1}V \supset \cdots \supset F^0 V = 0.$$

2. 2. 1. First of all we introduce the following

DEFINITION 2. 2. 1. *An algebraic skeleton of a tower (or simply, skeleton) on* V *is a triple* (E, G, ρ), *with* G *a* $\mathfrak{p}$-*Lie group,* $E = V \oplus \mathfrak{g}$, $\mathfrak{g}$ *the Lie algebra of* G, *and* ρ *a representaticon of* G *on* E, *such that the following conditions are satisfied :*

i) $\rho(a)A = Ad(a)A$ *for* $a \in G$, $A \in \mathfrak{g}$.
ii) $\rho(a)F^p V \subset F^p V \oplus \mathfrak{g}$ *for* $a \in G$, $p < 0$.
iii) *There exists an admissible filtration* $\{F^pG\}$ *of* G *such that* $F^p G = G$ *for* $p \le 0$, *and that the sequences :*

$$(2.1) \qquad 1 \longrightarrow F^{k+1}G \longrightarrow G \xrightarrow{\rho^{(k)}} F^0 GL(E^{(k-1)})/F^{k+1}$$

are exact and $\rho^{(k)}$ *are analytic for all* $k \ge 0$, *where*

$$F^p E = F^p V \oplus F^p \mathfrak{g}, \qquad E^{(p)} = E/F^{p+1}E$$

with $F^p \mathfrak{g}$ *the Lie algebra of* F^pG, *and* $\rho^{(k)}$ *denotes the homomorphism induced by* ρ.

It should be remarked that an admissible filtration $\{F^pG\}$ satisfying the above conditions is uniquely determined inductively by the exact sequences (2. 1).

Thus the condition iii) might be rephrased that the filtration of G defined by (2. 1) should be admissible. In particular, we have

$$(2.2) \qquad\qquad F^\infty G = \cap F^p G = \{e\}.$$

This is a sort of "effectiveness" condition for the action of G that does not depend on the filtration : In stead of the given filtration F of V one can use the trivial filtration F_{tr} of V ($F_{tr}^{-1}V = V$, $F_{tr}^0 V = 0$) to define another (or standard) filtration of G by the exact sequence (2. 1), which can be written in this case more simply as

(2. 3) $1 \longrightarrow F_{tr}^{k+1}G \longrightarrow G \longrightarrow GL(V \oplus \mathfrak{g}/F_{tr}^{k}\mathfrak{g}).$

Then it is easy to verify

(2. 4) $F^{p}G \supset F_{tr}^{p}G \supset F^{p\mu}G,$

where μ denotes the depth of V. Hence the filtration F of G is admissible if and only if so is F_{tr}.

We can therefore say that *a skeleton on a filtered vector space* (V, F) *is a skeleton on the trivial filtered vector space* (V, F_{tr}) *which leaves the filtration F invariant.*

As a typical example of skeletons, let us consider a homogeneous space L/G, where L is a finite-dimensional Lie group with Lie algebra $\mathfrak{l}$ and G is a closed subgroup of L with Lie algebra $\mathfrak{g}$. If we define a filtration of G by (2. 3) with $V \oplus \mathfrak{g}$ replaced by $\mathfrak{l}$ and set $G^{\infty} = \cap F_{tr}^{p}G$, then G^{∞} is a maximal closed normal subgroup of G such that the adjoint representation of G^{∞} on $\mathfrak{l}/\mathfrak{g}^{\infty}$ is trivial. In particular, $\mathfrak{g}^{\infty}$ is the maximal ideal of $\mathfrak{l}$ contained in $\mathfrak{g}$. Hence the action of L on L/G is almost effective if and only if G^{∞} is a discrete subgroup of G.

This in mind, let us say that the pair $(\mathfrak{l}, G)$ is formally effective if $G^{\infty} = \{e\}$. Then we have:

If the pair $(\mathfrak{l}, G)$ *is formally effective then the triple* $(\mathfrak{l}, G, Ad)$ *becomes a skeleton on* $V = \mathfrak{l}/\mathfrak{g}$ *by a choice of identification* $\mathfrak{l} = V \oplus \mathfrak{g}$. *Furthermore if there is a filtration F of V left invariant by the linear isotropy representation of G, then* $(\mathfrak{l}, G, Ad)$ *is a skeleton over* (V, F).

We remark that the filtration F_{tr} of G is nothing but the one derived from the Tayler expansion of the actions of G at the origin of L/G, while the filtration F of G defined by (2. 1) has the meaning of the weighted expansion with respect to the induced tangential filtration of L/G.

This example shows that the notion of skeleton (E, G, ρ) is an abstraction of some algebraic aspects of the homogeneous spaces. The crucial generalization is that G can be infinite dimensional and that E is not necessarily equipped with a Lie algebra structure. (As seen later on, if E has a Lie algebra structure extending the action of $\mathfrak{g}$ on E then the size of E becomes "relatively small".)

Let (E, G, ρ) be a skeleton on V. We denote by ρ_* the representation of $\mathfrak{g}$ on E, the differential of ρ. Let us see several properties of the skeletons which follow immediately from the definition.

PROPOSITION 2. 2. 1.

$\rho(a)(\rho_*(A)X) = \rho_*(\rho(a)A)(\rho(a)X)$ *for* $a \in G$, $A \in \mathfrak{g}$, $X \in E$.

PROOF: As in the case of finite-dimensional Lie groups, we have

$$\rho(a)\rho_*(A)\rho(a)^{-1}=\rho_*(Ad(a)A), \qquad a\in G, \ A\in\mathfrak{g}.$$

Then, since $Ad(a)A=\rho(a)A$, we have:

$$\rho(a)((\rho_*A)X)=\rho(a)\rho_*(A)\rho(a)^{-1}\rho(a)X$$
$$=\rho_*(Ad(a)A)\rho(a)X$$
$$=\rho_*(\rho(a)A)\rho(a)X.$$

PROPOSITION 2.2.2. *For $i\geq0$, we have*

(1) $a\in F^iG \Longleftrightarrow (\rho(a)-id)F^pV\subset F^{i+p}E$ *for all* $p<0$.

(2) $A\in F^i\mathfrak{g} \Longleftrightarrow \rho_*(A)F^pV\subset F^{i+p}E$ *for all* $p<0$.

PROOF: It suffices to show (1), since (2) follows immediately from (1). The implication ($\Rightarrow$) is clear by the exact sequences (2.1). Let us prove the converse by induction on i. The assertion holds clearly for $i=0$. Suppose that $i\geq1$ and that it holds for $i-1$. If $a\in G$ satisfies:

(*) $\qquad\qquad (\rho(a)-id)F^pV\subset F^{i+p}E$ for all $p<0$,

we want to show $a\in F^iG$, which is, in view of (2.1), equivalent to show

$$(\rho(a)-id)F^pE\subset F^{p+i}E+F^{i-1}E$$

for all $p\in\mathbf{Z}$. If $p<0$ this holds clearly by (*). If $p\geq0$ it amounts to saying

(**) $\qquad\qquad (\rho(a)-id)\mathfrak{g}\subset F^{i-1}\mathfrak{g}$.

But, by induction, $a\in F^{i-1}G$. Therefore, for $X\in\mathfrak{g}$, we have

$$\rho(a)X=Ad(a)X$$
$$\equiv X \pmod{F^{i-1}\mathfrak{g}},$$

which shows (**), hence proves (1).

PROPOSITION 2.2.3. *For $i, j,\in\mathbf{Z}$ we have:*

(1) $(\rho(a)-id)F^jE\subset F^{i+j}E$ *for* $a\in F^iG$.
(2) $\rho_*(F^i\mathfrak{g})F^jE\subset F^{i+j}E$.
(3) $[F^i\mathfrak{g}, F^j\mathfrak{g}]\subset F^{i+j}\mathfrak{g}$.

PROOF: Let us first prove (1). By Proposition 2.2.2, we have only to show it for $i, j\geq0$. For $a\in F^iG$ we write $\rho(a)=1+\alpha$. Assuming that

(*) $\qquad\qquad \alpha(F^pE)\subset F^{i+p}E$ for $p<j$,

we will show $a(F^j E) \subset F^{i+j} E$. In order that, by Proposition 2. 2. 2 (2), it suffices to show that if $B \in F^j \mathfrak{g}$ then

(**) $\rho_*(a(B))x^q \in F^{i+j+q} E$ for all $x^q \in F^q V$.

But, if we write $\rho(a)^{-1} = 1 + a'$, we have:

$$\rho_*(a(B))x^q = \rho_*(\rho(a)B - B)x^q$$
$$= \rho_*(Ad(a)B - B)x^q$$
$$= \{\rho(a)\rho_*(B)\rho(a)^{-1} - \rho_* B\}x^q$$
$$= \{a\rho_*(B) + \rho_*(B)a' + a\rho_*(B)a'\}x^q.$$

From this and (*) we deduce (**), which proves (1).

The assertion (2) follows immediately from (1). Note that since we have

$$\rho_*(A)B = [A, B]   \text{for } A, B \in \mathfrak{g},$$

the assertion (3) is a consequence of (2).

COROLLARY 2. 2. 1.

(1)   For $a \in G$, if $\rho(a)v = v$ for all $v \in V$ then $a = 1$.
(2)   For $A \in \mathfrak{g}$, if $\rho_*(A)v = 0$ for all $v \in V$ then $A = 0$.

Since $\cap F^i G = 1$ and $\cap F^i \mathfrak{g} = 0$, the corollary follows from Proposition 2. 2. 2.

In particular the representations ρ and ρ_* are faithful. In what follows, if there is no fear of confusion, we shall omit writing ρ and ρ_*.

Now we define a *morphism* (or *isomorphism*) from a skeleton (E, G) to another (E', G') (both on V) to be a homomorphism (resp. isomorphism) $\varphi : G \to G'$ of $\mathfrak{p}$-Lie groups such that

$$\varphi_*(aX) = \varphi(a)\varphi_*(X)   \text{for } a \in G, X \in E,$$

where we denote by φ_* not only the differential $\varphi_* : \mathfrak{g} \to \mathfrak{g}$ but also its trivial extension $id_V + \varphi_* : E(= V \oplus \mathfrak{g}) \to E'(= V \oplus \mathfrak{g}')$.

REMARK 2. 2. 1.   If $\varphi : (E, G) \to (E', G')$ is a morphism of skeletons, then, as easily seen from Proposition 2. 2. 2, it preserves the induced filtrations of the skeletons, namely

$$\varphi(F^p G) \subset F^p G'   \text{and}   \varphi_*(F^p E) \subset F^p E'   \text{for all } p \in \mathbf{Z}$$

REMARK 2. 2. 2.   Every morphism of skeletons is necessarily injective. This property will turn out to be clear in the course of the discussion in section 2. 2. 2.   Hence a morphism will be also referred to as an embed-

ding.

Now we state the following fundamental:

THEOREM 2.2.1. *For each filtered vector space* V *there exists a skeleton* $(E(V), G(V))$ *on* V *having the universal property: Any skeleton on* V *is uniquely embedded in* $(E(V), G(V))$.

Admitting this theorem for a while, we state a little generalized version of it. We set

$$(2.5) \qquad G^{(k)}(V) = G(V)/F^{k+1},$$

and we say that a Lie subgroup $G^{(k)}$ of $G^{(k)}(V)$ is adapted if there exists a skeleton (E, G) on V such that $G/F^{k+1} = G^{(k)}$.

Then we have:

THEOREM 2.2.2. *For an adapted subgroup* $G^{(k)}$ *of* $G^{(k)}(V)$ *there exists a unique skeleton on* V, *denoted by* $(E(V, \mathfrak{g}^{(k)}), G(V, G^{(k)}))$ *and called the universal skeleton prolonging* $G^{(k)}$, *such that*

     i) $G(V, G^{(k)})/F^{k+1} = G^{(k)}$,

     ii)   *If a skeleton* (E, G) *satisfies* $G/F^{k+1} \subset G^{(k)}$, *then* (E, G) *is embedded in* $(E(V, \mathfrak{g}^{(k)}), G(V, G^{(k)}))$.

In the next subsection, we will give an explicit construction of the universal skeletons as well as an explicit criterion for a subgroup of $G^{(k)}(V)$ to be adapted.

2.2.2.   Categories $\mathscr{A}^{(k)}$, $\mathscr{J}^{(k+1)}$, and functors #, $\widehat{\#}$.

We shall employ the following notation: For filtered Lie groups H, G, "$H < G$" will mean that H is a filtered Lie subgroup of G, that is, the inclusion $\iota: H \to G$ is a Lie homomorphism and $\iota(F^k H) = \iota(H) \cap F^k G$.

Let us define sets of filtered Lie groups $\mathscr{A}_V^{(k)}$, $\mathscr{J}_V^{(k)}$ (or simply written as $\mathscr{A}^{(k)}$, $\mathscr{J}^{(k)}$) $(k = -1, 0, 1, \cdots)$ and mappings #, $\widehat{\#}$ and q:

$$(2.6) \qquad
\begin{array}{ccc}
\mathscr{A}^{(k)} & \xrightarrow{\ \widehat{\#}\ } & \mathscr{J}^{(k+1)} \\
\big\| & & \big\downarrow q \\
\mathscr{A}^{(k)} & \xrightarrow{\ \#\ } & \mathscr{A}^{(k+1)}
\end{array}$$

so as to satisfy the following conditions (A.0) $\sim$ (A.4) for $k \geq 0$:

(A.0) $\mathscr{A}^{(-1)}=\mathscr{S}^{(-1)}=\{1_V(=$ the group consisting of the identity element$)\}$,

$\hat{\#}1_V=F^0GL(V)$, $\#1_V=F^0GL(V)/F^1$.

(A.1) $\hat{G}\in\mathscr{S}^{(k)}\iff\hat{G}/F^k\in\mathscr{A}^{(k-1)}$ and $\hat{G}<\hat{\#}(\hat{G}/F^k)$.

(A.2) $G\in\mathscr{A}^{(k)}\iff G/F^k\in\mathscr{A}^{(k-1)}$ and $G<\#(G/F^k)$.

(A.3)   The map $\hat{\#}:\mathscr{A}^{(k)}\to\mathscr{S}^{(k+1)}$ is defined by

$$\hat{\#}G^{(k)}=\left\{\alpha^{k+1}\in F^0GL(V\oplus\mathfrak{g}^{(k)})\,\middle|\,\begin{array}{l}\alpha^k\in\hat{\#}G^{(k-1)},[\alpha^k]\in G^{(k)}\\\alpha^{k+1}|_{\mathfrak{g}^{(k)}}=Ad([\alpha^k])\end{array}\right\}$$

for $G^{(k)}\in\mathscr{A}^{(k)}$, where we denote by $\mathfrak{g}^{(k)}$ the Lie algebra of $G^{(k)}$ and by α^k the induced map which makes the following diagram commutative:

$$\begin{array}{ccc}V\oplus\mathfrak{g}^{(k)} & \xrightarrow{\alpha^{k+1}} & V\oplus\mathfrak{g}^{(k)}\\\downarrow & & \downarrow\\V\oplus\mathfrak{g}^{(k-1)} & \xrightarrow{\alpha^k} & V\oplus\mathfrak{g}^{(k-1)}\end{array}$$

with $\mathfrak{g}^{(k-1)}$ the Lie algebra of $G^{(k-1)}=G^{(k)}/F^k$, and we denote by $[\alpha^k]$ the equivalence class of α^k mod F^{k+1}, i.e., an element in $\hat{\#}G^{(k-1)}/F^{k+1}(=\#G^{(k-1)})$.

(A.4)   The map $\#:\mathscr{A}^{(k)}\to\mathscr{A}^{(k+1)}$ is defined by $\#=q\circ\hat{\#}$, with $q:\mathscr{S}^{(k+1)}\to\mathscr{A}^{(k+1)}$ given by $q(\hat{G})=\hat{G}/F^{k+2}$ for $\hat{G}\in\mathscr{S}^{(k+1)}$.

To see that the above conditions allow one to determine $\mathscr{S}^{(k)},\mathscr{A}^{(k)}$, $\hat{\#}$, and $\#$ inductively, one has only to check that, once determind $\mathscr{S}^{(i)}$ and $\mathscr{A}^{(i)}(i\leq k)$, $\hat{\#}G^{(k)}$ becomes a filtered Lie subgroup of $F^0GL(V\oplus\mathfrak{g}^{(k)})$ with $\hat{\#}G^{(k)}/F^{k+1}=G^{(k)}$. The verification is easy and left to the reader. Here recall that the filtration of $G^{(k)}$ gives rise to that of $V\oplus\mathfrak{g}^{(k)}$ by

$$F^p(V\oplus\mathfrak{g}^{(k)})=F^pV\oplus F^p\mathfrak{g}^{(k)},$$

so that $F^0GL(V\oplus\mathfrak{g}^{(k)})$ is also filtered in the standard manner.

Notice that we have the following commutative diagram with rows and columms all exact:

$$
\begin{array}{ccc}
0 & & 1 \\
\downarrow & & \downarrow \\
0 \longrightarrow F^{k+2}\mathrm{Hom}(V, E^{(k)}) \longrightarrow F^{k+2}\widehat{\#}G^{(k)} \longrightarrow 1 & & \\
\downarrow & \downarrow & \downarrow \\
0 \longrightarrow F^{k+1}\mathrm{Hom}(V, E^{(k)}) \overset{\iota}{\longrightarrow} \widehat{\#}G^{(k)} \longrightarrow G^{(k)} \longrightarrow 1 & & \\
\downarrow & \downarrow & \downarrow \\
0 \longrightarrow \mathrm{Hom}(grV, grE^{(k)})_{k+1} \longrightarrow \#G^{(k)} \longrightarrow G^{(k)} \longrightarrow 1 & & \\
\downarrow & \downarrow & \downarrow \\
0 & 1 & 1
\end{array}
$$

(2.7)

where we difine ι by $\iota(\alpha)=id_{E^{(k)}}+\alpha$ for $\alpha\in F^{k+1}\mathrm{Hom}(V, E^{(k)})$, identifying $\mathrm{Hom}(V, E^{(k)})$ with a subspace of $\mathrm{Hom}(E^{(k)}, E^{(k)})$.

Now we have:

PROPOSITION 2.2.4. *If (E, G) is a skeleton on V then $G/F^{k+1}\in\mathscr{A}^{(k)}$.*

In fact, assuming that $G^{(k)}(=G/F^{k+1})\in\mathscr{A}^{(k)}$, we see that the image $\widehat{G}^{(k+1)}$ of $G\to F^0GL(V\oplus\mathfrak{g}^{(k)})$ is a filtered Lie subgroup of $\widehat{\#}\,G^{(k)}$ with $\widehat{G}^{(k+1)}/F^{k+1}=G^{(k)}$, so that

$$
G/F^{k+2}=\widehat{G}^{(k+1)}/F^{k+2}\in\mathscr{A}^{(k+1)}.
$$

Therefore the induction shows the proposition.

An important aspect of the mappings $\widehat{\#}, \#$ is that they have functorial properties if we regard $\mathscr{A}^{(k)}$ and $\mathscr{\widehat{A}}^{(k)}$ as categories whose morphisms consist of all adapted homomorphisms defined inductively by the following:

DEFINITION 2.2.2. For $H^{(k)}, G^{(k)}\in\mathscr{A}^{(k)}$, an *adapted homomorphism from $H^{(k)}$ to $G^{(k)}$* is a filtration preserving Lie homomorphism $\varphi^{(k)}: H^{(k)}\to G^{(k)}$ such that

  i) The induced map $\varphi^{(k-1)}: H^{(k)}/F^k \to G^{(k)}/F^k$ is an adapted homomorphism.

  ii) $(\#\varphi^{(k-1)})|_{H^{(k)}}=\varphi^{(k)}$, where $\#\varphi^{(k-1)}$ is given by the proposition below.

DEFINITION 2.2.3. For $\widehat{H}^{(k)}, \widehat{G}^{(k)}\in\mathscr{\widehat{A}}^{(k)}$, an *adapted homomorphism from $\widehat{H}^{(k)}$ to $\widehat{G}^{(k)}$* is a filtration preserving Lie homomorphism $\widehat{\varphi}^{(k)}: \widehat{H}^{(k)}\to \widehat{G}^{(k)}$ such that

i) The induced map $\varphi^{(k-1)}: \bar{H}^{(k)}/F^k \to \widehat{G}^{(k)}/F^k$ is an adapted homomorphism.

ii) $(\widehat{\#}\varphi^{(k-1)})|_{H^{(k)}} = \widehat{\varphi}^{(k)}$, where $\widehat{\#}\varphi^{(k-1)}$ is given by the proposition below.

The inductive definitions above work by virtue of the following:

PROPOSITION 2.2.5. *For an adapted homomorphism* $\varphi^{(k)}: H^{(k)} \to G^{(k)}$ *with* $H^{(k)}$, $G^{(k)} \in \mathscr{A}^{(k)}$, *there exist uniquely filtration preserving Lie homomorphisms*

$$\#\varphi^{(k)}: \#H^{(k)} \to \#G^{(k)}, \quad \widehat{\#}\varphi^{(k)}: \widehat{\#}H^{(k)} \to \widehat{\#}G^{(k)}$$

such that

i) *For* $a^{k+1} \in \widehat{\#}H^{(k)}$, *the following diagram is commutative:*

$$
\begin{array}{ccc}
V \oplus \mathfrak{h}^{(k)} & \xrightarrow{\varphi_*^{(k)}} & V \oplus \mathfrak{g}^{(k)} \\
\downarrow{\scriptstyle a^{k+1}} & & \downarrow{\scriptstyle (\widehat{\#}\varphi^{(k)})(a^{k+1})} \\
V \oplus \mathfrak{h}^{(k)} & \xrightarrow{\varphi_*^{(k)}} & V \oplus \mathfrak{g}^{(k)}
\end{array}
$$

ii) $\#\varphi^{(k)} = \widehat{\#}\varphi^{(k)}/F^{k+2}$.

PROOF: For $a^{k+1} \in \widehat{\#}H^{(k)}$, difine $\beta^{k+1}: V \oplus \mathfrak{g}^{(k)} \to V \oplus \mathfrak{g}^{(k)}$ by

$$\begin{cases} \beta^{k+1}(v) = \varphi_*^{(k)}a^{k+1}(v) & \text{for } v \in V, \\ \beta^{k+1}|_{\mathfrak{g}^{(k)}} = Ad(\varphi^{(k)}(a^k)), \end{cases}$$

where $a^k = a^{k+1}/F^k \in H^{(k)}$. Since $\varphi^{(k)}$ is adapted, it turns out that $\beta^{k+1} \in \widehat{\#}G^{(k)}$. Defining $\widehat{\#}\varphi^{(k)}$ by $\widehat{\#}\varphi^{(k)}(a^{k+1}) = \beta^{k+1}$, we see that $\widehat{\#}\varphi^{(k)}$ has the desired property. The uniqueness is obvious.

REMARK 2.2.3. As easily seen, every adapted homomorphism is necessarily injective.

Now we define $\widehat{G}^{(k)}(V)$ and $G^{(k)}(V)$ ($k=0, 1, 2, \cdots$) inductively by:

$$(2.8) \quad \begin{cases} \widehat{G}^{(0)}(V) = \mathrm{Aut}(V), \quad G^0(V) = \mathrm{Aut}(V)/F^1, \\ \widehat{G}^{(k+1)}(V) = \widehat{\#}G^{(k)}(V), \quad G^{(k+1)}(V) = \#G^{(k)}(V). \end{cases}$$

Here by abuse of notation we use the same symbol $G^{(k)}(V)$ for the one constructed above and the one defined by (2.5), since it soon turns out that both of them are isomorphic.

Then we have:

PROPOSITION 2.2.6.

(1)   *For any* $G^{(k)} \in \mathscr{A}^{(k)}$ *there exists a unique adapted homomorphim*

$$\iota : G^{(k)} \to G^{(k)}(V).$$

(2)   *For any* $\widehat{G}^{(k)} \in \mathscr{\hat{A}}^{(k)}$ *there exists a unique adapted homomorphism*

$$\iota : \widehat{G}^{(k)} \to \widehat{G}^{(k)}(V).$$

PROOF: If $k=0$ the assertion is obvious.   Assuming it valid up to $k-1$, for $G^{(k)} \in \mathscr{A}^{(k)}$ we have

$$G^{(k-1)} \xrightarrow{\iota} G^{(k-1)}(V),$$

where $G^{(k-1)} = G^{(k)}/F^k$.   Then by the definition of $\mathscr{A}^{(k)}$ and by Proposition 2.2.5, we have

$$G^{(k)} \to \# G^{(k-1)} \xrightarrow{\#\iota} \# G^{(k-1)}(V),$$

which proves (1).   The assertion (2) can be proved similarly.

From the construction, we have the following commutative diagram for $\ell > 0$:

$$
\begin{array}{ccc}
\widehat{G}^{(k+\ell)}(V) & \xrightarrow{\;\hat{\pi}\;} & \widehat{G}^{(k)}(V) \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\
G^{(k+\ell)}(V) & \xrightarrow{\;\pi\;} & G^{(k)}(V)
\end{array}
$$

Moreover we have :

PROPOSITION 2.2.7.   *If $\ell \geq \mu - 1$, where μ is the depth of V, then there is a canonical filtration preserving Lie homomorphism*

$$w : G^{(k+\ell)}(V) \to \widehat{G}^{(k)}(V)$$

which makes the following diagram commutative

$$
\begin{array}{ccc}
\widehat{G}^{(k+\ell)}(V) & \xrightarrow{\;\hat{\pi}\;} & \widehat{G}^{(k)}(V) \\
\downarrow & & \| \\
G^{(k+\ell)}(V) & \xrightarrow{\;w\;} & \widehat{G}^{(k)}(V)
\end{array}
$$

PROOF :   Recall that $G^{(k+\ell)}(V)=\widehat{G}^{(k+\ell)}(V)/F^{k+\ell+1}$. But

$$\widehat{\pi}(F^{k+\ell+1}\widehat{G}^{(k+\ell)}(V))\subset F^{k+\ell+1}GL(V\oplus\mathfrak{g}^{(k-1)}),$$

and

$$F^{k+\ell+1}GL(V\oplus\mathfrak{g}^{(k-1)})=1 \quad\text{for}\quad \ell\geq\mu-1.$$

Hence there exists such w.

2.2.3.   We are now in a position to construct a universal skeleton. We set

$$G(V)=\varprojlim G^{(k)}(V), \quad \widehat{G}(V)=\varprojlim \widehat{G}^{(k)}(V).$$

Then $G(V)$ and $\widehat{G}(V)$ become $\mathfrak{p}$-Lie groups, which are canonically isomorphic to each other as immediately seen from Proposition 2.2.7. Since $\widehat{G}^{(k)}(V)\subset F^0GL(V\oplus\mathfrak{g}^{(k-1)}(V))$, by passing to the limit, we obtain a representation

$$G(V)\cong\widehat{G}(V) \rightarrow F^0GL(V\oplus\mathfrak{g}(V)).$$

It is now easy to see that $(E(V), G(V))$ is a skeleton on V, where we put

$$E(V)=V\oplus\mathfrak{g}(V).$$

Note that the induced filtration of $G(V)$ is given by

$$F^pG(V)=\varprojlim_k F^pG^{(k)}(V)$$

and we have

$$G(V)/F^{k+1}\cong G^{(k)}(V)(=\#^k(\mathrm{Aut}(V)/F^1)).$$

To see that $(E(V), G(V))$ has the universal property, we note that if (E, G) is a skeleton on V then $G^{(k)}(=G/F^{k+1})$ belongs to $\mathscr{A}^{(k)}$, so that there is a unique injective adapted homomorphism $G^{(k)} \hookrightarrow G^{(k)}(V)$. This, by passage to limit, gives a canonical embedding of (E, G) into $(E(V), G(V))$. Thus we have proved Theorem 2.2.1.

Now we proceed to the proof of Theorem 2.2.2. For $G^{(k)}\in\mathscr{A}^{(k)}$, we construct a $\mathfrak{p}$-Lie group

$$G(V, G^{(k)})=\varprojlim_\ell \#^\ell G^{(k)},$$

whose Lie algebra will be denoted by $\mathfrak{g}(V, \mathfrak{g}^{(k)})$ as it depends only on the Lie algebra $\mathfrak{g}^{(k)}$ of $G^{(k)}$. Setting

$$E(V, \mathfrak{g}^{(k)})= V\oplus\mathfrak{g}(V, \mathfrak{g}^{(k)}),$$

we obtain a skeleton $(E(V, \mathfrak{g}^{(k)}), G(V, G^{(k)}))$. It is now easy to see that the skeleton $(E(V, \mathfrak{g}^{(k)}), G(V, G^{(k)}))$ has the universal property stated in Theorem 2.2.2. Thus we have proved Theorem 2.2.2.

This, combined with Proposition 2.2.4, also yields:

PROPOSITION 2.2.8. *A Lie subgroup $G^{(k)}$ of $G^{(k)}(V)$ is adapted (that is, there exists a skeleton (E, G) such that $G^{(k)} = G/F^{k+1}$) if and only if $G^{(k)} \in \mathscr{A}^{(k)}$.*

Recalling the definition of $\mathscr{A}^{(k)}$, we have thus:

(1) Any $G^{(0)} < G^{(0)}(V)(= \mathrm{Aut}(V)/F^1)$ is adapted.
(2) $G^{(k)} < G^{(k)}(V)$ is adapted $\Longleftrightarrow$ $G^{(k-1)}(= G^{(k)}/F^k)$ is adapted and $G^{(k)} < \# G^{(k-1)}$.

2.2.4. So far we have described basic properties of the skeletons. Now we will give a few supplements to it.

a) We can formulate the infinitesimal version of the preceding discussion in a quite parallel way. In particular, *an infinitesimal skeleton* on V is defined to be a triple $(E, \mathfrak{g}, \sigma)$, where $\mathfrak{g}$ is a Lie algebra, $E = V \oplus \mathfrak{g}$, and σ is a representation of $\mathfrak{g}$ on E such that the following conditions are satisfied:

i) $\sigma(A)B = ad(A)B$ for A, $B \in \mathfrak{g}$.
ii) $\sigma(A)F^p V \subset F^p V \oplus \mathfrak{g}$ for $A \in \mathfrak{g}$, $p < 0$.
iii) If we define a filtration $\{F^p \mathfrak{g}\}$ of $\mathfrak{g}$ by $F^p \mathfrak{g} = \mathfrak{g}$ for $p \leq 0$ and inductively by the exact sequences for $k \geq 0$:

$$0 \longrightarrow F^{k+1} \mathfrak{g} \longrightarrow \mathfrak{g} \overset{\sigma^{(k)}}{\longrightarrow} F^0 \mathfrak{gl}(E^{(k-1)})/F^{k+1} \quad \text{with } E^{(k-1)} = E/F^k$$

then $\mathfrak{g}$ is isomorphic to $\varprojlim \mathfrak{g}/F^k \mathfrak{g}$.

We can also define the categories of filtered Lie algebras, $\mathfrak{a}^{(k)}$, $\hat{\mathfrak{a}}^{(k)}$ and functors $\#$, $\hat{\#}$ corresponding to (2.6):

$$
\begin{array}{ccc}
\mathfrak{a}^{(k)} & \xrightarrow{\ \hat{\#}\ } & \hat{\mathfrak{a}}^{(k+1)} \\
\Big\| & & \Big\downarrow q \\
\mathfrak{a}^{(k)} & \xrightarrow{\ \#\ } & \mathfrak{a}^{(k+1)}
\end{array}
$$

For $\mathfrak{g}^{(k)} \in \mathfrak{a}^{(k)}$, $\hat{\#}\mathfrak{g}^{(k)}$ is given by:

$$\hat{\#}\mathfrak{g}^{(k)} = \{\hat{a}^{k+1} \in F^0 \mathfrak{gl}(V \oplus \mathfrak{g}^{(k)}) \mid a^k \in \mathfrak{g}^{(k)} \text{ and } \hat{a}^{k+1}|_{\mathfrak{g}^{(k)}} = ad(a^k)\},$$

where a^k denotes the projection of $\widehat{a}^{k+1}$ to $F^0\mathfrak{gl}(V \oplus \mathfrak{g}^{(k+1)})/F^{k+1}$.

We can then identify the Lie algebra of $G(V)$ with $\varprojlim \#^k(F^0\mathfrak{gl}(V)/F^1)$

and denote both by $\mathfrak{g}(V)$. We have also

$$\text{Lie alg}(G(V, G^{(k)})) = \varprojlim_\ell \#^\ell \mathfrak{g}^{(k)}$$

for an adapted subgroup $G^{(k)} \subset G^{(k)}(V)$ with Lie algegra $\mathfrak{g}^{(k)}$. We shall use freely such natural identifications without explicit mention.

b) Let us describe the structure of $(E(V), \mathfrak{g}(V))$ more explicitly.

Let $\mathfrak{u} = \oplus \mathfrak{u}_p$ be a graded vector space with $\mathfrak{u}_p = 0$ for $p < -\mu$ or $p \geq 0$. Define a graded vector space $\mathfrak{e}(\mathfrak{u}) = \oplus \mathfrak{e}_p(\mathfrak{u})$ inductively by setting

$$\begin{cases} \mathfrak{e}_p(\mathfrak{u}) = \mathfrak{u}_p & \text{for } p < 0, \\ \mathfrak{e}_p(\mathfrak{u}) = \text{Hom}(\mathfrak{u}, \oplus_{q<p}\mathfrak{e}_q(\mathfrak{u}))_p & \text{for } p \geq 0, \end{cases}$$

where for graded vector spaces $\mathfrak{u}, \mathfrak{w}$, $\text{Hom}(\mathfrak{u}, \mathfrak{w})_p$ denotes the space of linear maps $f: \mathfrak{u} \to \mathfrak{w}$ such that $f(\mathfrak{u}_i) \subset \mathfrak{w}_{p+i}$, for all i.

Let $\mathfrak{g}(\mathfrak{u}) = \oplus \mathfrak{g}_p(\mathfrak{u})$ be the non-negative part of $\mathfrak{e}(\mathfrak{u})$, i.e.,

$$\mathfrak{g}_p(\mathfrak{u}) = \mathfrak{e}_p(\mathfrak{u}) \quad (p \geq 0), \quad \mathfrak{g}_p(\mathfrak{u}) = 0 \quad (p < 0).$$

By recurrence we see that

$$\mathfrak{g}_k(\mathfrak{u}) = \bigoplus_{\substack{q-p_1-\cdots-p_\ell=k \\ \ell \geq 1, \quad q \geq p_1}} \mathfrak{u}_q \otimes \mathfrak{u}_{p_1}{}^* \otimes \cdots \otimes \mathfrak{u}_{p_\ell}^*.$$

We define a bracket operation:

$$[\ , \ ] : \mathfrak{e}(\mathfrak{u}) \times \mathfrak{e}(\mathfrak{u}) \to \mathfrak{e}(\mathfrak{u})$$

by the following conditions:

i) $[\mathfrak{e}_p(\mathfrak{u}), \mathfrak{e}_q(\mathfrak{u})] \subset \mathfrak{e}_{p+q}(\mathfrak{u})$ for $(p, q) \in Z \times Z$
ii) $[x, y] = -[y, x]$
iii) $[\mathfrak{u}, \mathfrak{u}] = 0$
iv) $[A, v] = A(v)$ for $A \in \mathfrak{g}(\mathfrak{u}), v \in \mathfrak{u}$
v) $\mathfrak{S}[[X, Y], z] = 0$ for $X, Y \in \mathfrak{g}(\mathfrak{u}), z \in \mathfrak{e}(\mathfrak{u})$.

where $\mathfrak{S}$ stands for the cyclie sum.

It is straightforward to see that the bracket operation is uniquely determined by induction. Moreover this makes $\mathfrak{g}(\mathfrak{u})$ a graded Lie algebra and act on $\mathfrak{e}(\mathfrak{u})$ by $AX = [A, X]$ for $A \in \bar{\mathfrak{g}}(\mathfrak{u})$, $X \in \mathfrak{e}(\mathfrak{u})$. Let $\bar{\mathfrak{e}}(\mathfrak{u})$ $\bar{\mathfrak{g}}(\mathfrak{u})$ be the completion of $\mathfrak{e}(\mathfrak{u})$ and $\mathfrak{g}(\mathfrak{u})$ respectively. The we see that $(\bar{\mathfrak{e}}(\mathfrak{u}), \bar{\mathfrak{g}}(\mathfrak{u}))$ is an infinitesimal skeleton on $\mathfrak{u}$ (the graded vector space $\mathfrak{u}$ being regarded as a filtered vector space in the obvious way).

For a filtered vector space V, if we identify grV with V (by taking complementary subspaces) then we have:

PROPOSITION 2. 2. 9.

$$(E(V), \mathfrak{g}(V)) \cong (\bar{\mathfrak{e}}(grV), \bar{\mathfrak{g}}(grV))$$

In fact, by Proposition 2.2.3, $gr\mathfrak{g}(V)$ becomes a graded Lie algebra and acts on $grE(V)$. Moreover, for $A \in gr\mathfrak{g}(V)$ if $AX = 0$ for all $X \in grV$, then $A = 0$. Hence we may regard as

$$gr_k\mathfrak{g}(V) \subset \mathrm{Hom}(grV, grE(V))_k$$

and hence as $gr_k\mathfrak{g}(V) \subset \mathfrak{g}_k(grV)$. But $\bar{\mathfrak{g}}(grV) \subset \mathfrak{g}(V)$ by the universal property. From this we conclude that the two skeletons must coincide.

2. 3. Towers.

We now introduce the notion of tower which will play a central rôle in our study of geometric stuctures.

DEFINITION 2.3.1. Let M be a filtered manifold of type V and let (E, G) be a skeleton on V. A tower on M with skeleton (E, G) is a $\mathfrak{p}$-principal fibre bundle (P, M, G, π) (with total space P, base space M, structure group G, and projection π) equipped with an E-valued 1-form θ having the following properties:

    i)   For all $z \in P$, $\theta_z : T_zP \to E$ is an isomorphism of filtered vector spaces.

    ii) $R_a^* = a^{-1}\theta$ for all $a \in G$.

    iii) $\theta(\tilde{A}) = A$ for all $A \in \mathfrak{g}$.

Here, as usual, R_a denotes the right translation and $\tilde{A}$ the vector field on P induced by the right translations $\{R_{\exp tA}\}$. The filtration of T_zP mentioned above is the natural one given by:

$$\begin{cases} F^pT_zP = \pi_*^{-1}(F^pT_{\pi(z)}M) & \text{for} \quad p \leq 0, \\ F^qT_zP = \{\tilde{A}_z \mid A \in F^q\mathfrak{g}\} & \text{for} \quad q \geq 0. \end{cases}$$

Thus, the tangential filtration of the base space is uniquely determined from the tower by:

$$F^pTM = \pi_* \circ \theta^{-1}(F^pV).$$

Let us define a morphism of towers. In general, by a bundle homomorphism of principal fibre bundles from (P, M, G, π) to (P', M', G', π') we mean a triple $(\varphi, \varphi^{(-1)}, \iota)$ consisting of differential maps $\varphi : P \to P'$, $\varphi^{(-1)} : M \to M'$ and a Lie homomorphism $\iota : G \to G'$ which satisfies:

$$\begin{cases} \varphi^{(-1)} \circ \pi = \pi' \circ \varphi, \\ \varphi(za) = \varphi(z)\iota(a) \qquad \text{for } z \in P, \ a \in G. \end{cases}$$

We may simply write as φ to denote the bundle homomorphism, since $\varphi^{(-1)}$ and ι are uniquely determined by φ.

Now let $(P, M, G\,;\theta)$ and $(P', M', G'\,;\theta')$ be towers on filtered manifolds M and M' respectively of same type V. We call a bundle homomorphism $(\varphi, \varphi^{(-1)}, \iota) : P \to P'$ a morphism of towers if

$$\varphi^* \theta = \iota_* \circ \theta,$$

where ι_* denotes the induced map $\iota_* : E(= V \oplus \mathfrak{g}) \to E'(= V \oplus \mathfrak{g}')$.

A morphism $(\varphi, \varphi^{(-1)}, \iota)$ will be referred to as an isomorphism if φ is a diffeomorphism, and as an embedding if $M = M'$ and $\varphi^{(-1)} = id_M$.

Immediately from the definitions above, we have:

PROPOSITION 2.3.1.   If $(\varphi, \varphi^{(-1)}, \iota) : (P, M, G) \to (P', M', G')$ is a morphism of towers, then

(1) $(\iota_*, \iota) : (E, G) \to (E', G')$ is a morphism of skeletons and hence injective.

(2) $\varphi^{(-1)} : M \to M'$ is a local isomorphism of filtered manifolds.

Now we state the following fundamental:

THEOREM 2.3.1.   For a filtered manifold M of type V there exists a tower $(\mathscr{R}(M), M, G(V), \theta_{\mathscr{R}})$ on M with skeleton $(E(V), G(V))$ which has the following universal property : Any tower on M is uniquely embedded in $\mathscr{R}(M)$.

We shall prove this theorem in the next section by constructing $\mathscr{R}(M)$ explicitly.

We set $\mathscr{R}^{(k)}(M) = \mathscr{R}(M)/F^{k+1}$, the quotient bundle by the action of $F^{k+1}G(V)$. It is a principal $G^{(k)}(V)$-bundle over M, and is referred to as the non-commutative frame bundle of M of (weighted) order $k+1$. (It will soon turn out that $\mathscr{R}^{(0)}(M)$ can be identified with the weighted 1-st order frame bundle of M introduced in Chapter I.) Geometrically, of importance is the following subbundles :

DEFINITION 2.3.2.   We say that a principal subbundle $P^{(k)}$ of $\mathscr{R}^{(k)}(M)$ is adapted if there exists a tower P on M such that $P^{(k)} = P/F^{k+1}$.

We shall also prove the following :

THEOREM 2.3.2.   For an adapted subbundle $P^{(k)}$ of $\mathscr{R}^{(k)}(M)$ with

*structure group $G^{(k)}$ there exists a unique tower $\mathcal{R}P^{(k)}$ with skeleton $(E(V,$
$\mathfrak{g}^{(k)})$, $G(V, G^{(k)}))$ satisfying: $\mathcal{R}P^{(k)}/F^{k+1}=P^{(k)}$ and having the following
universal property: Any tower Q on M such that $Q/F^{k+1}\subset P^{(k)}$ is embed-
ded in $\mathcal{R}P^{(k)}$.*

The tower $\mathcal{R}P^{(k)}$ is called the universal tower prolonging $P^{(k)}$ or the
universal prolongation of $P^{(k)}$.

2.4. Truncated towers.

Truncated towers are those principal fibre bundles which are obtained
by truncating towers up to finite orders. We shall characterize them
rather in a constructive way.

2.4.1. Categories $\widehat{\mathscr{B}}^{(k)}$ and $\mathscr{B}^{(k)}$.

We wish to define categories $\widehat{\mathscr{B}}^{(k)}$ and $\mathscr{B}^{(k)}(k=-1, 0, 1, \cdots)$ consisting
of principal fibre bundles over filtered manifolds whose structure groups
belong to $\widehat{\mathscr{A}}^{(k)}$ and $\mathscr{A}^{(k)}$ respectively and define at the same time functors

$$
\begin{array}{ccc}
\widehat{\mathscr{B}}^{(k)} & \xrightarrow{\ \widehat{\#}\ } & \widehat{\mathscr{B}}^{(k+1)} \\
\Big\| & & \Big\downarrow q \\
\mathscr{B}^{(k)} & \xrightarrow{\ \#\ } & \mathscr{B}^{(k+1)}
\end{array}
$$

(2.9)

by the following requirements (B.0) $\sim$ (B.4) for $k\geq 0$:

(B. 0) $\quad\left\{\begin{array}{l} \mathscr{B}^{(-1)}=\widehat{\mathscr{B}}^{(-1)}=\{\text{all filtered manifolds } M\}, \\ \widehat{\#}M=\widehat{\mathscr{B}}^{(0)}(M); \text{ the bundle of frames of } M \text{ of ordinary order 1,} \\ \#M=\mathscr{B}^{(0)}(M); \text{ the bundle of frames of } M \text{ of weighted order 1.} \end{array}\right.$

(B. 1) $\quad \widehat{P}\in\widehat{\mathscr{B}}^{(k)} \Longleftrightarrow \widehat{P}/F^k\in\widehat{\mathscr{B}}^{(k-1)}$ and $\widehat{P}$ is a subbundle of $\widehat{\#}(P/F^k)$.

(B. 2) $\quad P\in\mathscr{B}^{(k)} \Longleftrightarrow P/F^k\in\mathscr{B}^{(k-1)}$ and P is a subbundle of $\#(P/F^k)$.

(B. 3) $\quad$ The functor $\widehat{\#}: \mathscr{B}^{(k)} \to \widehat{\mathscr{B}}^{(k+1)}$ is defined as follows:

For $(P^{(k)}, G^{(k)})\in\mathscr{B}^{(k)}$, we set $\widehat{\#}P^{(k)}$ to be the set of all filtration
preserving linear isomtrphisms

$$\xi^{k+1}: V\oplus\mathfrak{g}^{(k)} \to T_{x^k}P^{(k)} \quad (x^k\in P^{(k)})$$

such that

   i) $\xi^{k+1}(A)=\widetilde{A}_{x^k}$ for $A\in\mathfrak{g}^{(k)}$,
   ii) $\xi^k\in\widehat{\#}P^{(k-1)}$,
   iii) $[\xi^k]=x^k$,

where we denote by ξ^k the induced map which makes the following diagram commutative:

$$
\begin{array}{ccc}
V \oplus \mathfrak{g}^{(k)} & \xrightarrow{\ \xi^{k+1}\ } & T_{x^k}P^{(k)} \\
\downarrow & & \downarrow{\scriptstyle \pi_*} \\
V \oplus \mathfrak{g}^{(k-1)} & \xrightarrow{\ \xi^k\ } & T_{x^{k-1}}P^{(k-1)}
\end{array}
$$

with $\mathfrak{g}^{(k-1)} = \mathfrak{g}^{(k)}/F^k$, $P^{(k-1)} = P^{(k)}/F^k$, π denoting the natural projection $P^{(k)} \to P^{(k-1)}$, and $x^{k-1} = \pi(x^k)$, and we denote by $[\xi^k]$ the projection of ξ^k to $\sharp P^{(k-1)}(=\widehat{\sharp}P^{(k-1)}/F^{k+1})$.

(B. 4)   The functor $\sharp : \mathscr{B}^{(k)} \to \mathscr{B}^{(k+1)}$ is defined by $\sharp = q \circ \widehat{\sharp}$, with $q : \mathscr{B}^{(k+1)} \to \mathscr{B}^{(k+1)}$ given by $q(\widehat{P}) = \widehat{P}/F^{k+2}$ for $\widehat{P} \in \mathscr{B}^{(k+1)}$.

To see that our inductive definition of $\mathscr{B}^{(k)}$, $\mathscr{B}^{(k)}$, $\widehat{\sharp}$, and $\sharp$ are well-defined, let us show that $\widehat{\sharp}P^{(k)}$ becomes a principal fibre bundle over M with structure group $\widehat{\sharp}G^{(k)}$ and that $\widehat{\sharp}P^{(k)}/F^{k+1} = P^{(k)}$. Let $\pi^k_{k+1} : \widehat{\sharp}P^{(k)} \to P^{(k)}$ be the map which sends ξ^{k+1} to $[\xi^k]$, and let $\pi_{k+1} = \pi_k \cdot \pi^k_{k+1}$, with $\pi_k : P^{(k)} \to M$ the natural projection. First of all, note that π^k_{k+1} is surjective. In fact given an $x^k \in P^{(k)}$; since $P^{(k)} \subset \sharp P^{(k-1)} = \widehat{\sharp}P^{(k-1)}/F^{k+1}$, there exists $\xi^k \in \widehat{\sharp}P^{(k-1)}$ such that $[\xi^k] = x^k$. Then take a lift ξ^{k+1} of ξ^k satisfying (B. 3) i), we see then $\xi^{k+1} \in \widehat{\sharp}P^{(k)}$ and $\pi^k_{k+1}(\xi^{k+1}) = x^k$. Thus we see, in particular $\pi_{k+1} : \widehat{\sharp}P^{(k)} \to M$ is surjective.

Next let us define the right action of $\widehat{\sharp}G^{(k)}$ on $\widehat{\sharp}P^{(k)}$. Let $\xi^{k+1} \in \widehat{\sharp}P^{(k)}$, $a^{k+1} \in \widehat{\sharp}G^{(k)}$ and denote their projections to $P^{(k)}$ and $G^{(k)}$ respectively by x^k and a^k. We define $\xi^{k+1} \cdot a^{k+1}$ by the following commutative diagram:

(2. 10)
$$
\begin{array}{ccc}
V \oplus \mathfrak{g}^{(k)} & \xrightarrow{\ \xi^{k+1}\ } & T_{x^k}P^{(k)} \\
\uparrow{\scriptstyle a^{k+1}} & & \downarrow{\scriptstyle (R_{a^k})_*} \\
V \oplus \mathfrak{g}^{(k)} & \xrightarrow{\ \xi^{k+1} \cdot a^{k+1}\ } & T_{x^k a^k}P^{(k)}.
\end{array}
$$

Assuming that for $\ell < k$ the action of $\widehat{\sharp}G^{(\ell)}$ on $\widehat{\sharp}P^{(\ell)}$ is defined in this way, we can easily verify that $\xi^{k+1} \cdot a^{k+1} \in \widehat{\sharp}P^{(k)}$ and that this indeed defines the action of $\widehat{\sharp}G^{(k)}$ on $\widehat{\sharp}P^{(k)}$.

Thus $\widehat{\sharp}P^{(k)}$ is a principal fibre bundle over M with structure group $\widehat{\sharp}G^{(k)}$, and $\sharp P^{(k)}$ is a principal fibre bundle over M with structure group $\sharp G^{(k)}$.

PROPOSITION 2.4.1.   *If* P *is a tower then the quotient bundle* P/F^{k+1} *belongs to* $\mathscr{B}^{(k)}$ *for each* $k \geq -1$.

PROOF: Write $P^{(k)} = P/F^{k+1}$. We proceed by induction. Assume that $P^{(k-1)} \in \mathscr{B}^{(k-1)}$. It suffices to show that there exists a canonical embedding $P^{(k)} \to \# P^{(k-1)}$. For each $z \in P$ define $\xi^{(k)}$ by the following commutative diagram

$$(2.11)$$

$$
\begin{array}{ccc}
T_z P & \xrightarrow{\ \theta\ } & E \\
\downarrow & & \downarrow \\
T_{z^{k-1}} P^{(k-1)} & \xleftarrow{\ \xi^k\ } & E^{(k-1)}
\end{array}
$$

then the assignment $z \to \xi^k$ yields a bundle map $P \to \widehat{\#} P^{(k-1)}$, which in turn by passage to quotient gives an embedding $P^{(k)} \to \# P^{(k-1)}$.

2.4.2.   Morphisms of $\mathscr{B}_V^{(k)}$ and $\widehat{\mathscr{B}}_V^{(k)}$.

We shall denote by $\mathscr{B}_V^{(k)}$ and $\widehat{\mathscr{B}}_V^{(k)}$ or $\mathscr{B}_M^{(k)}$ and $\widehat{\mathscr{B}}_M^{(k)}$ the subcategories of $\mathscr{B}^{(k)}$ and $\widehat{\mathscr{B}}^{(k)}$ consisting of bundles whose base spaces are filtered manifolds of type V or coincide with a filtered manifold M respectively. We define the notion of adapted bundle homomorphism by the following two definitions and the subsequent proposition.

DEFINITION 2.4.1.   For $P^{(k)}$, $P'^{(k)} \in \mathscr{B}_V^{(k)}$ a bundle homomorphism $\varphi^{(k)} : P^{(k)} \to P'^{(k)}$ is called *adapted* if and only if

  i) $\varphi^{(k-1)}(=\varphi^{(k)}/F^k) : P^{(k-1)}(=P^{(k)}/F^k) \to P'^{(k-1)}$ is adapted,
  ii) $(\#\varphi^{(k-1)})|_{P^{(k)}} = \varphi^{(k)}$.

DEFINITION 2.4.2.   For $\widehat{P}^{(k)}$, $\widehat{P}'^{(k)} \in \widehat{\mathscr{B}}_V^{(k)}$ a bundle homomorphism $\widehat{\varphi}^{(k)} : \widehat{P}^{(k)} \to \widehat{P}'^{(k)}$ is called *adapted* if and only if

  i) $\varphi^{(k-1)}(=\widehat{\varphi}^{(k)}/F^k) : P^{(k-1)}(=\widehat{P}^{(k)}/F^k) \to P'^{(k-1)}$ is adapted,
  ii) $(\widehat{\#}\varphi^{(k-1)})|_{\widehat{P}^{(k)}} = \widehat{\varphi}^{(k)}$.

Here we promise that an adapted bundle homomorphism $\varphi : M \to M'$ be a local isomorphism of filtered manifolds.

PROPOSITION 2.4.2.   *For an adapted bundle homomorphism* $\varphi^{(k)} : P^{(k)} \to P'^{(k)}$ *with* $P^{(k)}$, $P'^{(k)} \in \mathscr{B}_V^{(k)}$ *there exist unique adapted bundle homomorphisms*

$$
\# \varphi^{(k)} : \# P^{(k)} \to \# P'^{(k)}, \quad \widehat{\#} \varphi^{(k)} : \widehat{\#} P^{(k)} \to \widehat{\#} P'^{(k)}
$$

satisfying :

i) *For* $\xi^{k+1} \in \widehat{\#} P^{(k)}$ *the following diagram is commutative* :

$$
\begin{array}{ccc}
V \oplus \mathfrak{g}^{(k)} & \xrightarrow{\iota_*} & V \oplus \mathfrak{g}'^{(k)} \\
\downarrow{\scriptstyle \xi^{k+1}} & & \downarrow{\scriptstyle \widehat{\#}\varphi^{(k)}(\xi^{k+1})} \\
T_{x^k} P^{(k)} & \xrightarrow{\varphi_*^{(k)}} & T_{x'^k} P'^{(k)}
\end{array}
$$

with $x^k = \xi^{k+1} / F^{k+1}$, $x'^k = \varphi^{(k)}(x^k)$,

ii) $\#\varphi^{(k)} = \widehat{\#}\varphi^{(k)} / F^{k+2}$.

To see that the above inductive definitions is well-defined, it suffices to construct $\widehat{\#}\varphi^{(k)} : \widehat{\#} P^{(k)} \rightarrow \widehat{\#} P'^{(k)}$. For $\xi^{k+1} \in \widehat{\#} P^{(k)}$ we define $\widehat{\#}\varphi^{(k)}(\xi^{k+1})$ by

$$
\widehat{\#}\varphi^{(k)}(\xi^{k+1})(v) = \varphi_*^{(k)} \xi^{k+1}(v) \quad \text{for } v \in V
$$

and

$$
\widehat{\#}\varphi^{(k)}(\xi^{k+1})(A) = \tilde{A}_{x'^k} \quad \text{for } A \in \mathfrak{g}'^{(k)}.
$$

Them, taking account of the "adaptedness" of $\varphi^{(k)}$ and using the induction assumption, we see that $\widehat{\#}\varphi^{(k)}(\xi^{k+1}) \in \widehat{\#} P'^{(k)}$ and that $\widehat{\#}\varphi^{(k)}$ is also an adapted bundle homomorphism with $\widehat{\#}\varphi^{(k)} / F^{k+1} = \varphi^{(k)}$.

It should be remarked that if $(\varphi^{(k)}, \varphi^{(-1)}, \iota) : (P^{(k)}, M, G^{(k)}) \rightarrow (P'^{(k)}, M', G'^{(k)})$ is an adapted homomorphim of $\mathscr{B}_V^{(k)}$ then $\iota : G^{(k)} \rightarrow G'^{(k)}$ is an adapted homomorphism and $\varphi^{(-1)} : M \rightarrow M'$ is a local isomorphism of filtered manifolds. The same remark applies to the adapted homomorphisms of $\mathscr{B}_V^{(k)}$.

An adapted bundle homomorphism will be simply referred to as a morphism. If $\varphi^{(k)}$ is a diffeomorphism it is called an isomorphism, and if $M = M'$ and if $\varphi^{(-1)} = id$ it is called an embedding.

2. 4. 3.   The canonical form $\theta^{(k-1)}$ of $\widehat{P}^{(k)}$.

Let $(\widehat{P}^{(k)}, N, \widehat{G}^{(k)}) \in \mathscr{B}^{(k)}$ with $k \geq 0$. Since $\widehat{P}^{(k)} \subset \widehat{\#} P^{(k-1)}$ with $P^{(k-1)} = \widehat{P}^{(k)} / F^k$, $\widehat{P}^{(k)}$ may be regarded as a subbundle of the linear frame bundle of $P^{(k-1)}$, so that we can define the canonical form $\theta^{(k-1)}$ of $\widehat{P}^{(k)}$: Let $\mathfrak{g}^{(k-1)}$ denote the Lie algebra of $G^{(k-1)}(=\widehat{G}^{(k)} / F^k)$ and put $E^{(k-1)} = V \oplus \mathfrak{g}^{(k-1)}$. Define an $E^{(k-1)}$-valued 1-form $\theta^{(k-1)}$ on $\widehat{P}^{(k)}$ by the following commutative diagram:

$$T_{\xi^k} \widehat{P}^{(k)} \xrightarrow{\ \theta^{(k-1)}\ } E^{(k-1)}$$

$$T_{x^{k-1}} P^{(k-1)} \xleftarrow{\ \xi^k\ } E^{(k-1)}$$

for $\xi^k \in \widehat{P}^{(k)}$ with $x^{k-1} = \xi^k / F^k$.

Then we have easily :

PROPOSITION 2. 4. 3.

(1) $\theta^{(k-1)}(\widetilde{A}) = A / F^k$ *for* $A \in \widehat{\mathfrak{g}}^{(k)}$.

(2) $R_a^* \theta^{(k-1)} = a^{-1} \theta^{(k-1)}$ *for* $a \in \widehat{G}^{(k)}$.

(3) *If* $\varphi : \widehat{P}^{(k)} \to \widehat{P}'^{(k)}$ *is a morphism of* $\mathscr{B}_V^{(k)}$, *then* $\varphi^* \theta'^{(k-1)} = \theta^{(k-1)}$, *where* $\theta'^{(k-1)}$ *denotes the canonical form of* $P'^{(k)}$.

(4) *For* $\widehat{P}^{(k)}, \widehat{P}'^{(k)} \in \mathscr{B}_V^{(k)}$ *if a bundle homomorphism* $\varphi : \widehat{P}^{(k)} \to \widehat{P}'^{(k)}$ *satisfies* : $\varphi^* \theta'^{(k-1)} = \theta^{(k-1)}$, *then* φ *is adapted, that is a morphism of* $\mathscr{B}^{(k)}$.

2. 4. 4. $\mathscr{P}^{(k)}(M)$ *and* $\mathscr{\widehat{P}}^{(k)}(M)$.

We set

$$\mathscr{P}^{(k)}(M) = \# \mathscr{P}^{(k-1)}(M) (= \#^k \mathscr{P}^{(0)}(M) = \#^{k+1}(M))$$
$$\mathscr{\widehat{P}}^{(k)}(M) = \widehat{\#} \mathscr{P}^{(k-1)}(M).$$

Then it is straightforward to see the following :

PROPOSITION 2. 4. 4.

(1) *If* $P^{(k)} \in \mathscr{B}_M^{(k)}$, *then there exists a canonical embedding*
$$P^{(k)} \hookrightarrow \mathscr{P}^{(k)}(M).$$

(2) *If* $\widehat{P}^{(k)} \in \mathscr{B}_M^{(k)}$, *then there exists a canonical embedding*
$$\widehat{P}^{(k)} \hookrightarrow \mathscr{\widehat{P}}^{(k)}(M).$$

On the other hand we have :

PROPOSITION 2. 4. 5.   *If the depth of* M *is* μ, *there is a natural bundle homomrphism*

$$w : \mathscr{P}^{(k+\mu-1)}(M) \to \mathscr{\widehat{P}}^{(k)}(M)$$

which makes the following diagram commutative :

$$\begin{array}{ccc} \mathscr{\hat{R}}^{(k+\mu-1)}(M) & \xrightarrow{\ \hat{\pi}\ } & \mathscr{\hat{R}}^{(k)}(M) \\ \downarrow & & \| \\ \mathscr{R}^{(k+\mu-1)}(M) & \xrightarrow{\ w\ } & \mathscr{\hat{R}}^{(k)}(M) \end{array}$$

The proof is similar to that of Proposition 2. 2. 7.

We are now in a position to prove Theorem 2. 3. 1 and Theorem 2. 3. 2. We put

$$\mathscr{R}(M) = \varprojlim_{k} \mathscr{R}^{(k)}(M).$$

Then we see that $\mathscr{R}(M)$ is a $\mathfrak{p}$-principal fibre bundle over M with strucure group $G(V)$. Since we have the natural projection

$$\mathscr{R}(M) \to \mathscr{\hat{R}}^{(k+1)}(M),$$

we have a $E^{(k)}(V)$-valued 1-form $\theta_{\mathscr{R}}^{(k)}$ on $\mathscr{R}(M)$, the pull-back of the canonical form of $\mathscr{\hat{R}}^{(k+1)}(M)$. Clearly we have, for $k > \ell$,

$$\theta_{\mathscr{R}}^{(\ell)} = \theta_{\mathscr{R}}^{(k)}/F^{\ell+1},$$

where $\cdot/F^{\ell+1}$ denotes the projection $E^{(k)} \to E^{(\ell)}$. Passing to limit, we obtain an $E(V)$ valued 1-form:

$$\theta_{\mathscr{R}} = \varprojlim \theta_{\mathscr{R}}^{(k)}.$$

As easily seen from Proposition 2. 4. 3, this form $\theta_{\mathscr{R}}$ satisfies the conditions i) ~ iii) of Definition 2. 3. 1. Thus $(\mathscr{R}(M), G(V), \theta_{\mathscr{R}})$ is a tower. It is now immediate to see the universal property of $\mathscr{R}(M)$. In fact, if (P, M, G, θ) is a tower, then $P^{(k)} \in \mathscr{R}^{(k)}$ and is embedded in $\mathscr{R}^{(k)}(M)$. By passing to projective limit, P is embedded in $\mathscr{R}(M)$ as a principal subbundle. Moreover this embedding is uniquely determined by the condition $\iota^{*}\theta_{\mathscr{R}} = \theta$, which proves Theorem 2. 3. 1.

To prove Theorem 2. 3. 2, given $P^{(k)} \in \mathscr{R}_{M}^{(k)}$, we set

$$\mathscr{R}P^{(k)} = \varprojlim_{\ell} \#^{\ell} P^{(k)}.$$

It is immediate to see that $\mathscr{R}P^{(k)}$ is a tower with skeleton $G(V, G^{(k)})$ having the universal property, which completes the proof.

We have also shown

PROPOSITION 2. 4. 6.   *A principal subbundle $P^{(k)}$ is adapted if and only if $P^{(k)} \in \mathscr{B}^{(k)}$.*

The preceding discussion also yields:

PROPOSITION 2. 4. 7.

(1)   *For a morphism $\varphi^{(k)} : P^{(k)} \to P'^{(k)}$ there is a morphism of towers $\mathscr{R}\,\varphi^{(k)} : \mathscr{R}P^{(k)} \to \mathscr{R}P'^{(k)}$ such that $\mathscr{R}\,\varphi^{(k)}/F^{k+1} = \varphi^{(k)}$.*

(2)   *A morphism of towers $\varphi : P \to P'$ induces a morphism of $\mathscr{B}^{(k)}$, $\varphi^{(k)} : P^{(k)} \to P'^{(k)}$ and $\varphi = \mathscr{R}\,\varphi^{(k)}|_P$.*

PROOF: Set $\mathscr{R}\,\varphi^{(k)} = \varprojlim_l \#^l \varphi^{(k)}$.   Then the assertions are clear.

To close this section we notice one of the implications of Theorem 2. 3. 1.   Let L/G be a homogeneous space and suppose that the linear isotropy representation of G on $V = \mathfrak{l}/\mathfrak{g}$ leaves invariant a filtration F of V.   By left translation it then defines a left invariant tangential filtration F of $M = L/G$.   Now assume that $(\mathfrak{l}, G)$ is formally effective.   Then, as seen in section 2.2.1, $(\mathfrak{l}, G)$ becomes a skeleton on $V = (V, F)$ by a choice of identification $\mathfrak{l} = V \oplus \mathfrak{g}$.   It is then clear that (L, M, G, θ_L), with θ_L the Maurer-Cartan form of L, is a tower on M with skeleton $(\mathfrak{l}, G)$.   Hence, by Theorem 2.3.1, we have a unique embedding $\iota : L(M, G) \to \mathscr{R}(M)$ such that $\iota^* \theta_{\mathscr{R}(M)} = \theta_L$.   This universal property of $\mathscr{R}(M)$ holds not only for the homogeneous spaces but also for the Cartan connections (see § 3. 10).

Chapter III. Equivalence problems

3. 0.   Equivalence problems.

By a geometric structure on a filtered manifold M of order $k+1$ we shall mean an adapted subbundle $P^{(k)}$ of $\mathscr{R}^{(k)}(M)$.   (When we deal with intransitive geometric structures in section 3. 8, we need to consider generalized subbundles.)

Two geometric structures $(P^{(k)}, M, \pi)$ and $(P'^{(k)}, M', \pi')$ are said to be isomorphic (or equivalent) if there exists an isomorphism $\varphi^{(k)} : P^{(k)} \to P'^{(k)}$ of adapted subbundles.   In view of Definition 2. 4. 1, this is equivalent to saying that there exists an isomorphism $f : M \to M'$ of filtered manifolds such that the lift $\#^{k+1} f : \mathscr{R}^{(k)}(M) \to \mathscr{R}^{(k)}(M')$ sends $P^{(k)}$ onto $P'^{(k)}$.

We say that $P^{(k)}$ and $P'^{(k)}$ are locally isomorphic (or locally equivalent) at $(x, x') \in M \times M'$ if there exist neighbourhoods U, U' of x, x'

respectively and an isomorphism of filtered manifolds $f: U \to U'$ such that $f(x)=x'$ and that

$$\#^{k+1} f(P^{(k)}|_U) = P'^{(k)}|_{U'}.$$

In this chapter we shall address ourselves to the so-called equivalence problem:

"Given two geometric structures on filtered manifolds. Determine whether or not they are equivalent (or locally equivalent)."

The first half of this problem is to obtain necessary conditions:

"Given a geometric structure $P^{(k)}$. Find invariants (if possible the complete set of invariants) of $P^{(k)}$."

To treat these problems, our leading principle may be represented symbolically by the following tautology:

   geometric structure (of order $k+1$)
   =   adapted subbundle of $\mathscr{R}^{(k)}$
   =   truncated tower.

Geometric structures are not complete in a sense, especially from a group theoretical view point, and it is towers that we consider more complete. Therefore in studying a geometric structure $P^{(k)}$, we always keep in mind the towers P which complete it, that is $P/F^{k+1}=P^{(k)}$. Our main task is then to seek or construct a tower P which has as simple structure as can be among the towers completing the given geometric structure.

3.1. Structure functions.

3.1.1. We first introduce the structure function of a tower. Let (P, M, G, θ) be a tower on a filtered manifold M with skeleton (E, G). Since θ defines an absolute parallelism on P, there exists a unique $\mathrm{Hom}(\wedge^2 E, E)$-valued function γ on P which satisfies the following structure equation:

$$(3.1) \qquad\qquad d\theta + \frac{1}{2}\gamma(\theta, \theta) = 0.$$

This function γ is referred to as the structure function of the the tower P, and has the following properties:

PROPOSITION 3.1.1. *Let γ be the structure function of a tower (P, M, G, θ). Then*

  (1) $\gamma(z)(A, X)=A \cdot X$ *for* $z \in P$, $A \in \mathfrak{g}$, $X \in E$.

  (2) $\gamma(za)(X, Y)=a^{-1}\gamma(z)(aX, aY)$ *for* $z \in P$, $a \in G$, X, $Y \in E$.

  (3) *If* $\varphi: P \to P'$ *is a morphism of towers then* $\varphi^*\gamma' = \gamma$, *where* γ' *denotes the structure function of* P'.

PROOF : Since $R_a^*\theta=a^{-1}\theta$, we have

$$(3.2) \qquad L_{\tilde{A}}\theta=-A\cdot\theta \qquad \text{for} \quad A\in\mathfrak{g},$$

where $L_{\tilde{A}}$ stands for the Lie derivative with respect to $\tilde{A}$. Then we have

$$\tilde{A}\,\lrcorner\,d\theta=-d(\tilde{A}\,\lrcorner\,\theta)+L_{\tilde{A}}\theta=-A\cdot\theta,$$

from which follows (1). Applying R_a^* to (3.1), we obtain (2) immediately. The assertion (3) is obvious since $\varphi^*\theta'=\theta$.

If we denote ρ the natural representation of G on $\operatorname{Hom}(\wedge^2 E, E)$ defined by

$$(3.3) \qquad (\rho(a)\alpha)(X, Y)=a\alpha(a^{-1}X, a^{-1}Y)$$

for $a\in G$, $\alpha\in\operatorname{Hom}(\wedge^2 E, E)$, $X, Y\in E$, then the above formula (2) is written as

$$(3.4) \qquad R_a^*\gamma=\rho(a)^{-1}\gamma.$$

Since $E=V\oplus\mathfrak{g}$, we have the direct sum decomposition :

$$(3.5) \qquad \operatorname{Hom}(\wedge^2 E, E)=\operatorname{Hom}(\wedge^2 V, E)\oplus\operatorname{Hom}(\mathfrak{g}\otimes E, E),$$

where we identify $\operatorname{Hom}(\wedge^2 V, E)$ with the subspace of $\operatorname{Hom}(\wedge^2 E, E)$ consisting of all α such that $\alpha(A, X)=0$ for $A\in\mathfrak{g}$, $X\in E$, and also $\operatorname{Hom}(\mathfrak{g}\oplus E, E)$ similarly.

Note that $\operatorname{Hom}(\wedge^2 V, E)$ is a G-invariant subspace of $\operatorname{Hom}(\wedge^2 E, E)$, while $\operatorname{Hom}(\mathfrak{g}\otimes E, E)$ is not. Let β denote the element of $\operatorname{Hom}(\mathfrak{g}\otimes E, E)$ given by the action of $\mathfrak{g}$ on E :

$$(3.6) \qquad \beta(A, X)=A\cdot X \qquad \text{for} \quad A\in\mathfrak{g}, \ X\in E.$$

Then, by Proposition 2.2.1, we have

$$(3.7) \qquad (\rho(a)\beta)(A, X)=\beta(A, X) \qquad \text{for} \quad a\in G, \ A\in\mathfrak{g}, \ X\in E,$$

which means that the action of G on $\operatorname{Hom}(\wedge^2 E, E)/\operatorname{Hom}(\wedge^2 V, E)$ leaves fixed the equivalence class of β mod $\operatorname{Hom}(\wedge^2 V, E)$. Hence the representation ρ induces the affine representation of G on the affine subspace $\beta+\operatorname{Hom}(\wedge^2 V, E)$.

Thus the structure function γ is a G-equivariant map from P to the affine space $\beta+\operatorname{Hom}(\wedge^2 V, E)$ on which G acts as affine transformations. The $\operatorname{Hom}(\wedge^2 V, E)$-valued function c given by

$$(3.8) \qquad \gamma=\beta+c$$

is therefore the crucial part of γ and also called the structure function of P.

3. 1. 2.   We next introduce the structure function of a truncated tower. Let $(P^{(k)}, M, G^{(k)})$ be a truncated tower, that is, an adapted subbundle of $\mathscr{R}^{(k)}(M)$. Let (P, M, G, θ) be any tower prolonging $P^{(k)}$, i.e., $P/F^{k+1} = P^{(k)}$, and γ its structure function. Now recall that $\mathrm{Hom}(\wedge^2 E, E)$ has the natural filtration defined by

(3. 9) $a \in F^k \mathrm{Hom}(\wedge^2 E, E) \Longleftrightarrow a(F^p E \wedge F^q E) \subset F^{p+q+k} E$ $\forall p, q.$

First of all we have the following:

PROPOSITION 3. 1. 2.   *The structure function of a tower takes its values in $F^0 Hom(\wedge^2 E, E)$.*

PROOF: Let (P, M, G, θ) be a tower with structure function γ. By Proposition 3. 1. 1 (1)   and Proposition 2. 2. 3 (2), it suffices to show

(3. 10) $\gamma(z)(F^p E \otimes F^q E) \subset F^{p+q} E$

for $p, q < 0$ and $z \in P$.

Let $Pr^{(k)}$ denote the projection $E \to E^{(k)}$. Note that the V-valued form $Pr^{(-1)} \circ \theta$, which will be denoted by θ_V, may be regarded as a form on $P^{(\mu-1)}$, where μ is the depth of M, Let $x, z^{\mu-1}$ be the projections of z to M and $P^{(\mu-1)}$ respectively. Let σ be a local cross-section of $P^{(\mu-1)} \to M$ around x with $\sigma(x) = z^{\mu-1}$. Since $\sigma^* \theta_V$ defines an absolute parallelism of M around x, we define for $u \in V$ a local vector field $\tilde{u}$ by $<\sigma^* \theta, \tilde{u}> = u$. Then we have

(3. 11) $Pr^{(l)} \gamma(z)(u, v) = Pr^{(l)}(\sigma^* \theta_V)([\tilde{u}, \tilde{v}]_x)$

for $u, v \in V, l < 0.$

In fact, taking any $X, Y \in T_z P$ such that $\theta_V(X) = u, \theta_V(Y) = v$, we have

$$Pr^{(l)} \gamma(z)(u, v) = Pr^{(l)} \gamma(z)(\theta(X), \theta(Y))$$
$$= -Pr^{(l)} d\theta(X, Y)$$
$$= -Pr^{(l)} d\theta_V(\sigma_* \tilde{u}_x, \sigma_* \tilde{v}_x)$$
$$= Pr^{(l)}(\sigma^* \theta_V)([\tilde{u}, \tilde{v}]_x).$$

Now suppose that $u \in F^p V, v \in F^q$, then $[\tilde{u}, \tilde{v}]$ is a section of $T^{(p+q)} M$ by our basic assumption $[F^p TM, F^q TM] \subset F^{p+q} TM$. Hence we have

$$Pr^{(p+q-1)} \gamma(z)(u, v) = Pr^{(p+q-1)}(\sigma^* \theta_V)([\tilde{u}, \tilde{v}_x]) = 0,$$

which proves (3. 9), and hence the proposition.   q.e.d.

Now we set

$$\mathrm{Hom}(\wedge^2 E, E)^{(k)} = F^0 \mathrm{Hom}(\wedge^2 E, E)/F^{k+1}$$

and denote $\gamma^{(k)}$ the projectin of γ to this space. Since $F^{k+1}\mathrm{Hom}(\wedge^2 E, E)$ is obviously a G-invariant subspace, we have the induced representation $\rho^{(k)}$ of G on $\mathrm{Hom}(\wedge^2 E, E)^{(k)}$.

And we have:

$$\gamma^{(k)}(z \cdot a) = \rho^{(k)}(a^{-1})\gamma^{(k)}(z) \qquad \text{for} \quad a \in G, \ z \in P.$$

On the other hand, it is easy to see that

$$\rho^{(k)}(a) = 1 \qquad \text{for} \quad a \in F^{k+1}G.$$

Thus $\gamma^{(k)}$ may be regarded as a function on $P^{(k)}(=P/F^{k+1}G)$. By Proposition 3. 1. 1 (3), this $\gamma^{(k)}$ does not depend on the choice of the tower P prolonging $P^{(k)}$. Thus we have defined a mapping:

(3. 12) $$\gamma^{(k)} : P^{(k)} \to \mathrm{Hom}(\wedge^2 E, E)^{(k)}.$$

According to the decomposition; $\gamma = \beta + c$, we can also decompose

(3. 13) $$\gamma^{(k)} = \beta^{(k)} + c^{(k)},$$

where $\beta^{(k)}$ and $c^{(k)}$ take their values in $F^0\mathrm{Hom}(\mathfrak{g} \otimes E, E)/F^{k+1}$ and $F^0\mathrm{Hom}(\wedge^2 V, E)/F^{k+1}$ respectively. Note that

(3. 14) $$F^0\mathrm{Hom}(\wedge^2 V, E)/F^{k+1} = F^0\mathrm{Hom}(\wedge^2 V, E^{k+1})/F^{k+1}.$$

Therefore the crucial component $c^{(k)}$ is expressed only in terms of $E^{(k-1)}$. But the expression of $\beta^{(k)}$ involves the prolonged space E.

To adjust this, let us consider another filtration $\{I^k\mathrm{Hom}(\wedge^2 E, E)\}$ of $\mathrm{Hom}(\wedge^2 E, E)$. We employ the following notation: For $p \in \mathbf{Z}$, we set

(3. 15) $$p^* = \begin{cases} p & \text{for} \quad p < 0 \\ 0 & \text{for} \quad p \geq 0. \end{cases}$$

We then define:

(3. 16) $$a \in I^k\mathrm{Hom}(\wedge^2 E, E) \iff a(F^p E \wedge F^p E) \subset F^{p^* + q^* + k}E \qquad \forall p, q \in \mathbf{Z},$$

and we put

(3. 17) $$\mathrm{Hom}(\wedge^2 E, E)^{[k]} = F^0\mathrm{Hom}(\wedge^2 E, E)/I^{k+1}.$$

We denote by $\gamma^{[k]}$ the projection of γ to this space. Clearly $\gamma^{[k]}$ may be regarded as a function on $P^{(k)}$. Moreover we have the induced representation $\rho^{[k]}$ of $G^{(k)}$ on $\mathrm{Hom}(\wedge^2 E, E)^{[k]}$ and we have

(3. 18) $$\gamma^{[k]}(za) = \rho^{[k]}(a^{-1})\gamma^{[k]}(z) \qquad \text{for} \quad z \in P^{(k)}, \ a \in G^{(k)}.$$

Note that $\gamma^{[k]}$ is described in terms of $E^{(k)}$, since

(3. 19) $\operatorname{Hom}(\wedge^2 E, E)^{[k]} \cong F^0 \operatorname{Hom}(\wedge^2 E^{(k)}, E^{(k)})/I^{k+1}$.

Moreover we can decompose $\gamma^{[k]}$ as

$$\gamma^{[k]} = \beta^{[k]} + c^{(k)},$$

where $c^{(k)}$ is the $F^0 \operatorname{Hom}(\wedge^2 V, E^{(k-1)})/F^{k+1}$-component defined in (3. 13), and $\beta^{[k]}$ is the projection of β to $\operatorname{Hom}(\wedge^2 E, E)^{[k]}$.

The function $\gamma^{[k]}$, as well as $\gamma^{(k)}$ and $c^{(k)}$, will be referred to as the structure function of $P^{(k)}$. Summarizing the above discussion, we have:

PROPOSITION 3. 1. 3.   *The structure function $r^{[k]}$ of a truncated tower $(P^{(k)}, M, G^{(k)})$ is a $G^{(k)}$-equivariant map*

(3. 20) $\gamma^{[k]} : P^{(k)} \to \operatorname{Hom}(\wedge^2 E, E)^{[k]} \cong F^0 \operatorname{Hom}(\wedge^2 E^{(k)}, E^{(k)})/I^{k+1}$,

and if $\varphi^{(k)} : P^{(k)} \to P'^{(k)}$ is an adapted homomorphism then $(\varphi^{(k)})^* \gamma'^{[k]} = \gamma^{[k]}$.

In practice, the structure function $\gamma^{[k]}$ can be computed as follows: Consider the prolongation $\sharp P^{(k)}$, and choose a local cross-section σ of $\sharp P^{(k)} \to P^{(k)}$. Let $\theta^{(k)}$ be the canonical form of $\sharp P^{(k)}$ (which is an $E^{(k)}$-valued 1-form), then $\sigma^* \theta^{(k)}$ locally defines an absolute parallelism of $P^{(k)}$. Define an $F^0 \operatorname{Hom}(\wedge^2 E^{(k)}, E^{(k)})$-valued function Γ by

(3. 21) $d\sigma^* \theta^{(k)} + \dfrac{1}{2} \Gamma(\sigma^* \theta^{(k)}, \sigma^* \theta^{(k)}) = 0$,

then from our preceding discussion it is easy to see that the projection of Γ to $F^0 \operatorname{Hom}(\wedge^2 E^{(k)}, E^{(k)})/I^{k+1}$ just yields the structure function $\gamma^{[k]}$.

As an illustration, let us examine the structure function $\gamma^{[0]} = \beta^{[0]} + c^{(0)}$ of a first order truncated tower $(P^{(0)}, M, G^{(0)})$.

Recall that $gr T_x M$ has a graded Lie algebra structure, so that each $z \in P^{(0)}$ transports this Lie algebra structure to $gr V$ by the isomorphism $z : gr V \to gr T_x M$. Then, under the identification

$$F^0 \operatorname{Hom}(\wedge^2 V, V)/F^1 = \operatorname{Hom}(\wedge^2 gr V, gr V)_0,$$

we have

(3. 22) $c^{(0)}(z) = z^{-1}[z(u), z(u)]$ for $u, v \in gr V$.

We can verify this formula immediately by calculating the structure function as explained above. In fact, this is already implied by the formula (3. 11).

We note that the component $\beta^{[0]}$ just represents the action of $\mathfrak{g}^{(0)}$ on $gr V$.

3.2. Reviews on transitive filtered Lie algebras.

In [15] we have studied the structures of transitive filtered Lie algebras. In the next sections we shall find the geometric counterparts of those algebraic studies. In this section we review rapidly some algebraic notions needed later on. For more detail we refer to [15].

1. A graded Lie algebra is a Lie algebra $\mathfrak{l}$ equipped with a gradation $\mathfrak{l} = \bigoplus_{p \in \mathbf{Z}} \mathfrak{l}_p$ such that $[\mathfrak{l}_p, \mathfrak{l}_q] \subset \mathfrak{l}_{p+q}$. We denote by $\mathfrak{l}_-$ the negative part $\bigoplus_{p<0} \mathfrak{l}_p$. The graded Lie algebra is called transitive if the following conditions are satisfied:

$$(3.23) \qquad \begin{cases} \text{i}) & \dim \mathfrak{l}_p < \infty \\ \text{ii}) & \{x \in \mathfrak{l}_p \mid [x, \mathfrak{l}_-] = 0\} = 0 \qquad \text{for} \quad p \geq 0. \end{cases}$$

2. A truncated graded Lie algebra of order k is a graded vector space $\mathfrak{t} = \bigoplus_{p \leq k} \mathfrak{t}_p$ endowed with a bracket operation $[\,,\,] : \mathfrak{t}_p \otimes \mathfrak{t}_q \to \mathfrak{t}_{p+q}$ defined only for $p+q$, $p, q \leq k$ and satisfying the truncated Jacobi identity:

$$(3.24) \qquad \qquad \mathfrak{S}[[x_p, y_q], z_r] = 0$$

for $x_p \in \mathfrak{t}_p$, $y_q \in \mathfrak{t}_q$, $z_r \in \mathfrak{t}_r$, whenever the brackets make sense.

It is called transitive if the condition (3.23), with $\mathfrak{l}$ replaced by $\mathfrak{t}$, is satisfied.

PROPOSITION 3.2.1 [24]. *For a transitive truncated graded Lie algebra $\mathfrak{t} = \bigoplus_{p \leq k} \mathfrak{t}_p$ of order $k \geq -1$, there exists, uniquely up to isomorphism, a transitive graded Lie algebra $\mathfrak{l} = \bigoplus_{p \in \mathbf{Z}} \mathfrak{l}_p$ (called the prolongation of $\mathfrak{t}$ and denoted by $Prol(\mathfrak{t})$) such that*

    i)   *$\mathfrak{l}_p = \mathfrak{t}_p$ for $p \leq k$,*

    ii) *$\mathfrak{l}$ is maximal among the transitive graded Lie algebras satisfying i).*

3. Associated with the graded Lie algebra $\mathfrak{l}$, there is defined the cohomology group called the generalized Spencer cohomology groups: Define the coboundary operator

$$(3.25) \qquad \qquad \partial : \mathrm{Hom}(\wedge^p \mathfrak{l}_-, \mathfrak{l}) \to \mathrm{Hom}(\wedge^{p+1} \mathfrak{l}_-, \mathfrak{l})$$

by

$$(\partial \alpha)(X_1, \cdots X_{p+1}) = \sum (-1)^{i-1} [X_i \; \alpha(X_1, \cdots \widehat{X_i} \cdots X_{p+1})]$$
$$+ \sum (-1)^{i+j} \alpha([X_i, X_j], X_1, \cdots \widehat{X_i} \cdots \widehat{X_j} \cdots X_{p+1})$$

for $\alpha \in \mathrm{Hom}(\wedge^2 \mathfrak{l}_-, \mathfrak{l})$, $X_1, \cdots X_{p+1} \in \mathfrak{l}_-$. It is easy to verify that $\partial^2 = \partial \cdot \partial = 0$. We denote by

$$H(\mathfrak{l}_-, \mathfrak{l}) = \bigoplus_p H^p(\mathfrak{l}_-, \mathfrak{l})$$

the cohomology group defined by this complex. For each integer r, we have subcomplex

$$\mathrm{Hom}(\wedge \mathfrak{l}_-, \mathfrak{l})_r = \bigoplus_p \mathrm{Hom}(\wedge^p \mathfrak{l}_-, \mathfrak{l})_r,$$

where $\mathrm{Hom}(\wedge^p \mathfrak{l}_-, \mathfrak{l})_r$ consists of all homogeneous elements α of degree r, that is,

$$\alpha(\mathfrak{l}_{a_1} \wedge \cdots \wedge \mathfrak{l}_{a_p}) \subset \mathfrak{l}_{a_1 + \cdots + a_p + r}$$

for all $a_1, \cdots, a_p < 0$.
Denoting the associated cohomology group by

$$H(\mathfrak{l}_-, \mathfrak{l})_r = \bigoplus_p H^p(\mathfrak{l}_-, \mathfrak{l})_r,$$

we have the bi-gradation on $H(\mathfrak{l}_-, \mathfrak{l})$;

$$H(\mathfrak{l}_-, \mathfrak{l}) = \bigoplus_{p,r} H^p(\mathfrak{l}_-, \mathfrak{l})_r.$$

The follwing theorem is fundamental :

THEOREM 3. 4. 1 [15]. *For a transitive graded Lie algebra $\mathfrak{l} = \bigoplus_{p \in Z} \mathfrak{l}_q$, there exists an integer r_0 such that $H(\mathfrak{l}_-, \mathfrak{l})_r = 0$ for all $r \geq r_0$.*

For a transitive truncated graded Lie algebra t, we define its cohomology group by

$$H^p(\mathsf{t})_r = H^p(Prol(\mathsf{t})_-, Prol(\mathsf{t}))_r.$$

4. A Lie algebra L equipped with a filtration $\{L^p\}_{p \in Z}$ is called a transitive filtered Lie algebra (abbreviated TFLA) of depth μ (> 0) if it satisfies :

o) $L^p \supset L^{p+1}$,
i) $[L^p, L^q] \supset L^{p+q}$ $\forall p, p \in Z$,
ii) $\dim L^p / L^{p+1} < \infty$,
iii) $L = L^{-\mu}$,
iv) For $p > 0$, $L^p = \{x \in L^{p-1} | [x, L^a] \subset L^{p+a}$ $\forall a < 0\}$,
v) $\cap L^p = 0$,
vi) complete with respect to the uniform topology defined by the filtration.

For a TFLA, L, we put

$$gr\,L = \oplus gr_p L \quad \text{with} \quad gr_p L = L^p / L^{p+1},$$

which is a transitive graded Lie algebra.

5. Now let us recall the notion of truncated transitive filtered Lie algebra introduced in [15]. Let k be an integer with $k \geq 0$. Let T be a filtered vector space with the filtration $\{F^p\}(=\{F^p T\})$ satisfying:

o) $F^p \supset F^{p+1}$,

i) $\dim F^p / F^{p+1} < \infty$,

ii) $F^{-\mu} = T$ for some $\mu > 0$,

iii) $F^{k+1} = 0$.

As in (3.16), we have the filtration $\{I^p\}$ of $\mathrm{Hom}(\wedge^r T, T)$ defined by;

$$\alpha \in I^p \iff \alpha(F^{q_1} \wedge \cdots \wedge F^{q_r}) \subset F^{p+q_1^* + \cdots + q_r^*},$$

and we set

$$\mathrm{Hom}(\wedge^r T, T)^{[p]} = F^0 \mathrm{Hom}(\wedge^r T, T) / I^{p+1}.$$

Given an element $a^{[k]} \in \mathrm{Hom}(\wedge^2 T, T)^{[k]}$. Let $a \in F^0 \mathrm{Hom}(\wedge^2 T, T)$ be a representative of $a^{[k]}$. Let $a \circ a \in F^0 \mathrm{Hom}(\wedge^3 T, T)$ be defined by

$$(a \circ a)(x, y, z) = \underset{x,y,z}{\mathfrak{S}}\, a(a(x, y), z).$$

Then we see easily the equivalence class $a \circ a \bmod I^{k+1}$ does not depend on the choice of the representative a, which we denote $a^{[k]} \circ a^{[k]} (\in \mathrm{Hom}(\wedge^3 T, T)^{[k]})$.

We say that $a^{[k]}$ satisfies the truncated Jacobi identify if $a^{[k]} \circ a^{[k]} = 0$. If the filtered vector space T is endowed with such an $a^{[k]}$ satisfying the truncated Jacobi identify, we call $(T, a^{[k]})$ a truncated filtered Lie algebra of order k.

It should be remarked that if $(T, a^{[k]})$ is a truncated filtered Lie algebra then $F^0 T$ is always a filtered Lie algebra. Let

$$grT = \bigoplus_p gr_p T \quad \text{with} \quad gr_p T = F^p T / F^{p+1}$$

be the associated graded vector space. Then we have the homogeneous bilinear map of degree 0, $a_0 \in \mathrm{Hom}(\wedge^2 grT, grT)_0$, induced by $a^{[k]}$. Then (grT, a_0) is a truncated graded Lie algebra. We call $(T, a^{[k]})$ transitive if (grT, a_0) is transitive.

6. The main theorem of [15] may be roughly stated as follows:

THEOREM.   *The structure of a transitive filtered Lie algebra L is completely determined by the truncated transitive filtered Lie algebra L/L^{k+1} if $H^p((gr\,L)_-, gr\,L)_r=0$ for $r \geq k$, $p=1, 2$.*

3.3.   Towers and truncated towers with constant structure functions.

Let $(P, M, \pi\,;\,\theta)$ be a tower with skeleton (E, G). Let us see what the tower P looks like when the structure function γ is constant.

Assume that γ is constant. Applying the exterior differentiation to the structure equation $d\theta + \dfrac{1}{2}\gamma(\theta, \theta)=0$, we have

$$\gamma(\gamma(\theta, \theta), \theta)=0,$$

which implies $\gamma(\in \mathrm{Hom}(\wedge^2 E, E))$ satisfies the Jacobi identify

$$\mathfrak{S}\gamma(\gamma(x, y), z)=0, \qquad x, y, z \in E.$$

Hence the filtered vector space E, endowed with the bracket operation given by γ, becomes a Lie algebra. Moreover, as easily seen, it is a transitive filtered Lie algebra. Thus,

PROPOSITION 3.3.1.   *If a tower P with skeleton (E, G) has a constant structure function γ, then (E, γ) is a transitive filtered Lie algebra.*

Thus a tower $(P, M, G\,;\,\theta)$ with a constant structure function γ is an analogue of a homogeneous space $\tilde{G}/G$ with $\tilde{G}$ a Lie group and G its closed Lie subgroup. Though there is not defined a group structure, P might be regarded as underlying manifold of a Lie group $\tilde{G}$ (possibly infinite dimensional), θ as the Maurer-Cartan from, (E, γ) as its Lie algebra, G as the isotropy subgroup, and M as the homogeneous space.

We next show :

PROPOSITION 3.3.2.   *If a truncated tower $(P^{(k)}, M, G^{(k)})$ has a constant structure function $\gamma^{[k]}$, then $(E^{(k)}, \gamma^{[k]})$ is a truncated transitive filtered Lie algebra.*

PROOF: Let P be any tower prolonging $P^{(k)}$, with skeleton (E, G), say e.g. $\mathscr{R}P^{(k)}$. Thus $P^{(k)}=P/F^{k+1}$, $G^{(k)}=G/F^{k+1}$, $E^{(k)}=E/F^{k+1}$. Let γ be the structure function of P. Let us denote by

$$Pr^{[k]} : F^0\mathrm{Hom}(\wedge^i E, E) \to F^0\mathrm{Hom}(\wedge^i E, E)/I^{k+1}$$

the natural projection. In defining the structure function of $P^{(k)}$, we have seen that $Pr^{[k]}\circ\gamma$ is constant along the fibres of $P \to P^{(k)}$, and that $Pr^{[k]}\circ\gamma$ is identified with $\gamma^{[k]}$.

By differentiating the structure equation of P: $d\theta + \frac{1}{2}\gamma(\theta, \theta) = 0$,

we have

$$\gamma(\gamma(\theta, \theta), \theta) = d\gamma(\theta, \theta).$$

Hence we have

(3.26) $\gamma \circ \gamma = D\gamma$,

where the function $D\gamma : P \to F^0\mathrm{Hom}(\wedge^3 E, E)$ is given by

(3.27) $(D\gamma)(z)(\xi, \eta, \zeta) = \underset{\xi,\eta,\zeta}{\mathfrak{S}} \tilde{\xi}_z \gamma(\eta, \zeta)$ for $z \in P$, $\xi, \eta, \zeta \in E$.

Here $\tilde{\xi}$ denote the vector field on P determined by $\langle \theta, \tilde{\xi} \rangle = \xi$.

Now by assumption $\gamma^{[k]}$ being constant, we can write

$$\gamma = \varkappa + \text{const.},$$

with a certain $\varkappa : P \to I^{k+1}\mathrm{Hom}(\wedge^2 E, E)$.   Then we have

$$\begin{aligned}
\gamma^{[k]} \circ \gamma^{[k]} &= Pr^{[k]}(\gamma \circ \gamma) \\
&= Pr^{[k]}(D\gamma) \\
&= Pr^{[k]}(D(\varkappa + \text{const.})) \\
&= Pr^{[k]}(D\varkappa).
\end{aligned}$$

But since $\varkappa$ takes its values in $I^{k+1}\mathrm{Hom}(\wedge^2 E, E)$, it follows that $D\varkappa$ takes its values in $I^{k+1}\mathrm{Hom}(\wedge^3 E, E)$.   Thus $Pr^{[k]}(D\varkappa) = 0$, and therefore $\gamma^{[k]} \circ \gamma^{[k]} = 0$, which shows that $(E^{(k)}, \gamma^{[k]})$ is a truncated filtered Lie algebra.

On the other hand recall that

$$\gamma(A, \xi) = A \cdot \xi \qquad \text{for} \quad A \in \mathfrak{g}, \ \xi \in E.$$

Therefore we see that the "truncated action" of $\overset{k}{\underset{p=0}{\bigoplus}} gr_p E^{(k)}$ on $\underset{p \leq k}{\bigoplus} gr_p E^{(k)}$ comes from the action of $gr\mathfrak{g}$ on grE.   Hence the transitivity of $(E^{(k)}, \gamma^{[k]})$ is clear.   q.e.d.

3.4. Fundamental identities.

Let us take a closer look at the structure function of a truncated tower.   Let P be a tower on M with skeleton (E, G).   Write as usual, $P^{(k)} = P/F^{k+1}$, $G^{(k)} = G/F^{k+1}$, and $E^{(k)} = E/F^{k+1}$.   Let γ and $\gamma^{[k]}$ be the structure functions of P and $P^{(k)}$.   We wish to make clear the relation between $\gamma^{[k]}$ and $\gamma^{[k+1]}$.

We have the following commutative diagram:

$$P^{(k+1)} \xrightarrow{\quad \gamma^{[k+1]} \quad} \mathrm{Hom}(\wedge^2 E^{(k+1)}, E^{(k+1)})^{[k+1]}$$

(3.28)

$$P^{(k)} \xrightarrow{\quad \gamma^{[k]} \quad} \mathrm{Hom}(\wedge^2 E^{(k)}, E^{(k)})^{[k]}$$

so that $\gamma^{[k]}$ is determined by $\gamma^{[k+1]}$. Conversely, we want to know to what extent $\gamma^{[k]}$ determines $\gamma^{[k+1]}$.

To fix our discussion, choose complementary subspaces E_p so that

$$F^p E = E_p \oplus F^{p+1} E,$$

and identify $\oplus E_p$ with $gr\,E$, and $\prod E_p$ with E. Then we can also identify:

$$F^0 \mathrm{Hom}(\wedge^r E, E) = \prod_{i=0}^{\infty} \mathrm{Hom}(\wedge^r E, E)_i$$

$$F^0 \mathrm{Hom}(\wedge^r E, E)/F^{k+1} = \prod_{i=0}^{k} \mathrm{Hom}(\wedge^r E, E)_i,$$

where $\mathrm{Hom}(\wedge^r E, E)_i$ denotes the set of homogeneous elements α of degree i, that is, $\alpha(E_{p_1} \wedge \cdots \wedge E_{p_r}) \subset E_{p_1 + \cdots + p_r + i}$. Therefore we can write as

(3.29) $$\gamma = \sum_{i=0}^{\infty} \gamma_i$$

with γ_i being $\mathrm{Hom}(\wedge^2 E, E)_i$-valued functions on P. According to the decomposition $\gamma = \beta + c$ we can also write

$$c = \sum_{i=0}^{\infty} c_i,$$

where c_i are $\mathrm{Hom}(\wedge^2 V, E)_i$-valued functions. Then by the definition of $\gamma^{(k)}$, we can identify:

$$\gamma^{(k)} = \sum_{i=0}^{k} \gamma_i,$$

of which the crucial part is given by

$$c^{(k)} = \sum_{i=0}^{k} c_i.$$

The other part represents the structure of the truncated skeleton $(E^{(k)}, \mathfrak{g}^{(k)})$.

Now we recall that we have

$$\gamma \circ \gamma = D\gamma,$$

or more precisely

$$\underset{\xi,\eta,\zeta}{\mathfrak{S}} \gamma(z)(\gamma(z)(\xi, \eta), \zeta) = \underset{\zeta,\eta,\zeta}{\mathfrak{S}} \tilde{\xi}_z \gamma(\eta, \zeta)$$

for $z \in P$ and $\xi, \eta, \zeta \in E$. Substituting (3.29) into this equation, we have

$$(3.30) \qquad \begin{aligned} \sum_{i,j \geq 0} \gamma_i \circ \gamma_j &= \sum D\gamma_m \\ &= \sum_{l \in \mathbf{Z}, m \geq 0} D_l \gamma_m, \end{aligned}$$

where for $\varphi, \psi \in \mathrm{Hom}(\wedge^2 E, E)$ the multiplication $\varphi \circ \varphi \in \mathrm{Hom}(\wedge^3 E, E)$ is defined by

$$(3.31) \qquad (\varphi \circ \psi)(\xi, \eta, \zeta) = \mathfrak{S}\varphi(\psi(\xi, \eta), \zeta)$$

and D_l denote the differentiation with respect to the l-th component followed by the skewsymmetrization:

$$(D_l\gamma)(z)(\xi, \eta, \zeta) = \underset{\xi,\eta,\zeta}{\mathfrak{S}} \widetilde{\pi_l(\xi)}\gamma(\eta, \zeta)$$

with $\pi_l: E \to E_l$ standing for the projection. Note that

$$\deg(\gamma_i \circ \gamma_i) = i + j, \qquad \deg(D_l\gamma_m) = m - l.$$

Now specializing (3.30) as an identity in $F^0\mathrm{Hom}(\wedge^3 V, E)/F^{k+2}$-valued functions, we have

$$\sum_{i+j \leq k+1} (\gamma_i \circ \gamma_j)|_V = \sum_{m-l \leq k+1} (D_l\gamma_m)|_V,$$

where, for $\mathrm{Hom}(\wedge E, E)$-valued function ϕ, we denote by $\phi|_V$ the $\mathrm{Hom}(\wedge V, E)$-valued function obtained by restriction. Since

$$\gamma_i|_V = c_i, \qquad D_l\gamma_m|_V = 0 \qquad \text{for} \quad l \geq 0,$$

we have:

$$\begin{aligned} \gamma_0 \circ c_{k+1} &+ c_{k+1} \circ \gamma_0 \\ &= -\sum_{i+j \leq k} (\gamma_i \circ \gamma_j)|_V + \sum_{m-l \leq k+1, l < 0} (D_l\gamma_m)|_V, \end{aligned}$$

which can be written as

$$(3.32) \qquad \begin{aligned} \gamma_0 \circ c_{k+1} &+ c_{k+1} \circ \gamma_0 \\ &\equiv -(\gamma^{(k)} \circ \gamma^{(k)})|_V + \sum_{l<0}(D_l\gamma^{(k)})|_V \qquad (\mathrm{mod}\, F^{k+2}\mathrm{Hom}(\wedge^3 V, E)). \end{aligned}$$

This is the equation which describes how the structure function $c^{(k+1)}(= c^{(k)} + c_{k+1})$ of $P^{(k+1)}$ in constrained by the structure function $\gamma^{(k)}$ of $P^{(k)}$.

Now let us consider the case where $\gamma^{(k)}$ is constant. Then clearly the second member of the right-hand side of (3.32) vanishes. Moreover the left-hand side can be written as follows: Recall that by Proposition 3.3.2, $(E^{(k)}, \gamma^{[k]})$ is a transitive truncated filtered Lie algebra, and $(grE^{(k)}, \gamma_0)$ is a transitive truncated graded Lie algebra. Let

$$\partial : \mathrm{Hom}(\wedge^2 grV, grE^{(k)})_{k+1} \to \mathrm{Hom}(\wedge^3 grV, grE^{(k)})_{k+1}$$

be the coboundary operator of the complex associated to the graded Lie algebra $\mathrm{Prol}(grE^{(k)}, \gamma_0)$, the prolongation of $(grE^{(k)}, \gamma_0)$. Regarding c_{k+1} as $\mathrm{Hom}(\wedge^2 grV, grE^{(k)})_{k+1}$-valued function, we can then write the left-hand side of (3, 32) as

$$\gamma_0 \circ c_{k+1} + c_{k+1} \circ \gamma_0 = -\partial c_{k+1}.$$

Hence we get:

$$\partial c_{k+1} \equiv (\gamma^{(k)} \circ \gamma^{(k)})|_V \qquad (\mathrm{mod}\, F^{k+2}\mathrm{Hom}(\wedge^3 V, E)).$$

To summarize the above discussion, let $(P^{(k+1)}, M, G^{(k+1)})$ be a truncated tower and let $P^{(k)} = P^{(k+1)}/F^{k+1}G^{(k+1)}$. Let $\gamma^{[k+1]}$ and $\gamma^{[k]}$ be the structure function of $P^{(k+1)}$ and $P^{(k)}$ respectively. Let $\tilde{\gamma}^{[k]}$ be a lift of $\gamma^{[k]}$ to $P^{(k+1)}$, that is, a $\mathrm{Hom}(\wedge^2 E^{(k+1)}, E^{(k+1)})^{[k+1]}$-valued function obtained by pull-back and by choosing a splitting of $\mathrm{Hom}(\wedge^2 E^{(k+1)}, E^{(k+1)})^{[k+1]} \to \mathrm{Hom}(\wedge^2 E^{(k)}, E^{(k)})^{[k]}$. Set

$$c^{(k+1)} = \gamma^{[k+1]}|_V, \quad \tilde{c}^{(k)} = \tilde{\gamma}^{[k]}|_V.$$

Then we can write

$$c^{(k+1)} = \tilde{c}^{(k)} + c_{k+1}$$

with an $F^{k+1}\mathrm{Hom}(\wedge^2 V, E^{(k+1)})/F^{(k+2)}(= \mathrm{Hom}(\wedge^2 grV, grE^{(k)})_{k+1})$-valued function c_{k+1}, and we have:

PROPOSITION 3.4.1.

(3.33) $\gamma_0 \circ c_{k+1} + c_{k+1} \circ \gamma_0 \equiv -(\tilde{\gamma}^{[k]} \circ \tilde{\gamma}^{[k]})|_V + (D\tilde{\gamma}^{[k]})|_V.$
                                      $(\mathrm{mod}\, F^{k+2}\mathrm{Hom}(\wedge^3 V, E^{(k+1)})).$

In particular, if $\gamma^{[k]}$ is constant then

(3.34) $\partial c_{k+1} \equiv (\tilde{\gamma}^{[k]} \circ \tilde{\gamma}^{[k]})|_V \qquad (\mathrm{mod}\, F^{k+2}\mathrm{Hom}(\wedge^3 V, E^{(k+1)})).$

Next we study the behaviour of the structure function $\gamma^{[k+1]}$ of $P^{(k+1)}$ along each fibre of $P^{(k+1)} \to P^{(k)}$.

Note that we have the following exact sequence:

$$1 \to F^{k+1}G^{(k+1)} \to G^{(k+1)} \to G^{(k)} \to 1$$

and that $P^{(k+1)}$ is a principal fibre bundle over $P^{(k)}$ with structure group $F^{k+1}G^{(k+1)}$. Recall also that we have the canonical embedding

$$G^{(k+1)} \hookrightarrow F^0 GL(E^{(k)})/F^{k+2}.$$

Moreover the induced map

$$G^{(k+1)} \to F^0 \mathrm{Hom}(V, E^{(k)})/F^{k+2}$$

is injective. Therefore $F^{k+1}G^{(k+1)}$ can be identified with an abelian subgroup of

$$F^{k+1}\mathrm{Hom}(V, E^{(k)})/F^{k+2} = \mathrm{Hom}(grV, grE^{(k)})_{k+1}.$$

For an element $a^{k+1} \in F^{k+1}G^{(k+1)}$, if $\alpha_{k+1} \in \mathrm{Hom}(grV, grE^{(k)})_{k+1}$ is the corresponding element, we will write as

$$a^{k+1} = 1 + \alpha_{k+1}$$

since we have

$$a^{k+1}v_p \equiv v_p + \alpha_{k+1}v_p \qquad (\mathrm{mod}\, F^{p+k+2}E)$$

for $v_p \in F^p V$. Note here that $a^{k+1}v_p$ and $\alpha_{k+1}v_p$ are well-defined as elements of E/F^{p+k+2}. The above congruence will be written more simply as

$$a^{k+1}v \equiv (1+\alpha_{k+1})v \qquad (\mathrm{mod}\, F^{k+2}).$$

Now that we have

$$\gamma^{[k+1]}(za) = \rho^{[k+1]}(a^{-1})\gamma^{[k+1]}(z)$$

for $z \in P^{(k+1)}$, $a \in G^{(k+1)}$, let us examine this formula for $a = 1 + \alpha \in F^{k+1}G^{(k+1)}$. Let $c^{(k+1)} = \gamma^{[k+1]}|_V$ and denote γ_0 the projection of $\gamma^{[k+1]}$ to $\mathrm{Hom}(\wedge^2 E^{(k+1)}, E^{(k+1)})^{[0]}$. Then for $v_p \in F^p V$, $w_q \in F^q V$, we have

$$
\begin{aligned}
c^{(k+1)}(za)(v, w) &\equiv \gamma^{[k+1]}(za)(v, w)\\
&\equiv (1-\alpha)\gamma^{[k+1]}(z)((1+\alpha)v, (1+\alpha)w)\\
&\equiv c^{(k+1)}(z)(v, w)\\
&\quad + \gamma_0(z)(\alpha v, w) + \gamma_0(z)(v, \alpha w) - \alpha\gamma_0(z)(v, w)\\
&\qquad (\mathrm{mod}\, F^{p+q+k+2}E),
\end{aligned}
$$

which may be written more simply as:

$$c^{(k+1)}(z(1+\alpha)) = c^{(k+1)}(z) + \gamma_0(z) \circ \alpha - \alpha \circ \gamma_0(z),$$

where $\gamma_0(z) \circ \alpha - \alpha \circ \gamma_0(z)$ is an element of $F^{k+1}\mathrm{Hom}(\wedge^2 V, E^{(k)})/F^{k+2} \cong \mathrm{Hom}(\wedge^2 grV, grE^{(k)})_{k+1}$ given by

$$\begin{cases} \gamma_0(z) \circ \alpha(u, w) = \gamma_0(z)(\alpha(v), w) + \gamma_0(z)(v, \alpha(v)) \\ \alpha \circ \gamma_0(z)(v, w) = \alpha(\gamma_0(z)(v, w)) \end{cases}$$

for $v, w \in grV$.

In particular, if the structure function $\gamma^{[k]}$ of $P^{(k)}$ is constant, $\gamma_0(z)$ is the bracket of the truncated graded Lie algebra $grE^{(k)}$ and

$$\gamma_0(z) \circ \alpha - \alpha \circ \gamma_0(z) = \partial \alpha.$$

Summarizing the above discussion, we have

PROPOSITION 3.4.2. *Let $P^{(k+1)}$ be a truncated tower. The behaviour of the structure function $c^{(k+1)}$ along the fibres $P^{(k+1)} \to P^{(k)}$ is discribed as*

$$(3.35) \qquad c^{(k+1)}(z(1+\alpha)) = c^{(k+1)}(z) + \gamma_0(z) \circ \alpha - \alpha \circ \gamma_0(z),$$

for $z \in P^{(k+1)}$, $1 + \alpha \in F^{k+1}G^{(k+1)}$ with $\alpha \in \mathrm{Hom}(grV, grE^{(k)})_{k+1}$. If the structure function $\gamma^{[k]}$ of $P^{(k)}$ is constant, then

$$(3.36) \qquad c^{(k+1)}(z(1+\alpha)) = c^{(k+1)}(z) + \partial \alpha.$$

3.5.   Reductions.

The structure function of a truncated tower is, in general, not constant. Therefore of importance is the procedure of reduction that we are going to explain.

First we prepare the following:

LEMMA 3.5.1.   *Let M, N be differentiable manifolds and let $f : M \to N$ be a differentiable map. Assume that a Lie group G with countable open basis acts differentiablly on N and that the image $f(M)$ is contained in G-orbit S. Then the map $\bar{f} : M \to S$, defined by regarding f as a map from M to S, is also differentiable.*

PROOF: The orbit S has the natural diferentiable structure as a homogeneous space and is a immersed submanifold of N. It is easy to see that the differentiability of $\bar{f}$ follows immediately from the continuity of $\bar{f}$. So let us show $\bar{f}$ is continuous. For this purpose, given a point $q \in S$, first we are going to construct a suitable neighbourhood W of q in N. Let s, r be respectively the dimension and the codimension of S. Then there exist elements $A_1, \cdots, A_s$ of the Lie algebra $\mathfrak{g}$ of G such that $(\tilde{A}_1)_q$, $\cdots, (\tilde{A}_s)_q$ are linearly independent, where $\tilde{A}$ denotes the vector field on N induced by the infinitesimal action of $A \in \mathfrak{g}$. Choose a regular submanifold Λ through q of N of dimension r such that

$$T_q\Lambda \oplus \langle (\tilde{A}_1)_q, \cdots, (\tilde{A}_s)_q \rangle = T_q N.$$

Let D be an open disk in $\boldsymbol{R}^s$ and define a map $\Phi : D \times \Lambda \to N$ by

$$\Phi((a_1, \cdots, a_s), \lambda) = \exp(\Sigma a_i A_i)\lambda.$$

Since the differential of Φ at $(0, q)$ is non-singular, if necessary by shrinking D and Λ, we may assume that Φ is a diffeomorphism of $D \times \Lambda$ onto an open set W of N. For $\lambda \in \Lambda$ we set $W_\lambda = \Phi(D \times \{\lambda\})$. Clearly each W_λ is contained in a single G-orbit. Therefore we can write

$$S \cap W = \bigcup_{\lambda \in S \cap \Lambda} W_\lambda \quad \text{(disjoint union).}$$

Note also that $W_\lambda(\lambda \in S \cap \Lambda)$ is an open submanifold of S. Since S has a countable open basis, $S \cap \Lambda$ is a countable set. It follows from this that the connected component of $S \cap W$ containing q with respect to the topology induced from N is just W_q.

Now it is straightforward to show the continuity of $\bar{f}$: Let $p \in M$ and $\bar{f}(p) = q$. For an open set $U \ni q$ in S, we may choose a neighbourhood W of q in N as constructed above so as to satisfy $U \supset W_q$. Since $f : M \to N$ is continuous, there exists an open set $V \ni p$ of M such that V is connected and that $f(V) \subset W$, but $f(V) \subset S \cap W$ and connected, therefore $f(V) \subset W_q$. This proves the continuity and hence the differentiability of $\bar{f}$. q.e.d.

Let $(P^{(k)}, \boldsymbol{M}, G^{(k)})$ be a truncated tower. Recall that the structure function

$$\gamma^{[k]} : P^{(k)} \to \operatorname{Hom}(\wedge^2 E^{(k)}, E^{(k)})^{[k]}$$

is a $G^{(k)}$-equivariant map. Therefore the image of $\gamma^{[k]}$ is a disjoint union of $G^{(k)}$-orbits. If it consists of a single $G^{(k)}$-orbit, we can reduce $P^{(k)}$ to a smaller one:

PROPOSITION 3.5.1. *Let $(P^{(k)}, \boldsymbol{M}, G^{(k)})$ be a truncated tower with structure function $\gamma^{[k]}$. If the image of $\gamma^{[k]}$ consists of a single $G^{(k)}$-orbit, then for each $\overset{\circ}{\gamma} \in \gamma^{[k]}(P^{(k)})$, the inverse image $Q^{(k)} = (\gamma^{[k]})^{-1}(\overset{\circ}{\gamma})$ is a principal subbundle of $P^{(k)}$, of which the structure group is the isotropy subgroup of $G^{(k)}$ at $\overset{\circ}{\gamma}$.*

PROOF: Set theoretically the assertion is obvious. We have only to verify the differentiability, but by Lemma 3.5.1 we see that the map $\gamma^{[k]} : P^{(k)} \to \gamma^{[k]}(P^{(k)})$ is differentiable. Moreover clearly it is a surjective submersion. Hence $Q^{(k)}$ is a regular submanifold of $P^{(k)}$.

Now several remarks are in order.

REMARK 3.5.1. The reduced bundle $Q^{(k)}$ is not necessarily adapted.

If $Q^{(k)} \to P^{(k-1)}$ is a surjective submersion, then $Q^{(k)}$ is adapted. In particular, if $k=0$ it is always adapted.

REMARK 3.5.2. The structure function of $Q^{(k)}$ is obviously equal to $\overset{\circ}{\gamma}$ and hence constant. Another choice of $\overset{\circ}{\gamma}$ gives rise to another subbundle of $G^{(k)}$ which is conjugate to $Q^{(k)}$ by an element of $G^{(k)}$. Thus the reduction is not canonical but "semi-canonical" in the sence that $Q^{(k)}$ is determined up to conjugate.

REMARK 3.5.3. If the image of $\gamma^{[k]}$ is not a single $G^{(k)}$-orbit, then the structure is intransitive. To treat the intransitive structures systematically, we need such a formulation as done in [14]. We will consider this problem in section 3.8.

Let us apply the procedure of reduction to first order frame bundles. Let $\mathfrak{m} = \underset{p<0}{\oplus} \mathfrak{m}_p$ be a graded Lie algebra. We say that a filtered manifold M is regular (of type $\mathfrak{m}$) if the symbol algebra $gr T_x M$, $x \in M$, are all isomorphic (to $\mathfrak{m}$) as graded Lie algebras.

Let $(\mathscr{R}^{(0)}(M), M, G^{(0)}(V))$ be the first order frame bundle of M. Then by (3.22), Proposition 3.1.1, and the fact that $G^{(0)}(V) = \mathrm{Aut}(grV)$, we have immediately:

PROPOSITION 3.5.2. *A filtered manifold is regular if and only if the structure function of* $\mathscr{R}^{(0)}(M)$ *takes its values in a single* $G^{(0)}(V)$-*orbit.*

Given a filtered manifold M of regular of type $\mathfrak{m}$, we shall always identify $\mathfrak{m}$ with V and also with grV, where V is the modeled filtered vector space used to define $\mathscr{R}(M)$. Let $\gamma^{[0]} = \beta^{[0]} + c^{(0)}$ be the structure function of $\mathscr{R}^{(0)}(M)$. Then $c^{(0)}$ may be considered as taking values in $\mathrm{Hom}(\wedge^2 \mathfrak{m}, \mathfrak{m})_0$. Let $c_{\mathfrak{m}}^{(0)}$ be the bilinear map which defines the bracket operation of $\mathfrak{m}$. We set

$$\mathscr{R}^{(0)}(M, \mathfrak{m}) = \{z \in \mathscr{R}^{(0)}(m) | c^{(0)}(z) = c_{\mathfrak{m}}^{(0)}\}.$$

Then, by Proposition 3.5.2, $\mathscr{R}^{(0)}(M, \mathfrak{m})$ is a principal subbundle of $\mathscr{R}^{(0)}(M)$. Clearly its structure group, denoted by $G^{(0)}(\mathfrak{m})$, consists of all automorphisms of the graded Lie algebra $\mathfrak{m}$.

In view of (3.22), $\mathscr{R}^{(0)}(M, \mathfrak{m})$ is nothing but the set of all isomorphisms $z : \mathfrak{m} \to gr T_x M$ of graded Lie algebras.

We shall denote by $\mathscr{R}(M, \mathfrak{m})$ the universal tower $\mathscr{R}\mathscr{R}^{(0)}(M, \mathfrak{m})$ prolonging $\mathscr{R}^{(0)}(M, \mathfrak{m})$ and by $(E(\mathfrak{m}), G(\mathfrak{m}))$ its skeleton. Hence $E(\mathfrak{m}) = \mathfrak{m} \oplus \mathfrak{g}(\mathfrak{m})$, where $\mathfrak{g}(\mathfrak{m})$ is the Lie algebra of $G(\mathfrak{m})$. We set $\mathscr{R}^{(k)}(M, \mathfrak{m}) = \mathscr{R}(M, \mathfrak{m})/F^{k+1}$ and call it the reduced frame bundle of M of order $k+1$.

Now let us examine the structure function of $\mathscr{R}(M, \mathfrak{m})$. We define a bilinear map

(3. 37) $\qquad\qquad \beta_{\mathfrak{m}}=[\quad,\quad]: E(\mathfrak{m})\times E(\mathfrak{m})\to E(\mathfrak{m})$

by

$$\begin{cases} [u, v]=[u, v]_{\mathfrak{m}} & \text{(the bracket of } \mathfrak{m}) \\ [A, B]=[A, B]_{\mathfrak{g}(\mathfrak{m})} & \text{(the bracket of } \mathfrak{g}(\mathfrak{m})) \\ [A, x]=Ax & \text{(the action of } \mathfrak{g}(\mathfrak{m}) \text{ on } E(\mathfrak{m})) \end{cases}$$

for $u, v\in\mathfrak{m}$, $x\in E(\mathfrak{m})$, and $A, B\in\mathfrak{g}(\mathfrak{m})$. It should be remarked that this bracket does not satisfy the Jacobi identity.

Recall that $G(\mathfrak{m})$ has the natural representation on $\mathrm{Hom}(\wedge^2E(\mathfrak{m}), E(\mathfrak{m}))$, and note that the subspace $F^1\mathrm{Hom}(\wedge^2\mathfrak{m}, E(\mathfrak{m}))$ is $G(\mathfrak{m})$-invariant. Moreover it is easy to see that $\beta_{\mathfrak{m}} \mod F^1\mathrm{Hom}(\wedge^2\mathfrak{m}, E(\mathfrak{m}))$ is fixed by the induced action of $G(\mathfrak{m})$ on the quotient space. Hence $G(\mathfrak{m})$ has the induced affine representation on the affine space $\beta_{\mathfrak{m}}+F^1\mathrm{Hom}(\wedge^2\mathfrak{m}, E(\mathfrak{m}))$. Then it is straightforward to see:

PROPOSITION 3. 5. 3. *The structure function* $\gamma_{\mathscr{P}(M,\mathfrak{m})}$ *of* $\mathscr{P}(M, \mathfrak{m})$ *is a* $G^{(k)}$-*equivariant map from* $\mathscr{P}(M, \mathfrak{m})$ *to the affine space* $\beta_{\mathfrak{m}}+F^1\mathrm{Hom}(\wedge^2\mathfrak{m}, E(\mathfrak{m}))$.

We can therefore write

$$\gamma_{\mathscr{P}(M,\mathfrak{m})}=\beta_{\mathfrak{m}}+\hat{c}$$

with $\hat{c}$ an $F^1\mathrm{Hom}(\wedge^2\mathfrak{m}, E(\mathfrak{m}))$-valued function on $\mathscr{P}(M, \mathfrak{m})$.

In applications, most of the first order geometric structures are defined as subbundles $P^{(0)}$ of the reduced frame bundle $\mathscr{P}^{(0)}(M, \mathfrak{m})$. Thus the prolongation $\mathscr{P}P^{(0)}$ is contained in $\mathscr{P}(M, \mathfrak{m})$ as an adapted subbundle. Note that clearly the structure function of an adapted subbundle of $\mathscr{P}(M, \mathfrak{m})$ satisfies the same properties as in Proposition 3. 5. 3.

3. 6. Involutive truncated towers.

Given a truncated tower $P^{(k)}$ with constant strucure function. Is it possible to construct a tower P with constant structure function prolonging $P^{(k)}$? In general, it is not the case, but it holds for the involutive truncated towers which we are now going to define.

DEFINITION 3. 6. 1. An adapted subbundle $(P^{(k)}, M, G^{(k)})$ of $\mathscr{P}^{(k)}(M)$, namely a truncated tower is called *involutile* if the following conditions are satisfied:

   i) The structure function $\gamma^{[k]}$ is constant.
   ii) $H^2(grE^{(k)})_r=0 \qquad$ for $r\geq k+1$.

Note that, in the definition above, since $\gamma^{[k]}$ is constant, $grE^{(k)}$

becomes a transitive truncated graded Lie algebra, so that it makes sense to speak of its cohomology group $H(grE^{(k)})$ (see § 3. 2).

THEOREM 3. 6. 1.   *For an involutive truncated tower $P^{(k)}$, we can construct, in a natural manner, a tower P with constant structure function such that $P/F^{k+1}=P^{(k)}$.*

PROOF: Let $P^{(k)}$ be an involutive tower. Let P be the universal prolongation of $P^{(k)}$, that is $P=\mathscr{R}P^{(k)}$, and let $P^{(k+1)}=P/F^{k+2}$.

First let us show that the image $\gamma^{[k+1]}(P^{(k+1)})$ of the structure function $\gamma^{[k+1]}$ of P^{k+1} consists of a single $G^{(k+1)}$-orbit.

To see this it suffices to show that, for any $z, z'\in P^{(k+1)}$, there exists $a\in G^{(k+1)}$ such that $c^{(k+1)}(z')=c^{(k+1)}(za)$. Since the structure function $\gamma^{[k]}$ of $P^{(k)}$ is constant, by Proposition 3. 4. 1 we have

$$\partial(c^{(k+1)}(z)-\tilde{c}^{(k)})\equiv(\tilde{\gamma}^{[k]}\circ\tilde{\gamma}^{[k]})|_V \quad \mathrm{mod}F^{k+2}\mathrm{Hom}(\wedge^3 V, E^{(k)})$$

for all $z\in P^{(k+1)}$. Hence

$$\partial(c^{(k+1)}(z')-c^{(k+1)})(z))=0.$$

Now, since $H^2(grE^{(k)})_{k+1}=0$, we have the following exact sequence:

$$\mathrm{Hom}(grV, grE^{(k)})_{k+1}\xrightarrow{\ \partial\ }\mathrm{Hom}(\wedge^2 grV, grE^{(k)})_{k+1}$$

$$\xrightarrow{\ \partial\ }\mathrm{Hom}(\wedge^3 grV, grE^{(k)})_{k+1}$$

Therefore there exists $a_{k+1}\in\mathrm{Hom}(grV, grE^{(k)})_{k+1}$ such that

$$c^{(k+1)}(z')=c^{(k+1)}(z)+\partial a_{k+1}$$

Note that the injection

$$F^{k+1}G^{(k+1)}\hookrightarrow\mathrm{Hom}(grV, grE^{(k)})_{k+1}$$

is an isomorphism in the case when P is the universal prolongation of $P^{(k)}$. Hence we see that

$$1+a_{k+1}\in F^{k+1}G^{(k+1)}$$

and by Proposition 3.4.2 we have

$$c^{(k+1)}(z(1+a_{k+1}))=c^{(k+1)}(z)+\partial a_{k+1},$$

which proves the assertion required.

Now choose an element $\overset{\circ}{\gamma}{}^{[k+1]}$ in $\gamma^{[k+1]}(P^{(k+1)})$, and let $Q^{(k+1)}=(\gamma^{[k+1]})^{-1}(\overset{\circ}{\gamma}{}^{[k+1]})$. It is clear that $Q^{(k+1)}\to P^{(k)}$ is surjective, so that $Q^{(k+1)}$ is

an adapted subbundle of $\mathscr{R}^{(k+1)}(M)$. It is immediate to see that $Q^{(k+1)}$ is involutive. By iterating this construction, we obtain, as the limit, a tower with constant structure function prolonging $P^{(k)}$.   q.e.d.

Now, assuming the analyticity, we solve the local equivalence problem of involutive truncated towers.

THEOREM 3. 6. 2.   *Let M and M' be filtered manifolds of type V, and let $(P^{(k)}, M, G^{(k)})$ and $(P'^{(k)}, M', G'^{(k)})$ be involutive subbundle of $\mathscr{R}^{(k)}(M)$ and $\mathscr{R}^{(k)}(M')$ with structure functions $\gamma^{[k]}$ and $\gamma'^{[k]}$ respectively. Then under the assumption of analyticity the following two conditions are equivalent :*

(1) $G^{(k)} = G'^{(k)}$ *and* $\gamma^{[k]} = \gamma'^{[k]}$

(2)   *For any $(p, p') \in P^{(k)} \times P'^{(k)}$, there exist open neighbourhoods U and U' of $\pi(p)$ and $\pi'(p')$ respectively, and an analytic isomorphism of filtered manifolds $f : U \to U'$ such that $\#^{k+1} f(P^{(k)}|_U) = P'^{(k)}|_{U'}$ and that $\#^{k+1} f(p) = p'$.*

PROOF :   The implication $(2) \Rightarrow (1)$ holds even without the analyticity. Our task is thus to prove $(1) \Rightarrow (2)$.

If the filtration of V is trivial, this theorem coincides exactly with Theorem 8. 1 in [14], and can be proved by using the Cartan-Kähler theorem.

In the general case, we will prove this theorem, first building towers with constant structure functions and then cutting them to obtain truncated towers with trivial filtered manifolds as base spaces to which the Cartan-Kähler theorem applies.

According to Theorem 3. 6. 1, we construct towers (P, G) and (P', G') with constant structure functions γ and γ' prolonging $P^{(k)}$ and $P'^{(k)}$ respectively. We can arrange the construction so as to have $\gamma = \gamma'$, $G = G'$. Then clearly $P^{(k)}$ and $P'^{(k)}$ are isomorphic if and only of so are P and P'.

Now we regard the skeleton (E, G) as the skeleton on the trivial filtered vector space V, and we denote $\{F_{\mathrm{tr}}^p\}$ the standard filtration of G and E associated with the trivial filtration of V defined by (2. 3). Let $P_{\mathrm{tr}}^{(p)}$ denote the quotient of P by $F_{\mathrm{tr}}^{p+1} G$. Then, in view of (2. 4), we have the natural bundle maps :

$$P^{(p\mu)} \to P_{\mathrm{tr}}^{(p)} \to P^{(p)}.$$

Since the structure function γ is constant, $P_{\mathrm{tr}}^{(k)}$ are truncated towers with constant structure functions on the trivial filtered manifold M.

Let $gr_{\mathrm{tr}}E$ be the graded Lie algebra associated with $\{F_{\mathrm{tr}}^p E\}$.   Then

$H^p(\mathrm{gr_{tr}}E)_r$ is the usual Spencer cohomology group and there exists an integer l_0 such that

$$H^p(\mathrm{gr_{tr}}E)_r = 0 \qquad \text{for } p = 1, 2 \quad r \geq l_0 + 1.$$

It follows then that $P_{\mathrm{tr}}^{(l)}$ are involutive for $l \geq l_0$. Applying the Cartan–Kähler theorem, we obtain a local isomorphism of $P_{\mathrm{tr}}^{(l)}$ to $P'^{(\ell)}_{tr}$ for an l with $l \geq k$, which gives the local isomorphism of $P^{(k)}$ and $P'^{(k)}$. This completes the proof of Theorem 2. 6. 2.

As a consequence of Theorem 3. 6. 2, we have in particular

COROLLARY 3. 6. 1. *Let* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{m}_p$ *be a graded Lie algebra such that the cohomology group associated with the prolongation of* $\mathfrak{m}$ *satisfies* :

$$H^2(\mathfrak{m})_r = 0 \qquad \text{for } r \geq 1.$$

Then every analytic filtered manifold M *regular of type* $\mathfrak{m}$ *is locally isomorphic to a standard filtered manifold* $\mathfrak{M}$ *of type* $\mathfrak{m}$.

PROOF: The reduced frame bundle $\mathscr{R}^{(0)}(M, \mathfrak{m})$ has the constant structure function $\beta_{\mathfrak{m}}^{[0]}$. By the assumption on the cohomology group, $\mathscr{R}(M, \mathfrak{m})$ is therefore involutive. Hence the assertion follows from Theorem 3. 6. 2.

A contact manifold is a simple example to which the above corollary applies. Its symbol algebras are all isomorphic to the Heisenberg Lie algebra. As we calculated in [15] (Theorem 5. 2), the associated cohomology group H_r^p vanishes if $(p, r) \neq (0, -2)$. In this case the above corollary gives a formal aspect of Darboux' theorem. Other examples are also found among the higher order contact structures (see [15] Corollary 5. 3).

3. 7. Transitive geometric structures.

Restricting ourselves to the transitive geometric structures of the 1-st order, we will explain how to find out all the invariants of a given geometric structure.

We say that a truncated tower $(P^{(k)}, M, G^{(k)})$ is transitive (or more precisely, locally transitive) if for any $x, y \in M$ there exists a local automorphism f of $P^{(k)}$ such that $f(x) = y$.

THEOREM 3. 6. 1. *Let* $(P^{(0)}, M, G^{(0)})$ *be a transitive truncated tower of the 1-st order on a filtered manifold* M. *Then we can construct an involutive truncated tower* $(Q^{(l)}, M, H^{(l)})$ *associated with* $P^{(0)}$ *up to conjugate in a way compatible with the equivalence relation.*

PROOF. Given a transitive truncated tower of the 1-st order $(P^{(0)}, \boldsymbol{M}, G^{(0)})$. We first construct a sequence $(Q_i, \boldsymbol{M}, H_i)_{i=1,2,\dots}$ of transitive truncated towers of order k_i+1 with constant structure function $\overset{\circ}{\gamma}_i$.

Let $\gamma^{[0]}$ denote the structure function of $P^{(0)}$. Since $P^{(0)}$ is transitive, $\gamma^{[0]}(P^{(0)})$ consists of a single $G^{(0)}$-orbit. Choose an element $\overset{\circ}{\gamma}_1 \in \gamma^{[0]}(P^{(0)})$ and let

$$Q_1 = (\gamma^{[0]})^{-1}(\overset{\circ}{\gamma}_1).$$

Then Q_1 is a truncated tower of the 1-st order. Now supposing that Q_i is constructed, we construct Q_{i+1} as follows: First form $\#Q_i$, and let γ denote its structure function. Note that since Q_i is transitive, so is $\#Q_i$. By choosing an element $\overset{\circ}{\gamma}_{i+1}$ in $\gamma(\#Q_i)$, we set

$$Q'_{i+1} = \gamma^{-1}(\overset{\circ}{\gamma}_{i+1}).$$

It can happen that this subbundle is no more adapted. If Q'_{i+1} is not adapted, we set

$$Q_{i+1} = \pi^{(0)} Q'_{i+1},$$

where $\pi^{(0)}$ denotes the projection $\mathscr{R}^{(k_{i+1})}(M) \to \mathscr{R}^{(0)}(M)$. If Q'_{i+1} is adapted, we set $Q_{i+1} = Q'_{i+1}$. In both cases Q_{i+1} is a transitive truncated tower with constant structure function. Thus by induction we obtain a sequence $\{(Q_i, \boldsymbol{M}, H_i)\}_{i=1,2,\dots}$ of adapted subbundles of $\mathscr{R}^{(k_i)}(\boldsymbol{M})$.

From the construction we see that there exists a subsequence $\{(Q^{(j)}, \boldsymbol{M}, H^{(j)})\}_{j=0,1,2,\dots}$ of adapted subbundles of $\mathscr{R}^{(j)}(\boldsymbol{M})$ such that the projection $Q^{(j+1)} \to Q^{(j)}$ is surjective for all $j \geq 0$, and that the structure function of $Q^{(j)}$ is constant for $j \geq 0$.

Let $(Q, \boldsymbol{M}, H)$ be the tower obtained as the projective limit of $Q^{(j)}$. Clearly the structure function of Q is constant. Therefore $E = V \oplus \mathfrak{h}$ is a transitive filtered Lie algebra. The associated cohomology group $H^p(grE)_r$ vanishes for all p if $r \geq r_0$ (r_0 is an integer determined by grE, see Theorem 3.4.1). Then $Q^{(j)}$ is involutive if $j \geq r_0$.

Note that the construction of Q_i depends on the choice of $\overset{\circ}{\gamma}_i$. A different choice gives a conjugate bundle $R_a(Q_i)$ with $a \in \#H_{i-1}$.

Note also that for two transitive geometric structures $P^{(0)}, P'^{(0)}$, we can construct the associated truncated towers $Q^{(l)}, Q'^{(l)}$ in such a way that $P^{(0)}$ and $P'^{(0)}$ are equivalent if and only if $Q^{(l)}$ and $Q'^{(l)}$ are equivalent. q.e.d.

The above theorem and Theorem 3.6.2 give a general principle to treat the equivalence problem of transitive geometric structures of order 1.

Higher order geometric structures can be treated by regarding them

as 1-st order geometric structures on some enlarged base spaces, since to a $(k+1)$-th order geometric structure $P^{(k)} \subset \mathscr{R}^{(k)}(M)$ there corresponds in a natural manner a 1-st order geometric structure on $P^{(k-1)}$.

For the intransitive case see the next section.

3.8. Intransitive geometric structures.

3.8.0. To study the equivalence problem of geometric structures in full generality one cannot avoid encountering intransitive geometric structures and needs to extend the formulation developed in the preceding sections.

To illustrate this, let us consider a filtered manifold M whose symbol algebras $gr_x TM$ are not all isomorphic to each other. This is equivalent to saying that the image of the structure function $c^{(0)}$ of $\mathscr{R}^{(0)}(M)$ contains more than one $G^{(0)}(V)$-orbits, that is, the induced map from M to the moduli space of the orbits is not a constant map. Working locally in a neighbourhood of a regular point, we have a surjective submersion $\pi_N : M \to N$ such that $gr_x TM$ and $gr_y TM$ are isomorphic if and only if $\pi_N(x) = \pi_N(y)$. Clearly π_N makes a part of the invariants of the structure.

In order to get further invariants, we proceed as follows: Let $\mathscr{\hat{R}}^{(0)}(M)$ be the ordinary first order frame bundle of M and consider a bundle map

$$\chi : \mathscr{\hat{R}}^{(0)}(M) \to \mathrm{Hom}(V, TN)$$

defined by

$$\chi(\zeta)v = (\pi_N)_* \zeta(v) \qquad \text{for} \quad \zeta \in \mathscr{\hat{R}}^{(0)}(M), \ v \in V.$$

The map χ is also called the structure function and satisfies:

$$\chi(\zeta a) = \chi(\zeta) \circ a \qquad \text{for} \quad a \in \hat{G}^{(0)}(V).$$

We therefore have the induced map from M to the space of $\hat{G}^{(0)}(V)$-orbits in $\mathrm{Hom}(V, TN)$, which may be viewed, in a neighbourhood of a generic point, as giving a surjective submersion $\pi_{N'} : M \to N'$ with N' a certain submanifold of the orbit space. If the surjection $N' \to N$ is not bijective, $\pi_{N'}$ is a new invariant and we can repeat the same procedure for $\pi_{N'}$ and successively until it stabilizes.

With N being eventually replaced by the enlarged one, we may thereby assume that the map χ sends each fibre of $\mathscr{\hat{R}}^{(0)}(M) \to N$ to a single $\hat{G}^{(0)}(V)$-orbit. Now choose a section $\overset{\circ}{\chi} : N \to \mathrm{Hom}(V, TN)$ contained in the image of χ, and define $\hat{P}^{(0)}$ to be the inverse image of $\overset{\circ}{\chi}(N)$ by χ. The subbundle $\hat{P}^{(0)}$ thus obtained of $\mathscr{\hat{R}}^{(0)}(M)$ is a new invariant determined up to conjugate depending on the choice of $\overset{\circ}{\chi}$.

It should be noted that $\hat{P}^{(0)}$ may not be a principal subbundle of $\mathscr{R}^{(0)}(M)$ in the usual sense, but a generalized one in the following sense; its structure group $G^{(0)}$ over $x \in M$ is the isotropy subgroup of $\hat{G}^{(0)}(V)$ at $\overset{\circ}{\chi}(\pi_N(x))$ and may therefore change depending on $t = \pi_N(x) \in N$.

It should be also noted that since the restriction of χ to $\hat{P}^{(0)}$ sends each fibre of $\hat{P}^{(0)} \to N$ to a single point, it makes N a filtered manifold with the tangential filtration $\{F^p TN\}$ given by $F^p T_t N = \chi(\zeta) F^p V$ for any $\zeta \in \hat{P}^{(0)}$ projecting to $t \in N$.

Finally, by projecting $\hat{P}^{(0)}$ to $\mathscr{R}^{(0)}(M)$, we obtain in an invariant way a generalized principal subbundle $P^{(0)}$ of $\mathscr{R}^{(0)}(M)$, which becomes an object of the next study.

In this way we are led to consider subbundles of $\mathscr{R}^{(0)}(M)$ whose structure group may vary with some parameters and to generalize the prolongation scheme to a class of such bundles. In the case when the filtration is trivial, a thorough study was developed in [14]. Here we will not enter into details, but we will content ourselves only to give a rapid account how this generalization can be done.

3.8.1. We need the notions of N-Lie group and N-principal fibre bundle ([28], [14]).

DEFINITION 3.8.1. *An N-Lie group* is a fibred manifold $\varepsilon_N : G \to N$ such that

    i) Each fibre $G(t) = \varepsilon_N^{-1}(t)$ is a Lie group for $t \in N$.

    ii) The mapping $G \times_N G \ni (a, b) \mapsto ab^{-1} \in G$ is differentiable, where $G \times_N G$ denotes the fibre product.

Let (G, N, ε_N) and $(\bar{G}, \bar{N}, \varepsilon_{\bar{N}})$ be N and $\bar{N}$-Lie groups. We say that (G, N, ε_N) is an N-subgroup of $(\bar{G}, \bar{N}, \varepsilon_{\bar{N}})$ if there are given a differentiable mapping $h : N \to \bar{N}$ and an immersion $\iota : G \to \bar{G} \times_{\bar{N}} N$ such that $\varepsilon_N \circ \iota = \varepsilon_N$ and that $\iota|_{G(t)} : \bar{G}(t) \to \bar{G}(h(t))$ is an injective Lie homomorphism for all $t \in N$.

Analogously we have the notions of N-Lie algebra and N-Lie subalgebra. Clearly to an N-Lie group (G, N, ε_N) one can associate an N-Lie algebra $(\mathfrak{g}, N, \varepsilon_N)$ by defining $\mathfrak{g}(t)$ to be the Lie algebra of $G(t)$.

DEFINITION 3.8.2. Let (G, N, ε_N) be an N-Lie group. *An N-principal fibre bundle $P(M, N, G)$ over M with structure group (G, N, ε_N)* is a fibred manifold $\pi : P \to M$ endowed with a surjective submersion $\pi_N : M \to N$ and a right action of G on $P : P \times_N G \ni (p, a) \mapsto pa \in P$ such that the action of $G(\pi_N(x))$ on $\pi^{-1}(x)$ is simply transitive for all $x \in M$.

The notion of N-principal subbundle is also defined similarly.

3.8.2.　Let M be a filtered manifold of depth μ and of type V. Let N be a filtered manifold and suppose that there is given a surjection π_N : $M \to N$ such that $(\pi_N)_* T^p M = T^p N$. We wish to define a class of N-principal subbundles of $\mathscr{R}^{(k)}(M)$ called adapted.

First we note that the structure function $\chi : \mathscr{R}^{(0)}(M) \to \text{Hom}(V, TN)$ induces the structure function

$$\chi^{(k)} : \mathscr{R}^{(k)}(M) \to \text{Hom}(V, TN)^{(k)} = F^0 \text{Hom}(V, TN)/F^{k+1}.$$

In fact, for $z^k \in \mathscr{R}^{(k)}(M)$ take $\zeta^k \in \hat{\mathscr{R}}^{(k)}(M)$ which projects to z^k, and let $\zeta^0 \in \hat{\mathscr{R}}^{(0)}(M)$ be the projection of ζ^k. Clearly $\chi(\zeta^0)$ preserves the filtration, that is, belongs to $F^0 \text{Hom}(V, TN)$. We then set

$$\chi^{(k)}(z^k) \equiv \chi(\zeta^0) \qquad \mod F^{k+1},$$

which is, as easily seen, well-defined and hence defines the map $\chi^{(k)}$.

It should be remarked that if $k \geq \mu - 1$, where μ denotes the depth of M, then $\text{Hom}(V, TN)^{(k)} = F^0 \text{Hom}(V, TN)$ and $\chi^{(k)}$ coincides with the pullback of the structure function χ of $\hat{\mathscr{R}}^{(0)}(M)$ by the projection $\mathscr{R}^{(k)}(M) \to \hat{\mathscr{R}}^{(0)}(M)$.

We also note that $G^{(k)}(V)$ acts on $\text{Hom}(V, TN)^{(k)}$ to the right in the natural manner and we have :

$$\chi^{(k)}(za) = \chi^{(k)}(z) \cdot a \qquad \text{for} \quad a \in G^{(k)}(V), \ z \in \mathscr{R}^{(k)}(M).$$

If $P^{(k)}$ is a subbundle of $\mathscr{R}^{(k)}(M)$, the restriction of $\chi^{(k)}$ to $P^{(k)}$ is also denoted by $\chi^{(k)}$ and called the structure function (of the first kind) of $P^{(k)}$. We say that the structure function $\chi^{(k)}$ is N-constant if it is constant along each fibre of $P^{(k)} \to N$.

DEFINITION 3.8.3.　*An N-principal subbundle $P^{(k)}$ of $\mathscr{R}^{(k)}(M)$ is called adapted (abbreviated adapted N-bundle) if the following conditions are satisfied ((ii) and (iii) being void if $k=0$) :*

　i)　The structure function $\chi^{(k)}$ of $P^{(k)}$ is N-contant.
　ii)　$P^{(k-1)}(=P^{(k)}/F^k)$ is an adapted N-principal subbundle of $\mathscr{R}^{(k-1)}(M)$.
　iii)　$P^{(k)}$ is an N-principal subbundle of $\# P^{(k-1)}$.

To complete the definition above we have to define the functor $\#$. Let $P^{(k)}(M, G^{(k)})$ be an adapted N-subbundle of $\mathscr{R}^{(k)}(M)$. By definition the structure function $\chi^{(k)}$ of $P^{(k)}$ is N-constant and therefore considered as a section of the bundle $\text{Hom}(V, TN)^{(k)}$ over N. Choose a section χ of the bundle $F^0 \text{Hom}(V, TN)$ such that χ projects to $\chi^{(k)}$. Now we define $\widehat{\#} P^{(k)}$

to be the subset of $\mathscr{R}^{(k+1)}(M)$ consisting of all

$$\xi^{k+1} : V \oplus \mathfrak{g}^{(k)}(V) \to T_{x^k}\mathscr{R}^{(k)}(M)$$

with $\xi^{k+1} \in \mathscr{R}^{(k+1)}(M)$ and $x^k \in P^{(k)}$ such that

i) $\xi^{k+1}(V \oplus \mathfrak{g}^{(k)}(t)) \subset T_{x^k}P^{(k)}$,
ii) $\chi_{\mathscr{R}^{(0)}(M)}(\xi^0) = \chi(t)$,

where ξ^0 and t denote the projections of ξ^{k+1} to $\mathscr{R}^{(0)}(M)$ and N respectively.

We note that if $k \geq \mu - 1$ then the condition ii) above is automatically satisfied. If $k < \mu - 1$, the construction of $\widehat{\#}P^{(k)}$ depends on the choice of the lift χ of $\chi^{(k)}$.

We then define : $\#P^{(k)} = \widehat{\#}P^{(k)}/F^{k+2}$.

Now let us describe the structure group of $\#P^{(k)}$ and $\widehat{\#}P^{(k)}$. We fix an surjection

$$\chi : V \times N \to TN$$

in view of the fact that the structure function of an adapted N-subbundle is N-constant.

DEFINITION 3.8.4. An N-Lie subgroup $(G^{(k)}, N, \varepsilon_N)$ of $G^{(k)}(V)$ is *adaped* if the following conditions are satisfied ((ii) and (iii) being void if $k=0$) :

i) $\chi^{(k)}(t) \cdot a^{(k)} = \chi^{(k)}(t)$ for $t \in N$, $a^{(k)} \in G^{(k)}(t)$.
ii) $G^{(k-1)}(=G^{(k)}/F^k)$ is an adapted N-Lie subgroup of $G^{(k-1)}(V)$.
iii) $G^{(k)}$ is an N-Lie subgroup of $\#G^{(k-1)}$.

To define the functor $\#$ we prepare the following two formulas.

Let G be a Lie group, and let $\underline{G \times N}$ denote the sheaf of sections of the trivial group bundle $G \times N \to N$. We define a differential operator

$$D : \underline{G \times N} \to \mathrm{Hom}(\underline{TN, \mathfrak{g}})$$

by

$$D\underline{a} = \underline{a}^* \omega \quad \text{for} \quad a \in \underline{G \times N},$$

where ω denotes the Maurer-Cartan from of G. Composed with $\chi : V \times N \to TN$, it gives an operator

$$D_\chi : \underline{G \times N} \to \mathrm{Hom}(V, \mathfrak{g}) \times N.$$

Then we have :

(*) $D_\chi(\underline{a} \cdot \underline{b}) = Ad(\underline{b})^{-1} \circ D_\chi \underline{a} + D_\chi \underline{b}$

for $\underline{a}, \underline{b} \in \underline{G \times N}$.

Next let us define a map

$$\delta : \widehat{G}^{(k+1)}(V) \to \operatorname{Hom}(V, \mathfrak{g}^{(k)}(V)).$$

Recalling that $\widehat{G}^{(k+1)}(V) \subset F^0 GL(V \otimes \mathfrak{g}^{(k)}(V))$, we set

(**) $(\delta a^{k+1})(v) = Ad(a^k)^{-1}(\pi_\mathfrak{g} a^{k+1}(v))$

for $v \in V$, $a^{k+1} \in \widehat{G}^{(k+1)}(V)$, where $\pi_\mathfrak{g}$ denotes the projection $V \oplus \mathfrak{g}^{(k)}(V) \to$ $\mathfrak{g}^{(k)}(V)$ and a^k denotes the image of a^{k+1} by the projection $\widehat{G}^{(k+1)}(V) \to$ $G^{(k)}(V)$. Then we have:

(***) $\delta(a^{k+1} \cdot \beta^{k+1}) = Ad(b^k)^{-1} \circ \delta a^{k+1} \circ \beta^0 + \delta \beta^{k+1}$

for $a^{k+1}, \beta^{k+1} \in \widehat{G}^{(k+1)}(V)$, where b^k and β^0 denote the projections of β^{k+1} to $G^{(k)}(V)$ and $\widehat{G}^{(0)}(V)$ respectively.

Now let $(G^{(k)}, N, \varepsilon_N)$ be an adapted N-Lie subgroup of $G^{(k)}(V)$. We define a subsheaf $\#\underline{G}^{(k)}$ of $\widehat{G}^{(k+1)}(V) \times N$ as follows: $\underline{a}^{k+1} \in \#\underline{G}^{(k)}$ for $\underline{a}^{k+1} \in$ $\underline{\widehat{G}^{(k+1)}(V) \times N}$ if and only if

  i) $\underline{a}^k \in \underline{G}^{(k)}$,
  ii) $\delta \underline{a}^{k+1} - D_\chi \underline{a}^k \in \underline{\operatorname{Hom}(V, \mathfrak{g}^{(k)})}$,
  iii) $\chi \cdot \underline{a}^{k+1} = \chi$,

where $\underline{a}^k$ denotes the section of $\underline{G^{(k)}(V) \times N}$ obtained as projection of $\underline{a}^{k+1}$.

By virtue of (*) and (***), we see that $\#\underline{G}^{(k)}$ is a sheaf of groups. Moreover, on account of the exact sequence (recall (2. 7))

$$0 \longrightarrow F^{k+1}\operatorname{Hom}(V, E^{(k)}(V)) \overset{\iota}{\longrightarrow} \widehat{G}^{(k+1)}(V) \longrightarrow G^{(k)}(V) \longrightarrow 1$$

with ι given by $\iota(a) = 1 + a$, we have the following exact sequence:

$$0 \longrightarrow \underline{F^{(k+1)}\operatorname{Hom}(V, E^{(k)})^\chi} \longrightarrow \#\underline{G}^{(k)} \longrightarrow \underline{G}^{(k)} \longrightarrow 1,$$

where $E^{(k)} = V \oplus \mathfrak{g}^{(k)}$ and $\operatorname{Hom}(V, E^{(k)})^\chi$ denotes the vector bundle over N whose fibre over $t \in N$ consists of all $a \in \operatorname{Hom}(V, E^{(k)}(t))$ such that $\chi(t) \circ$ $Pr_V \circ a = 0$, where Pr_V denotes the projecfion $E^{(k)} \to V$. Note that F^{k+1} $\operatorname{Hom}(V, E^{(k)})^\chi = F^{k+1}\operatorname{Hom}(V, E^{(k)})$ if $k \geq \mu - 1$.

It then follows that there exists uniquely an N-Lie subgroup $\widehat{\#}G^{(k)}$ of $\widehat{G}^{(k+1)}(V)$ such that $\#\underline{G}^{(k)} = \widehat{\#}\underline{G}^{(k)}$. Finally we define: $\#G^{(k)} = \widehat{\#}G^{(k)}/F^{k+2}$, which is an N-Lie subgroup of $G^{(k+1)}(V)$. Note that we have the following commutative diagram with exact rows.

$$0 \longrightarrow F^{k+1}\mathrm{Hom}(V, E^{(k)})^{\chi} \longrightarrow \widehat{\#}G^{(k)} \longrightarrow G^{(k)} \longrightarrow 1$$

$$\Big\downarrow \qquad\qquad\qquad \Big\downarrow \qquad\qquad \Big\|$$

$$0 \longrightarrow \mathrm{Hom}(grV, grE^{(k)})^{\chi}_{k+1} \longrightarrow \#G^{(k)} \longrightarrow G^{(k)} \longrightarrow 1 .$$

It should be remarked that the action of $(\widehat{\#}G^{(k)})(t)$ on $E^{(k)}(V)$ does not necessarily leave invariant the subspace $E^{(k)}(t)(= V \oplus \mathfrak{g}^{(k)}(t))$, while $\mathrm{Ker}\chi(t) \oplus \mathfrak{g}^{(k)}(t)$ is an invariant subspace.

If $P^{(k)}(M, G^{(k)})$ is an adapted N-Lie subgroup of $\mathcal{R}^{(k)}(M)$, then the structure groups of $\widehat{\#}P^{(k)}$ and $\#P^{(k)}$ are $\widehat{\#}G^{(k)}$ and $\#G^{(k)}$ respectively.

To understand that the inductive definitions above are all consistent, it would be enough to show that $\widehat{\#}G^{(k)}$ acts on $\widehat{\#}P^{(k)}$. For $(\xi^{k+1}, \alpha^{k+1}) \in \widehat{\#}P^{(k)} \times_N \widehat{\#}G^{(k)}$, let η^{k+1} be given by the following commutative diagram:

$$
\begin{array}{ccc}
V \oplus \mathfrak{g}^{(k)}(V) & \xrightarrow{\ \xi^{k+1}\ } & T_{z^k}\mathcal{R}^{(k)}(M) \\[4pt]
{\scriptstyle \alpha^{k+1}}\Big\uparrow & & \Big\downarrow {\scriptstyle R_{a^k *}} \\[4pt]
V \oplus \mathfrak{g}^{(k)}(V) & \xrightarrow{\ \eta^{k+1}\ } & T_{z^k a^k}\mathcal{R}^{(k)}(M),
\end{array}
$$

where as usual z^k and a^k denote the projections of ξ^{k+1} and α^{k+1} on $P^{(k)}$ and $G^{(k)}$ respectively. We then define: $\xi^{k+1} \cdot \alpha^{k+1} = \eta^{k+1}$. To see that this actually defines the action of $\widehat{\#}G^{(k)}$ on $\widehat{\#}P^{(k)}$, we have to verify, among others, that $\eta^{k+1} \in \widehat{\#}P^{(k)}$. For that, let us show

(*) $$\eta^{k+1}(V) \subset T_{z^k a^k}P^{(k)}.$$

Take a section $\underline{\alpha}^{k+1} \in \widehat{\#}G^{(k)}$ such that $\underline{\alpha}^{k+1}(t_0) = \alpha^{k+1}$, where $t_0 = \pi_N(z^k)$. Then

$$(\delta\underline{\alpha}^{k+1} - D_\chi\underline{a}^k)(V) \subset \mathfrak{g}^{(k)}.$$

Therefore

$$Ad(\underline{a}^k)(\delta\underline{\alpha}^{k+1} - D_\chi\underline{a}^k)(V) \subset \mathfrak{g}^{(k)},$$

and therefore

$$(\underline{\alpha}^{k+1} - Ad(\underline{a}^k)D_\chi\underline{a}^k)(V) \subset V \oplus \mathfrak{g}^{(k)}.$$

Putting

$$\tilde{\alpha}^{k+1} = \underline{\alpha}^{k+1} - Ad(\underline{a}^k)D_\chi\underline{a}^k,$$

we have

$$(R_{\underline{a}^{k}*} \circ \xi^{k+1} \circ \tilde{a}^{k+1}(t_0))v \in T_{z^k a^k} P^{(k)} \quad \text{for} \quad v \in V.$$

On the other hand, we have

$$(R_{\underline{a}^k*} \circ \xi^{k+1} \circ \tilde{a}^{k+1}(t_0))v$$
$$= R_{\underline{a}^k*} \circ \xi^{k+1} \circ \{\underline{a}^{k+1}(t_0) - Ad(\underline{a}^k(t_0))(D_\chi \underline{a}^k)(t_0)\}v$$
$$= R_{\underline{a}^k*} \{(\xi^{k+1} \circ \underline{a}^{k+1}(t_0))v - [Ad(\underline{a}^k(t_0))(D_\chi \underline{a}^k)(t_0)(v)]^{\sim}{}_{z^k}\}$$
$$= (R_{a^k*} \circ \xi^{k+1} \circ a^{k+1})v + [(D_\chi \underline{a}^k)(t_0)(a^0 v)]^{\sim}{}_{z^k a^k} - [(D_\chi \underline{a}^k)(t_0)v]^{\sim}{}_{z^k a^k}$$
$$= (R_{a^k*} \circ \xi^{k+1} \circ a^{k+1})v \quad (\text{since } a^0 v = v),$$

which proves (*). The rest of the verification is now straightforward.

3.8.3   The prolongation scheme being well constructed, we can develop the theory analogously as in the transitive case. We give the outline without proof.

Let $(P^{(k)}, M, N)$ be an adapted N-principal subbundle of $\mathscr{R}^{(k)}(M)$. Let

$$\gamma^{[k]} : \mathscr{R}^{(k)}(M) \to \text{Hom}(\wedge^2 E^{(k)}(V), E^{(k)}(V))^{[k]}$$
$$\chi^{(k)} : \mathscr{R}^{(k)}(M) \to \text{Hom}(V, TN)^{(k)}$$

be the structure functions of $\mathscr{R}^{(k)}(M)$. Let $\iota : P^{(k)} \hookrightarrow \mathscr{R}^{(k)}(M)$ be the canonical injection. The pull-backs $\iota^* \gamma^{[k]}$ and $\iota^* \chi^{(k)}$ are called the structure functions of $P^{(k)}$ and denoted also by $\gamma^{[k]}$ and $\chi^{(k)}$.

PROPOSITION 3.8.1.   *If the structure functions $\gamma^{[k]}$ of an adapted N-principal subbundle $P^{(k)}(M, G^{(k)}, N)$ is N-constant, then $gr(E^{(k)}(t), \gamma^{[k]}(t))$ is a truncated transitive graded Lie algebra for all $t \in N$.*

In this case, we can therefore define the cohomology group $H_p(grE^{(k)}(t))_r$ for $t \in N$, and we can introduce the notion of involutivity:

DEFINITION 3.8.5.   An adapted N-principal subbundle $P^{(k)}(M, G^{(k)}, N)$ is called *involutive* if the following conditions are satisfied:

i)   The structure function $\gamma^{[k]}$ is N-constant.
ii) $H^2(grE^{(k)}(t))_r = 0$ for $r > k+1$ and for all $t \in N$.
iii) $k \geq \mu - 1$

We remark that in the above condition iii), μ can be replaced by

$$\lambda = \text{Min}\{k \geq 1 \mid F^k \text{Hom}(V, TN) = 0\}.$$

THEOREM 3.8.1.   *For an involutive adapted N-principal subbundle $P^{(k)}$ of $\mathscr{R}^{(k)}(M)$, we can construct, uniquely up to conjugate, an N-principal subbundle P of $\mathscr{R}(M)$ with N-constant structure functions such that*

$$P/F^{k+1} = P^{(k)}.$$

This theorem gives a solution (at least formally) to the equivalence problem of involutive adapted N-principal subbundles $P^{(k)}$. The invariants of $P^{(k)}$ are completely determined by the structure group $G^{(k)}$ and the structure functions $c^{[k]}$ and χ.

However, contrary to the transitive case or to the case of trivial filtration, the analytic theorem such as Theorem 3.6.2. or Theorem 8.1 in [14] does not seem to hold. The proof of Theorem 3.6.2 breaks in this intransitive case, since the dimensions of the fibres $P_{tr}^{(k)} \to N$ may not be constant. The proper category will be the formal Gevrey class (see [18]).

The importance of the notion of involutivity and Theorem 3.8.1 also lies in the fact that a theorem similar to Theorem 9.1 of [14] holds in this case of filtered manifolds: Roughly speaking, the equivalence problem of any geometric structure on a filtered manifold M can be generically reduced to that of a certain involutive N-subbundle of $\mathscr{R}^{(k)}(M)$.

3.9. An application to Monge-Ampère equations.

To illustrate the general procedure to the equivalence problem we will prove the following:

THEOREM 3.9.1. *A Monge-Ampère equation is locally contact equivalent to the equation* $\dfrac{\partial^2 z}{\partial x \partial y} = 0$ *if its two characterstic systems are not confounded and if each of them has two independent first integrals.*

The origin of this theorem goes back to S. Lie (cf. [7], [10]). Following our general method developed in the preceding sections, we will give another proof of the theorem.

First of all recall that, according to Morimoto [13], to consider a Monge-Ampère equation having two distinct characteristic systems is equivalent to giving a data (M, D, ξ, η), where M is a differentiable manifold of dimension 5, D a contact structure on M, and ξ, η subbundles of D of rank 2 satisfying:

    i) $D = \xi \oplus \eta$

    ii) $d\omega(\xi, \eta) = 0$ for a local contact form ω defining D.

The subbundles ξ, η are the characteristic systems associated with the Monge-Ampère equation. Note that under the above conditions the derived systems of ξ, η are of rank 3, that is, there exist subbundles ξ', η' of TM of rank 3 such that $\underline{\xi'} = \underline{\xi} + [\underline{\xi}, \underline{\xi}]$, $\underline{\eta'} = \underline{\eta} + [\underline{\eta}, \underline{\eta}]$. Therefore saying that each of the characteristic systems has two independent first integrals is

equivalent to saying:

    iii)   The derived systems ξ', η' are completely integrable.

Let us call (M, D, ξ, η) a M-A system of *hyperbolic type* if it satisfies the above two conditions i)-ii) and of *wave type* if it satisfies moreover iii). Then we can reformulate the above theorem as follws:

THEOREM 3.9.2. *Any two M-A systems of wave type are locally equivalent, that is, for such (M, D, ξ, η) and $(M', D' \xi', \eta')$ there exists a local diffeomorphism from M to M' such that $f_* D = D'$, $f_* \xi = \xi'$, $f_* \eta = \eta'$.*

Since the M-A system corresponding to the equation $\dfrac{\partial^2 z}{\partial x \partial y} = 0$ is, as easily seen, of wave type, this theorem is equivalens to the preceding one.

    To prove the theorem we first show that to each M-A system corresponds a first order geometric structure on a contact manifold.

    Let (M, D, ξ, η) be a M-A system of hyperbolic type. Denote by $\boldsymbol{M}$ the contact manifold (M, D) regarded as a filtered manifold (i.e., $T^{-2}\boldsymbol{M} = TM$, $T^{-1}\boldsymbol{M} = D$). Let $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ be the Heisenberg Lie algebra of dimension 5 and take bases $\{e_0\}$ of $\mathfrak{g}_{-2}$ and $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}_{-1}$ such that

$$[e_1, e_2] = [e_3, e_4] = e_0$$

with the other brackets trivial. Set

$$\mathfrak{g}'_{-1} = \mathrm{span}\{e_1, e_2\}, \quad \mathfrak{g}''_{-1} = \mathrm{span}\{e_3, e_4\}.$$

Then for any $x \in M$ there exists an isomorphism (of the graded Lie algebras)

$$z : \mathfrak{g}_- \to gr T_x \boldsymbol{M}$$

such that $z(\mathfrak{g}'_{-1}) = \xi_x$, $z(\mathfrak{g}''_{-1}) = \eta_x$. Let $P^{(0)}$ be the set of all such isomorphisms. Then $P^{(0)}$ can be regarded as a subbundle of the first order frame bundle $\mathscr{R}^{(0)}(\boldsymbol{M})$, whose structure group is given by

$$G_0 = \{\phi \in \mathrm{Aut}(\mathfrak{g}_-) \mid \phi(\mathfrak{g}'_{-1}) \subset \mathfrak{g}'_{-1}, \ \phi(\mathfrak{g}''_{-1}) \subset \mathfrak{g}''_{-1}\}.$$

Its Lie algebra $\mathfrak{g}_0$ has the following matrix representation:

$$\begin{pmatrix} 2\lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda+a & b & 0 & 0 \\ 0 & c & \lambda-a & 0 & 0 \\ 0 & 0 & 0 & \lambda+a' & b' \\ 0 & 0 & 0 & c' & \lambda-a' \end{pmatrix}$$

Thus we have a direct sum decomposition:

$$\mathfrak{g}_0 = sp(\mathfrak{g}'_{-1}) \oplus sp(\mathfrak{g}''_{-1}) \oplus \mathfrak{z}_0,$$

where $\mathfrak{z}_0$ is the center of dimension 1.

Note that two M-A systems are equivalent if and only if so are the corresponding geometric structures.

Now let (M, D, ξ, η) be a M-A system of wave type and $P^{(0)}(M, G_0)$ the corresponding subbundle of $\mathcal{R}^{(0)}(M)$. Let us determine the invariants of $P^{(0)}$.

From the construction of $P^{(0)}$, it is clear that the structure function $\gamma^{(0)}$ of $P^{(0)}$ just represents the structure of the graded Lie algebra $\mathfrak{g}_- \oplus \mathfrak{g}_0$. Thus it is constant.

Next we consider prolongation $\#P^{(0)} \subset \mathcal{R}^{(1)}(M)$. Let $\gamma^{(1)}$ be the structure function of $\#P^{(0)}$. As in (3.13) we decompose $\gamma^{(1)}$ as $\gamma^{(1)} = \beta^{(1)} + c^{(1)}$ with $c^{(1)} : \#P^{(0)} \to \mathrm{Hom}(\wedge^2 \mathfrak{g}_-, E(\mathfrak{g}_0))^{(1)}$. We make a further decomposition of $c^{(1)}$ into homogeneous components:

$$c^{(1)} = c_0 + c_1,$$

where c_i takes its values in $\mathrm{Hom}(\wedge^2 \mathfrak{g}_-, \mathfrak{g}_-)_i$. Clearl c_0 represents the bracket of $\mathfrak{g}_-$.

LEMMA. *For any* $z^0 \in P^{(0)}$ *there exists* $z^1 \in \#P^{(0)}$ *with* $\pi(z^1) = z^0$ *such that* $c_1(z^1) = 0$.

PROOF: Let us denote by $\tilde{G}_1$ the strucure group of the fibring $\#P^{(0)} \to P^{(0)}$ and by $\tilde{\mathfrak{g}}_1$ its Lie algebra. On account of (2.7) they are abelian and

$$\tilde{\mathfrak{g}}_1 \cong \mathrm{Hom}(\mathfrak{g}_-, \mathfrak{g}_- \oplus \mathfrak{g}_0)_1.$$

Recalling (3.36), we have, for $A \in \tilde{\mathfrak{g}}_1$,

$$c^{(1)}(z(1+A)) = c^{(1)}(z) + \partial A,$$

where

$$(\partial A)(x, y) = [x, A(y)] + [A(x), y] - A([x, y])$$

for $x, y \in \mathfrak{g}_-$.

We first show that there exists $z^1 \in \#P^{(0)}$ with $\pi(z^1) = z^0$ such that $c_1(z^1)(\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1}) = 0$. Take any point $w \in \#P^{(0)}$ over z^0, and write

$$\begin{cases} c_1(w)(e_1 \wedge e_2) = u' + u'' \\ c_1(w)(e_3 \wedge e_4) = v' + v'' \end{cases}$$

with $u', v' \in \mathfrak{g}'_{-1}$ and $u'', v'' \in \mathfrak{g}''_{-1}$. Define $A \in \mathrm{Hom}(\mathfrak{g}_{-2}, \mathfrak{g}_{-1})$ by

$$A(e_0) = u'' + v'.$$

Then we have

$$\begin{cases} c_1(w(1+A))(\mathfrak{g}'_{-1}\wedge\mathfrak{g}'_{-1})\subset\mathfrak{g}'_{-1} \\ c_1(w(1+A))(\mathfrak{g}''_{-1}\wedge\mathfrak{g}''_{-1})\subset\mathfrak{g}''_{-1}. \end{cases}$$

Now we define $\mathfrak{g}_0^{(1)}$ by the following exact sequence:

$$0 \to \mathfrak{g}_0^{(1)} \to \mathrm{Hom}(\mathfrak{g}_{-1},\,\mathfrak{g}_0) \xrightarrow{\ \partial\ } \mathrm{Hom}(\wedge^2\mathfrak{g}_{-1},\,\mathfrak{g}_{-1}),$$

where ∂ is given by

$$(\partial\varphi)(x,\,y)=\varphi(x)y-\varphi(y)x$$

for $\varphi\in\mathrm{Hom}(\mathfrak{g}_{-1},\,\mathfrak{g}_0)$, $x,\,y\in\mathfrak{g}_{-1}$. Since $\mathfrak{g}_0=sp(\mathfrak{g}'_{-1})\oplus sp(\mathfrak{g}''_{-1})\oplus\mathring{\mathfrak{z}}_0$, an easy calculation shows that

$$\mathfrak{g}_0^{(1)}=sp(\mathfrak{g}'_{-1})^{(1)}\oplus sp(\mathfrak{g}''_{-1})^{(1)},$$

where

$$sp(\mathfrak{g}'_{-1})^{(k)}=\mathrm{Hom}(S^2\mathfrak{g}'_{-1},\,sp(\mathfrak{g}'_{-1})^{(k-2)})\cap\mathrm{Hom}(\mathfrak{g}'_{-1},\,sp(\mathfrak{g}'_{-1})^{(k-1)}),$$

the usual k-th prolongation of $sp(\mathfrak{g}'_{-1})$ (see e.g., [21]). In particular we have dim $\mathfrak{g}_0^{(1)}=8$. On the other hand the image of ∂ is contained in

$$W=\{\alpha\in\mathrm{Hom}(\wedge^2\mathfrak{g}_{-1},\,\mathfrak{g}_{-1})\,|\,\alpha(\wedge^2\mathfrak{g}'_{-1})\subset\mathfrak{g}'_{-1},\,,\,\alpha(\wedge^2\mathfrak{g}''_{-1})\subset\mathfrak{g}''_{-1}\}.$$

Noting that dim $W=20$ and dim $\mathrm{Hom}(\mathfrak{g}_{-1},\,\mathfrak{g}_0)=28$, we conclude that $\partial\mathrm{Hom}(\mathfrak{g}_{-1},\,\mathfrak{g}_0)=W$, Hence there exists $B\in\mathrm{Hom}(\mathfrak{g}_{-1},\,\mathfrak{g}_0)$ such that

$$c_1(w(1+A)(1+B))(\mathfrak{g}_{-1}\wedge\mathfrak{g}_{-1})=0.$$

Thus we have found $z^1\in\#P^{(0)}$ with $\pi(z^1)=z^0$ such that $c_1(z^1)(\mathfrak{g}_{-1}\wedge\mathfrak{g}_{-1})=0$.

Next we show that this implies moreover $c_1(z^1)(\mathfrak{g}_{-1}\wedge\mathfrak{g}_{-2})=0$ by using the fundamental identity. In fact, by Proposition 3. 4. 1. we have

$$\partial c_1(z^1)\equiv(\tilde{\gamma}^{[0]}(z^0)\circ\tilde{\gamma}^{[0]}(z^0))|_V \qquad (\mathrm{mod}\,F^2).$$

But in this case clearly the right hand side vanishes, hence $\partial c_1(z^1)=0$. In particular we have

$$\mathfrak{S}c_1(z^1)([x_1,\,x_2],\,x_3)=0$$

for $x_i\in\mathfrak{g}_{-1}$, from which follows immediately that $c_1(z^1)(\mathfrak{g}_{-1}\wedge\mathfrak{g}_{-2})=0$. Hence $c_1(z_1)=0$. Thus the lemma is proved.

By the above lemma, we see that the image of $\gamma^{(1)}$ is contained in a single $G^{(1)}$-orbit through $\tilde{\gamma}^{(1)}$, where we set $\mathring{c}^{(1)}=c_0$ and $\tilde{\gamma}^{(1)}=\beta^{(1)}+\mathring{c}^{(1)}$. If we set

$$P^{(1)} = \{z^1 \in \#P^{(0)} \mid \gamma^{(1)}(z^1) = \overset{\circ}{\gamma}{}^{(1)}\},$$

then it is an adapted subbundle of $\#P^{(0)}$ and $P^{(1)} \to P^{(0)}$ is a principal fibre bundle. Let G_1 denote its structure group and $\mathfrak{g}_1$ the Lie algebra of G. Then we see easily that

$$\mathfrak{g}_1 \cong sp(\mathfrak{g}'_{-1})^{(1)} \oplus sp(\mathfrak{g}''_{-1})^{(1)}.$$

Now we proceed to the second order structure $\#P^{(1)}$. Denote by $c^{(2)}$ its structure function, and decompose it into homogeneous components:

$$c^{(2)} = c_0 + c_1 + c_2.$$

By construction, c_0 represents the bracket of $\mathfrak{g}_-$, $c_1 = 0$, and c_2 takes it values in $\mathrm{Hom}(\wedge^2 \mathfrak{g}_-, \mathfrak{g}_- \oplus \mathfrak{g}_0)_2$. Writing down the structure equation of $\mathscr{R}P^{(2)}$ and taking account of our assumption that ξ', η' are completely integrable, we see that

(*)
$$c_2(z)(\mathfrak{g}_{-2} \wedge \mathfrak{g}'_{-1}) \subset \mathfrak{g}'_{-1}, \quad c_2(z)(\mathfrak{g}_{-2} \wedge \mathfrak{g}''_{-1}) \subset \mathfrak{g}''_{-1}$$

for any $z \in P^{(2)}$.

Let us denote by $\tilde{G}_2$ the structure group of the fibring $\#P^{(1)} \to P^{(1)}$ and by $\tilde{\mathfrak{g}}_2$ its Lie algebra. Then

$$\tilde{\mathfrak{g}}_2 \cong \mathrm{Hom}(\mathfrak{g}_-, \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1)_2$$
$$= \mathrm{Hom}(\mathfrak{g}_{-2}, \mathfrak{g}_0) \oplus \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_1).$$

On account of (*) and the formula (3.36), we see that for any $z^1 \in P^{(1)}$ there exists z^2 with $\pi(z^2) = z^1$ such that

$$\begin{cases} c_2(z^2)(e_0, x) = 0 & x \in \mathfrak{g}'_{-1} \\ c_2(z^2)(e_0, y) = \lambda y & y \in \mathfrak{g}''_{-1} \end{cases}$$

for some $\lambda \in \mathbf{R}$. (In fact, for any $w^2 \in \#P^{(1)}$, we can find $\alpha \in \mathrm{Hom}(\mathfrak{g}_{-2}, \mathfrak{g}_0)$ so that $z^2(= w^2(1+\alpha))$ satisfies the above condition.)

Now in the fundamental identity:

$$\partial c_2(z^2) \equiv (\tilde{\gamma}^{[1]}(z^2) \circ \tilde{\gamma}^{[1]}(z^2))|_V \quad (\mathrm{mod}\, F^3),$$

again the right-hand side vanishes because $c_1 = 0$. Hence $\partial c_2(z^2) = 0$.

From $\partial c_2(z^2)(x, y, e_0) = 0$, we deduce easily that

$$c_2(z^2)(x, y) \in \mathfrak{h}_0 = sp(\mathfrak{g}'_{-1}) \oplus sp(\mathfrak{g}''_{-1})$$

for any $x, y \in \mathfrak{g}'_{-1}$. From $\partial c_2(z^2)(x_1, x_2, x_3) = 0$ for $x_i \in \mathfrak{g}_{-1}$, we see easily that $\lambda = 0$. Therefore

$$c_2(z^2)(\mathfrak{g}_{-2}\otimes\mathfrak{g}_{-1})=0.$$

Moreover this implies $c_2(z^2)\in\mathrm{Hom}(\wedge^2\mathfrak{g}_{-1},\mathfrak{h}_0)$.

Now consider the following sequence:

$$\mathrm{Hom}(\mathfrak{g}_{-1},\mathfrak{g}_1)\xrightarrow{\ \partial\ }\mathrm{Hom}(\wedge^2\mathfrak{g}_{-1},\mathfrak{h}_0)\xrightarrow{\ \partial\ }\mathrm{Hom}(\wedge^3\mathfrak{g}_{-1},\mathfrak{g}_{-1}),$$

which is, as easily seen, exact. By the above argument $c_2(z^2)\in\mathrm{Hom}$ $(\wedge^2\mathfrak{g}_{-1},\mathfrak{h}_0)$ and closed. Hence $c_2(z^2)=\partial\beta$ for some $\beta\in\mathrm{Hom}(\mathfrak{g}_{-1},\mathfrak{g}_1)$. This shows that for any $z^1\in P^{(1)}$ we can choose $z^2\in\#P^{(1)}$ with $\pi(z^2)=z^1$ so as to get $c_2(z^2)=0$.

Put

$$\overset{\circ}{c}{}^{(2)}=c_0\quad\text{and}\quad\overset{\circ}{\gamma}{}^{(2)}=\beta^{(2)}+\overset{\circ}{c}{}^{(2)},$$

and set

$$P^{(2)}=\{z^2\in\#P^{(1)}\,|\,\gamma^{(2)}(z^2)=\overset{\circ}{\gamma}{}^{(2)}\}.$$

Then $P^{(2)}\to P^{(1)}$ is surjective and $P^{(2)}$ is an adapted subbundle of $\mathscr{R}^{(2)}(M)$ with constant structure function $\overset{\circ}{\gamma}{}^{(2)}$.

LEMMA. $P^{(2)}$ *is involutive.*

PROOF: Let us denote by $G^{(2)}$ the structure group of $P^{(2)}\to M$, and by $\mathfrak{g}^{(2)}$ its Lie algebra. It is easy to see that the prolongation of the truncated graded Lie algebra $gr(\mathfrak{g}_-\oplus\mathfrak{g}^{(2)})$ is isomorphic to $\mathfrak{g}=\underset{p\geq-2}{\oplus}\mathfrak{g}_p$, where

$$\mathfrak{g}_0=sp(\mathfrak{g}'_{-1})\oplus sp(\mathfrak{g}''_{-1})\oplus\mathfrak{z}_0$$
$$\mathfrak{g}_p=sp(\mathfrak{g}'_{-1})^{(p)}\oplus sp(\mathfrak{g}''_{-1})^{(p)}\qquad(p>0).$$

Now we claim

$$H^2(\mathfrak{g}_-,\mathfrak{g})_r=0\qquad\text{for}\quad r\geq3.$$

In fact, let $\omega\in\mathrm{Hom}(\wedge^2\mathfrak{g}_-,\mathfrak{g})_r$ and suppose that $\partial\omega=0$. Write

$$\omega=\alpha+\beta$$

with $\alpha\in\mathrm{Hom}(\mathfrak{g}_{-2}\otimes\mathfrak{g}_{-1},\mathfrak{g}_{r-3})$ and $\beta\in\mathrm{Hom}(\wedge^2\mathfrak{g}_{-1},\mathfrak{g}_{r-2})$. From $\partial\omega(x,y,e_0)=0$ for $x,y\in\mathfrak{g}_{-1}$, it follows that

$$[x,\alpha(y,e_0)]+[y,\alpha(e_0,x)]+[e_0,\beta(x,y)]=0.$$

If $r\geq3$, we have $[e_0,\beta[x,y)]=0$. From this we see that there exists $\alpha'\in\mathrm{Hom}(\mathfrak{g}_{-2},\mathfrak{g}_{r-2})$ such that

$$\alpha'(e_0)=\alpha(e_0,\cdot).$$

Put $\omega'=\omega-\partial\alpha'$, then $\omega'\in\mathrm{Hom}(\wedge^2\mathfrak{g}_{-1},\mathfrak{g}_{-2})$ and $\partial\omega'=0$. By the well-known fact that $sp(V)$ is involutive ([21]), we can find $\beta'\in\mathrm{Hom}(\mathfrak{g}_{-1},\mathfrak{g}_{r-1})$ such that $\partial\beta'=\omega'$. This proves that $H^2(\mathfrak{g}_-,\mathfrak{g})_r=0$ for $r\geq3$. Hence $P^{(2)}$ is involutive.

Thus, starting from any M-A system of wave type, we have constructed an involutive truncated tower $P^{(2)}$ with same constant structure function $\overset{\circ}{\gamma}{}^{(2)}$. Therefore, under the assumption of analyticity, Theorem 3. 9. 2 follows from Theorem 3. 6. 2.

REMARK 3. 9. 1.   In [13], we gave an outline of the proof of Theorem 3. 9. 2. on the basis of the usual method of G-structures. The proof given here based on the weighted frame bundles is more natural.

REMARK 3. 9. 2.   The classical proof of Theorem 3. 9. 1 (see [7], [10]) only uses the contact geometry and the Hamilton-Jacobi theory for the first-order PDEs. Thus the theorem is also valid in C^∞-category. Systematic studies of C^∞-integrability for transitive geometric structures have been made by many authors, in particular D. C. Spencer, V. W. Guillemin, H. Goldschmidt, P. Molino (see e.g. [6], [11]). However, as far as the author knows, there is not yet a general theory enough to cover the above theorem in C^∞-category.

3. 10.   Cartan connections.

3. 10. 1.   Definition of a Cartan connection. Let $\mathfrak{l}$ be a Lie algebra and $\mathfrak{k}$ a Lie subalgebra of $\mathfrak{l}$. Let K be a Lie group with Lie algebra $\mathfrak{k}$ equipped with a representation

$$\rho: K \to GL(\mathfrak{l})$$

such that the differential $\rho_*:\mathfrak{k}\to\mathfrak{gl}(\mathfrak{l})$ coincides with the adjoint representation of $\mathfrak{k}$ on $\mathfrak{l}$. By abuse of notation this representation ρ will be denoted by Ad.

Let $P(M,K)$ be a principal fibre bundle over a manifold M with structure group K. A Cartan connection in P of type $(\mathfrak{l},K)$ is a 1-form θ on P with values in $\mathfrak{l}$ satisfying the following conditions:

    i) $\theta: T_zP \to \mathfrak{l}$ is an isomorphism for all $z\in P$.
    ii) $R_a^*\theta=Ad(a)^{-1}\theta$ for $a\in K$.
    iii) $\theta(\tilde{A})=A$ for $A\in\mathfrak{k}$.

REMARK 3. 10. 1.   In the usual definition of a Cartan connection (cf. [9]), there is given moreover a Lie group L with Lie algebra $\mathfrak{l}$ containing K as a closed Lie subgroup, and the representation ρ of K on $\mathfrak{l}$ is the one

induced by the adjoint representation of L.

REMARK 3. 10. 2.   Our frame bundle $\mathscr{R}(M)$ has the universal property also for the Cartan connection: Assume that the pair $(\mathfrak{l}, K)$ is formally effective (see § 2. 2. 1).   By choosing a complementary subspace V of $\mathfrak{l}$ to $\mathfrak{k}$ we can view $(\mathfrak{l}, K, Ad)$ as a skeleton on V.   Then it is clear that a Cartan connection (P, M, K, θ) of type $(\mathfrak{l}, K)$ is a tower on M.   Hence, by Theorem 2. 3. 1, there exists a unique embedding $\iota : P \to \mathscr{R}(M)$ such that $\iota^* \theta_{\mathscr{R}} = \theta$.

EXAMPLE.   Let L be a Lie group and K a closed Lie subgroup.   The Maurer-Cartan form θ of L defines in the principal fibre bundle $L(L/K, K)$ a Cartan connection called the standard Cartan connection.

Let (P, θ) be a Cartan connection of type $(\mathfrak{l}, K)$.   Since θ defines an absolute parallelism on P, we can write the structure equation as follows:

$$d\theta + \frac{1}{2}[\theta, \theta] + \frac{1}{2} c(\theta \wedge \theta) = 0,$$

where c is a $\mathrm{Hom}(\wedge^2 \mathfrak{l}, \mathfrak{l})$-valued function on P called the curvature.   Note that, as easily seen from the conditions ii), iii) of the definition of a Cartan connection, we have

$$c(z)(\mathfrak{k}, \mathfrak{l}) = 0 \qquad \text{for} \quad z \in P.$$

Therefore we may view the function c as taking values in $\mathrm{Hom}(\wedge^2 \mathfrak{l}/\mathfrak{k}, \mathfrak{l})$.

Given two Cartan connections $(P, \theta), (P', \theta')$ of type $(\mathfrak{l}, K)$, we call a bundle isomorphism (or local diffeomorphism) $f : P \to P'$ an isomorphism (resp. local isomorphism) of the Cartan connections if $f^* \theta' = \theta$.   Note that, as a consequence of the definition, a local isomorphism commutes locally with the right translations.

All the local invariants of a Cartan connection can be, in principle, obtained from the curvature and its higher order derivatives (with respect to the absolute parallelism).   In paricular, if the curvature vanishes identically it is locally isomorphic to the standard Cartan connection.

Thus Cartan connections may be considered as geometric structures rather nice to deal with, and naturally arises the following question:

Given a geometric structure Γ on a manifold M, is it possible to construct a principal bundle over M and a Cartan connection θ in P in the way that (P, θ) is canonically associated with Γ?

We know various examples hitherto obtained: Riemannian, conformal or projective structures (cf. [9]), strongly pseudo-convex CR- struc-

tures [25], and more generally certain geometric structures associated with simple graded Lie algebras [26].

In the next subsections we shall give a general criterion and a unified method to construct Cartan connections.

3. 10. 2.　Criterion for the existence of Cartan connections. Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{m}_p$ be a graded Lie algebra, and M a filtered manifold regular of type $\mathfrak{m}$. Let $(\mathscr{R}^{(0)}(M, \mathfrak{m}), M, G^{(0)}(\mathfrak{m}))$ be the first order reduced frame bundle of M (see § 3. 5). Given a principal subbundle $(P^{(0)}, M, G_0)$ of $\mathscr{R}^{(0)}(M, \mathfrak{m})$ with G_0 a Lie subgroup of $G^{(0)}(\mathfrak{m})$, we ask whether there exists a Cartan connection (P, θ) naturally associated with $P^{(0)}$.

Let $\mathscr{R}P^{(0)}$ be the universal prolongation of $P^{(0)}$ and denote $G(\mathfrak{m}, G_0)$ its structure group. The canonical from θ of $\mathscr{R}P^{(0)}$ takes values in

$$E(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m} \oplus \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0),$$

where $\mathfrak{g}_0$ and $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ denote the Lie algebras of G_0 and $G(\mathfrak{m}, G_0)$ respectively. Then our $(\mathscr{R}P^{(0)}, \theta)$ is already similar to a Cartan connection, but only lacking in the condition that the space $E(\mathfrak{m}, \mathfrak{g}_0)$ in which θ takes values be a Lie algebra. We therefore seek for some nice subbundle of $\mathscr{R}P^{(0)}$ satisfying this requirement.

Let us first introduce, for a given Lie subgroup $G_0 \subset G^{(0)}(\mathfrak{m})$, a subskeleton

$$(\mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0), K(\mathfrak{m}, G_0)) \subset (E(\mathfrak{m}, \mathfrak{g}_0), G(\mathfrak{m}, G_0)).$$

Let $\mathfrak{M}$ be a standard filtered manifold of type $\mathfrak{m}$, that is a Lie group having $\mathfrak{m}$ as its Lie algebra. Then the 1-st order reduced frame bundle $\mathscr{R}^{(0)}(\mathfrak{M}, \mathfrak{m})$ is identified with the trivial bundle $\mathfrak{M} \times G^{(0)}(\mathfrak{m})$, and the trivial bundle $\mathfrak{M} \times G_0$ represents a standard geometric structure, which is clearly transitive. Applying Theorem 3. 6. 1 to this geometric structure, we obtain a tower P with constant structure function $\hat{c} = 0$. We denote by $K(\mathfrak{m}, G_0)$ its structure group and set $\mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m} \oplus \mathfrak{k}(\mathfrak{m}, \mathfrak{g}_0)$ with $\mathfrak{k}(\mathfrak{m}, \mathfrak{g}_0)$ the Lie algebra of $K(\mathfrak{m}, G_0)$

Then we have:

1)　$(\mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0), K(\mathfrak{m}, G_0)$ is a subskeleton of $(E(\mathfrak{m}, \mathfrak{g}_0), G(\mathfrak{m}, G_0))$.

2)　$a[X, Y] = [aX, aY]$　for　$a \in K(\mathfrak{m}, \mathfrak{g}_0)$,　$X, Y \in \mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0)$.

3)　$\mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0)$ is a transitive filtered Lie algebra with respect to the bracket operation defined by (3. 37), and isomorphic to the completion of $gr\mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0)$.

4)　$gr\mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0)$ is canonically isomorphic to the prolongation of the truncated graded Lie algebra $\mathfrak{m} \oplus \mathfrak{g}_0$.

5) G_0 is embedded in $K(\mathfrak{m}, G_0)$ as a closed subgroup.

Now we consider the following complex:

$$\cdots \to \operatorname{Hom}(\wedge^k\mathfrak{m}, \mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0)) \xrightarrow{\partial} \operatorname{Hom}(\wedge^{k+1}\mathfrak{m}, \mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0)) \xrightarrow{\partial} \cdots$$

Since $\mathfrak{m}$ is a Lie subalgebra of $\mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0)$, the coboundary operator ∂ is defined as usual. Note that the group $K(\mathfrak{m}, G_0)$ acts on $\operatorname{Hom}(\wedge^{\cdot}\mathfrak{m}, \mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0))$ by the induced representation of (3. 3), which evidently preserves the filtration $\{F^p\operatorname{Hom}(\wedge^{\cdot}\mathfrak{m}, \mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0))\}$.

It being prepared,

DEFINITION 3. 10. 1. We say that a Lie subgroup $G_0 \subset G^{(0)}(\mathfrak{m})$ satisfies the condition (C) if there exists a subspace

$$W \subset F^1\operatorname{Hom}(\wedge^2\mathfrak{m}, \mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0))$$

such that

i) $F^1\operatorname{Hom}(\wedge^2\mathfrak{m}, \mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0)) = W \oplus \partial F^1\operatorname{Hom}(\wedge^1\mathfrak{m}, \mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0))$,

ii) W is stable under the actions of $K(\mathfrak{m}, G_0)$.

THEOREM 3. 10. 1. *Let M be a filtered manifold regular of type $\mathfrak{m}$, and let G_0 be a Lie subgroup of $G^{(0)}(\mathfrak{m})$ satisfying the condition (C). Then for each principal subbundle $P^{(0)}$ of $\mathscr{R}^{(0)}(M, \mathfrak{m})$ with structure group G_0, we can construct a tower $P \subset \mathscr{R}(M, \mathfrak{m})$ in such a way that*:

i) *P is a tower on M with skeleton $(\mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0), K(\mathfrak{m}, G_0))$.*

ii) *The structure function $\widehat{c}$ of P takes values in W.*

iii) *The assignment $P^{(0)} \rightarrowtail P$ is compatible with equivalences.*

Thus (P, θ) is a Cartan connection of type $(\mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0), K(\mathfrak{m}, G_0))$ associated with $P^{(0)}$, where θ is the canonical form of P.

PROOF: Let us construct subbundles $P^{(l)} \subset \mathscr{R}^{(l)}(M, \mathfrak{m})$ by induction on l, with $P^{(0)}$ the given one.

For an $l \geq 0$, suppose that we have constructed $P^{(l)}$ so as to satisfy:

1) $P^{(l)}$ is an adapted subbundle of $\mathscr{R}^{(l)}(M, \mathfrak{m})$ with structure group $K^{(l)}$, where $K^{(l)} = K(\mathfrak{m}, G_0)/F^{l+1}$.

2) The structure function $\widehat{c}^{(l)}$ of $P^{(l)}$ takes values in $W^{(l)} = W/F^{l+1}$, where $\widehat{c}^{(l)}$ is the $F^1\operatorname{Hom}(\wedge^2\mathfrak{m}, \mathfrak{m} \oplus \mathfrak{k}(\mathfrak{m}, \mathfrak{g}_0))^{(l)}$-valued function defined by decomposing the total structure function: $\gamma^{[l]} = \beta_{\mathfrak{m}}^{[l]} + \widehat{c}^{(l)}$.

Now we construct $P^{(l+1)}$ as follows: Let $\#P^{(l)}$ be the prolongation of $P^{(l)}$. Then its structure funtion $\widehat{c}^{(l+1)}$ takes values in $F^1\operatorname{Hom}(\wedge^2\mathfrak{m}, \mathfrak{m} \oplus \mathfrak{g}(\mathfrak{k}^{(l)}))^{(l+1)}$. This space is identified with $F^1\operatorname{Hom}(\wedge^2\mathfrak{m}, \mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0))^{(l+1)}$(because

of the truncation), which has the $K^{(l)}$-invariant direct sum decomposition:

$$F^1\mathrm{Hom}(\wedge^2\mathfrak{m}, \mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0))^{(l+1)} = W^{(l+1)} \oplus \partial F^1\mathrm{Hom}(\mathfrak{m}, \mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0))^{(l+1)}.$$

We then set:

$$P^{(l+1)} = \{z \in \#P^{(l)} | \, \hat{c}^{(l+1)}(z) \in W^{(l+1)}\}.$$

By Proposition 3. 4. 2 it is easy to see that $P^{(l+1)}$ is a principal fibre bundle over $P^{(l)}$ with structure group $F^{k+1}K(\mathfrak{m}, G_0)/F^{l+2}$ ($\cong gr_{l+1}\mathfrak{l}$). Moreover, since $W^{(l+1)}$ is $K^{(l)}$-invariant and $\beta_\mathfrak{m}(aX, aY) = a\beta_\mathfrak{m}(X, Y)$ for $a \in K$, $X, Y \in \mathfrak{m} \oplus \mathfrak{k}$, it follows easily from the formula

$$\beta_\mathfrak{m}^{[l+1]} + \hat{c}^{(l+1)}(za) = (\rho(a^{-1})\beta_\mathfrak{m})^{[l+1]} + \rho(a^{-1})\hat{c}^{(l+1)}(z)$$
$$\text{for} \quad z \in \#P^{(l)}, \; a \in \#K^{(l)}$$

that $K^{(l+1)}$ acts on $P^{(l+1)}$. It then turns out that $P^{(l+1)}$ is an adapted sub-bundle of $\mathscr{R}^{(l+1)}(M, \mathfrak{m})$ with structure group $K^{(l+1)}$, which completes the inductive construction of $P^{(l)}$. Putting $P = \lim P^{(l)}$, we obtain a tower satisfying the required conditions.

REMARK 3. 10. 3. If the dimension of $\mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0)$ is finite, say $F^{k+1}\mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0) = 0$, then in constructing the Cartan connection (P, θ), we have

$$P^{(k)} \cong P^{(k+1)} \cong \cdots \cong P^{(k+\mu)} \cong \#P^{(k+\mu)} \cong \cdots \cong P.$$

Moreover, since θ ($= \pi^k \circ \theta$, with $\pi^k : \mathfrak{l} \to \mathfrak{l}^{(k)}$ the canonical projection) is defined already on $P^{(k+\mu)}$ and yields an absolute parallelism of $P^{(k)}$ and hence of $P^{(k+\mu)}$, we can identiby (P, θ) with $(P^{(k+\mu)}, \theta^{(k)})$. Thus the contruction actually finishes at $P^{(k+\mu)}$.

3. 10. 3. Various examples of Cartan connections are hitherto known. The most general result has been the theorem of Tanaka [26] which states that one can construct a Cartan connection associated with a subbundle $P^{(0)}(M, G_0)$ of $\mathscr{R}^{(0)}(M, \mathfrak{m})$ if $\mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0)$ is finite dimensional and simple.

Theorem 3. 10. 1 gives a new proof of the above theorem if we admit the following algebraic fact: The condition (C) is satisfied if $\mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0)$ is finite dimensional and simple. This statement is essentially shown in [26]. Mereover its proof suggests us the following more general sufficient condition for G_0 to satisfy the condition (C).

PROPOSITION 3. 10. 1. *Let* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{m}_p$ *be a graded Lie algebra and let* G_0 *be a Lie subgroup of* $G^{(0)}(\mathfrak{m})$. *Assume that* $\mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0)$ (*simply denoted by* $\mathfrak{l}$ $= \oplus \mathfrak{l}_p$) *is finite dimensional and that there exists a positive definite symmet-*

ric bilinear form

$$(\, , \,) : \mathfrak{l} \times \mathfrak{l} \to \boldsymbol{R}$$

satisfying :

i) $(\mathfrak{l}_p, \mathfrak{l}_q) = 0$ *if* $p \neq q$.

ii) *There exists* $\tau : \mathfrak{k}(\mathfrak{m}, \mathfrak{g}_0) \to \mathfrak{l}$ *such that*

$$\begin{cases} \tau(\mathfrak{l}_p) \subset \mathfrak{l}_{-p} & \textit{for} \quad p \geq 0 \\ ([A, x], y) = (x, [\tau(A), y]) & \textit{for} \quad x, y \in \mathfrak{l}, \; A \in \mathfrak{k}. \end{cases}$$

iii) *There exists* $\tau_0 : G_0 \to G_0$ *such that*

$$(ax, y) = (x, \tau_0(a)y) \qquad \textit{for} \quad x, y \in \mathfrak{l}, \; a \in G_0.$$

Then G_0 satisfies the condition (C).

PROOF: We extend the inner product of $\mathfrak{l}$ to that of $\mathrm{Hom}(\wedge^p \mathfrak{m}, \mathfrak{l})$ by the following formula :

$$(\phi, \psi) = n! \sum_{1 \leq i_1 < \cdots < i_p \leq n} (\phi(e_{i_1}, \cdots, e_{i_p}), \psi(e_{i_1}, \cdots, e_{i_p}))$$

for $\phi, \psi \in \mathrm{Hom}(\wedge^p \mathfrak{m}, \mathfrak{l})$, where $n = \dim \mathfrak{m}$ and $\{e_1, \cdots, e_n\}$ is an orthonormal basis of $\mathfrak{m}$.

We then define the formal adjoint ∂^* of ∂

$$\cdots \xleftarrow{\ \partial^*\ } \mathrm{Hom}(\wedge^p \mathfrak{m}, \mathfrak{l}) \xleftarrow{\ \partial^*\ } \mathrm{Hom}(\wedge^{p+1} \mathfrak{m}, \mathfrak{l})) \xleftarrow{\ \partial^*\ } \cdots$$

by

$$(\phi, \partial\psi) = (\partial^*\phi, \psi)$$

for $\phi \in \mathrm{Hom}(\wedge^{p+1} \mathfrak{m}, \mathfrak{l})$, $\psi \in \mathrm{Hom}(\wedge^p \mathfrak{m}, \mathfrak{l})$. Then as is well-known, we have the direct sum decomposition :

$$\mathrm{Hom}(\wedge^p \mathfrak{m}, \mathfrak{l})) = \partial \mathrm{Hom}(\wedge^{p-1} \mathfrak{m}, \mathfrak{l})) \oplus Ker \, \partial^*.$$

Now let us show that $Ker \, \partial^*$ is an invariant subspace by the actions of $K(\mathfrak{m}, G_0)$.

If we denote by ρ the representation of K on $\mathrm{Hom}(\wedge^{\cdot} \mathfrak{m}, \mathfrak{l}))$, it suffices to show

$$\partial^* \circ \rho(a) = \rho(a) \circ \partial^* \qquad \text{for} \quad a \in K.$$

But since any element $a \in K$ is written as

$$a = a_0 \cdot exp A$$

with $a_0 \in G_0$, $A \in F^1 \mathfrak{k}(\mathfrak{m}, \mathfrak{g}_0)$, it suffices to show

(a) $\partial^* \circ \rho(a_0) = \rho(a_0) \circ \partial^*$ for $a_0 \in G_0$

(b) $\partial^* \circ \rho_*(A) = \rho_*(A) \circ \partial^*$ for $A \in \mathfrak{k}$,

where ρ_* denote the representation of $\mathfrak{k}$ on $\mathrm{Hom}(\wedge \cdot \mathfrak{m}, \mathfrak{l})$

On the other hand, we have in general

$$\partial \circ \rho(a_0) = \rho(a_0) \circ \partial \quad \text{for} \quad a_0 \in G_0$$

and

$$\partial \circ \lambda(B) = \lambda(B) \circ \partial \quad \text{for} \quad B \in \mathfrak{l}^{(0)} = \bigoplus_{p \leq 0} \mathfrak{l}_p,$$

where λ denotes the representation of $\mathfrak{l}^{(0)}$ on $\mathrm{Hom}(\wedge \cdot \mathfrak{m}, \mathfrak{l})$. Now it follows easily from our assumption that

$$(\rho(a_0)\phi, \psi) = (\phi, \rho(\tau_0(a_0))\psi)$$
$$(\rho_*(A)\phi, \psi) = (\phi, \lambda(\tau(A))\psi)$$

for $a_0 \in G_0$, $A \in \mathfrak{k}$, $\phi, \psi \in \mathrm{Hom}(\wedge^p \mathfrak{m}, \mathfrak{l})$.

Hence we have

$$(\partial^* \rho_*(A)\phi, \psi) = (\rho_*(A)\phi, \partial \psi)$$
$$= (\phi, \lambda(\tau(A))\partial \psi)$$
$$= (\phi, \partial \lambda(\tau(A))\psi)$$
$$= (\partial^* \phi, \lambda(\tau(A))\psi)$$
$$= (\rho_*(A)\partial_* \phi, \psi)$$

which shows (b). Similarly we can verify (a).

If we set $W = Ker\, \partial_* \cap F^1 \mathrm{Hom}(\wedge^2 \mathfrak{m}, \mathfrak{l})$, then W has the desired property.

REMARK 3. 10. 4. There are non-simple Lie algebras which satisfy the assumption of Proposition 3. 10. 1 (and hence the condition (C)) among the Lie algebras $\mathfrak{l}$ obtained as the semi-direct product $\mathfrak{l} = \mathfrak{s} \ltimes \mathfrak{v}$ of a semi-simple graded Lie algebra $\mathfrak{s} = \oplus \mathfrak{s}_p$ and a graded vector space $\mathfrak{v} = \bigoplus_{p<0} \mathfrak{v}_p$ by an irreducible representation ρ of $\mathfrak{s}$ on $\mathfrak{v}$ such that $\rho(\mathfrak{s}_p)\mathfrak{v}_q, \subset \mathfrak{v}_{p+q}$. A typical example of this type has first appeared in the geometrie study of ordinary differential equations due to N. Tanaka [27], where $\mathfrak{s} = \mathfrak{sl}(n)$ and $\mathfrak{v}$ is the space of the homogeneous polynomials in n-variables of degree μ. It is through this example that Proposition 3. 10. 1 was noticed independently by N. Tanaka and the author. In the near future Tanaka will publish a ditailed treatise on the geometric study of ordinary differential equations. For other examples see T. Yatsui [30].

3. 11. Example (Conformal structures)

It is quite well-known that associated with a riemannian structure there exists a unique Levi-Civita connection. This is one of the simplest examples of Cartan connections. It is also well-known but seems less familifar that there exists a Cartan connection associated with a conformal structure ([3], [23], [9]). We will explain this classical fact from our point of view, which will serve as a good illustration of the general method given in the preceding section.

A conformal structure on a differentiable manifold M is a conformal equivalence class $[g]$ of a pseudo-riemannian metric g on M, where two metrics g and g' on M are said to be equivalent if there exists a function ρ such that $g' = \rho g$.

Let $(\mathscr{R}^{(0)}(M), M, G^{(0)}(V))$ be the first order frame bundle of M. (Here M is viewed as a trivial filtered manifold, so that V is simply a vector space of the same dimension as M, $G^{(0)}(V) = GL(V)$, and $\mathscr{R}^{(0)}(M)$ is the usual linear frame bundle of M.)

Let us fix an inner product $(,)$ of V of signature (r, s). Let $CO(V)$ be the linear conformal transformation group, that is,

$$CO(V) = \{a \in GL(V) \mid \exists \lambda \ s.t. \ (ax, ay) = \lambda(x, y) \ \text{ for } \ x, y \in V\},$$

and let $\mathrm{co}(V)$ donote its Lie algebra.

It is fundamental that there is a bijective correspondence between the conformal structures on M of signature $\{r, s\}$ and the $CO(V)$-subbundles of $\mathscr{R}^{(0)}(M)$.

We know by a simple calculation (see [21]) that if dim $V \geq 3$ then the prolongation of the truncated graded Lie algegra $V \oplus \mathrm{co}(V)$ is isomorphic to

$$(3.38) \qquad\qquad V \oplus \mathrm{co}(V) \oplus V^*.$$

The bracket operation is defined, for $v \in V$, $\alpha \in V^*$, by

$$[v, \alpha] = \langle v, \alpha \rangle id_V + v \otimes \alpha - (v \otimes \alpha)^\dagger$$

where $\varphi^\dagger$ denotes the adjoint of $\varphi \in \mathrm{Hom}(V, V)$ determined by

$$(\varphi(v), \omega) = (v, \varphi^\dagger(w)).$$

The brackets for the other pairs are defined by the natural actions of $\mathrm{co}(V)$ on V, V^* and $\mathrm{co}(V)$.

We remark that if $\dim V \leq 2$ then the prolongation of $V \oplus \mathrm{co}(V)$ is infinite dimensional.

The Lie algebra (3.38) can be represented as the infinitesimal conformal transformation group of the Möbius space. Let $\boldsymbol{R}^{n+2}$ be a pseudo-

Euclidean space endowed with an inner product $(\ , \ )$ of signature $(r+1, s+1)$, say

$$(x, y) = {}^t x \hat{J} y \qquad \text{for} \quad x, y \in \boldsymbol{R}^{n+2}$$

with

$$\hat{J} = \begin{pmatrix} & & -1 \\ & J & \\ -1 & & \end{pmatrix}, \quad J = \begin{pmatrix} I_r & \\ & -I_s \end{pmatrix}.$$

Let Q^n be the quadric in the projective space $P(\boldsymbol{R}^{n+2})$ defined by the homogeneous equation: $(x, x) = 0$. The inner product of $\boldsymbol{R}^{n+2}$ induces on Q a natural conformal structure. The linear group

$$O(n+2, \hat{J}) = \{a \in GL(n+2, \boldsymbol{R}) \mid {}^t a \hat{J} a = \hat{J}\}$$

acts on P^{n+1} as projective transformations and induces on Q^n conformal transformations. Moreover the action of $L = O(n+2, \hat{J})/Z$ on Q^n is effective and transitive, where Z is the center of $O(n+2, \hat{J})$ consisting of $\{E, -E\}$. Thus if we denote by H the isotroy subgroup of L at $[1, 0, \cdots, 0] \in Q$, we have the Möbius space $L/H = Q$.

The Lie algebra $\mathfrak{l}$ $(= \mathfrak{o}(n+2, \hat{J}))$ of L has a gradation $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ given by the following matrix decomposition:

$$l_{-1} = \left\{ \begin{pmatrix} 0 & & \\ v & 0 & \\ 0 & {}^t v J & 0 \end{pmatrix} \ ; \ v \in \boldsymbol{R}^n \right\}$$

$$l_0 = \left\{ \begin{pmatrix} \lambda & & \\ & A & \\ & & -\lambda \end{pmatrix} \ ; \ \begin{matrix} \lambda \in \boldsymbol{R} \\ {}^t A J + J A = 0 \end{matrix} \right\}$$

$$l_1 = \left\{ \begin{pmatrix} 0 & \alpha & 0 \\ & 0 & J^t \alpha \\ & & 0 \end{pmatrix} \ ; \ \alpha \in (\boldsymbol{R}^n)^* \right\}$$

The Lie algebra $\mathfrak{h}$ of H is then given by: $\mathfrak{h} = \mathfrak{l}_0 \oplus \mathfrak{l}_1$. One can verify easily that the graded Lie algebra $V \oplus \mathfrak{co}(V) \oplus V^*$ is isomorphic to $\oplus \mathfrak{l}_p$.

In section 3.8.2 we have defined a skeleton $(\mathfrak{l}(\mathfrak{m}, \mathfrak{g}_0), K(\mathfrak{m}, G_0))$ for a given Lie subgroup $G_0 \subset G^{(0)}(\mathfrak{m})$. In the present case where $\mathfrak{m} = V$, $G_0 = CO(V) \subset G^{(0)}(V)$, we see that $(\mathfrak{l}(V, \mathfrak{co}(V)), K(V, CO(V)))$ is isomorphic to the skeleton $(\mathfrak{l}, H)$ just constructed above.

To see that $(\mathfrak{l}, H)$ satisfies the conditions of Proposition 3.10.1, we notice that in general if we define an inner product of $\mathfrak{gl}(n, \boldsymbol{R})$ by

$$(X, Y) = Tr X^t Y \qquad X, Y \in \mathfrak{gl}(n, \mathbf{R})$$

and set : $\tau(X) = {}^t X$, then we have

$$([A, X], Y) = (X, [\tau(A), Y]) \qquad X, Y, A \in \mathfrak{gl}(n, \mathbf{R})$$
$$(Ad(a)X, Y) = (X, Ad((\tau(a))Y \qquad X, Y \in \mathfrak{gl}(n, \mathbf{R}), \quad a \in GL(n, \mathbf{R}).$$

Since $O(n+2, \tilde{J})$ is invariant by τ, it is clar that $(\mathfrak{l}, H)$ with the induced inner product satisfies the conditions of Proposition 3. 10. 1. (This is a special case of the fact that the conditions are satisfied if $\mathfrak{l}$ is simple.)

According to the general prescription, let

$$\partial_* : \mathrm{Hom}(\wedge^2 \mathfrak{l}_{-1}, \mathfrak{l}) \to \mathrm{Hom}(\mathfrak{l}_{-1}, \mathfrak{l}),$$

be the adjoint of ∂ defined by using the inner product of $\mathfrak{l}$ and set

$$W = F^1 \mathrm{Hom}(\wedge^2 \mathfrak{l}_{-1}, \mathfrak{l}) \cap Ker \, \partial_*$$

then we have

$$F^1 \mathrm{Hom}(\wedge^2 \mathfrak{l}_{-1}, \mathfrak{l}) = \partial F^1 \mathrm{Hom}(\mathfrak{l}_{-1}, \mathfrak{l}) \oplus W.$$

In homogeneous components

$$\mathrm{Hom}(\wedge^2 \mathfrak{l}_{-1}, \mathfrak{l}_p) = \partial \mathrm{Hom}(\mathfrak{l}_{-1}, \mathfrak{l}_{p+1}) \oplus W_{p+2} \qquad (p = -1, 0, 1)$$
$$W = W_1 \oplus W_2 \oplus W_3.$$

But here $W_1 = 0$ because $\partial \mathrm{Hom}(V, \mathfrak{o}(V)) = \mathrm{Hom}(\wedge^2 V, V)$ (the reason for the existence of Levi-Civita connection), and obviously $W_3 = \mathrm{Hom}(\wedge^2 \mathfrak{l}_{-1}, \mathfrak{l}_1)$. By definition W_2 is the kernel of $\partial_* : \mathrm{Hom}(\wedge^2 \mathfrak{l}_{-1}, \mathfrak{l}_0) \to \mathrm{Hom}(\mathfrak{l}_{-1}, \mathfrak{l}_1)$ and can be expressed in terms of components as follows: Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of $\mathfrak{l}_{-1}$. For $a \in \mathrm{Hom}(\wedge^2 \mathfrak{l}_{-1}, \mathfrak{l}_0)$ write

$$a(e_i, e_j) e_k = \sum a^l_{ijk} e_l.$$

Then a simple calculation shows that $\partial^* a = 0$ if and only if

$$\sum_a a^a_{iaj} = 0.$$

Now given a $CO(V)$-subbundle $P^{(0)}$ of $\mathscr{R}^{(0)}(M)$, let us construct a Cartan connection of type $(\mathfrak{l}, H)$ associated with $P^{(0)}$.

Let $\#P^{(0)} \subset \mathscr{R}^{(1)}(M)$ be the prolongation of $P^{(0)}$. The structure function $\hat{c}^{(1)}$ takes values in $\mathrm{Hom}(\wedge^2 \mathfrak{l}_{-1}, \mathfrak{l}_{-1})$. Since $W_1 = 0$, putting

$$P^{(1)} = \{z \in \#P^{(0)} | \hat{c}^{(1)}(z) = 0\},$$

we get a principal subbundle of $\mathscr{R}^{(1)}(M)$ with structure group $H \subset G^{(1)}(V)$. Next consider the structure function $\hat{c}^{(2)}$ of $\#P^{(1)}$ and set

$$P^{(2)} = \{z \in \#P^{(1)} \mid \hat{c}^{(2)}(z) \in W_2\}.$$

The fact that $\mathfrak{l}_2 = 0$ implies that $P^{(2)} \to P^{(1)}$ is bijective and the structure group of $P^{(2)} \to M$ is identified with H. Then the successive prolongation only yields isomorphic bundles

$$P^{(1)} \cong P^{(2)} \cong \#P^{(2)} \cong \cdots \cong \#^l P^{(2)}.$$

Let $P = \lim \#^l P^{(2)} \subset \mathscr{R}(M)$ and θ the canonical form of P, which takes values in $\mathfrak{l}$. Thus (P, θ) is a Cartan connection of type $(\mathfrak{l}, H)$. We remark that since $P^{(2)} \subset \mathscr{R}^{(2)}(M) = \mathscr{R}^{(2)}(M)$, the canonical form θ is already defined on $P^{(2)}$. Thus the construction is actually finished at $P^{(2)}$.

The structure equation of (P, θ) is written as

$$d\theta + \frac{1}{2}[\theta, \theta] + \frac{1}{2}c(\theta_{-1}, \theta_{-1}) = 0,$$

where θ_{-1}, denotes the $\mathfrak{l}_{-1}$-component of θ and c is a function on P taking values in W. According to the decomposition $W = W_2 \oplus W_3$, we have the decomposition

(3. 39) $c = c_2 + c_3,$

where c_2 is a $\mathrm{Hom}(\wedge^2 \mathfrak{l}_{-1}, \mathfrak{l}_0)$-valued function satisfying

(3. 40) $\partial^* c_2 = 0$

and c_3 is a $\mathrm{Hom}(\wedge \mathfrak{l}_{-1}, \mathfrak{l}_1)$-valued function.

Since the total structure function, $\gamma = [\ , \ ] + c$, satisfies $\gamma \circ \gamma = D\gamma$, by equating with zero the homogeneous parts of degree 2 and 3, we have

(3. 41) $\begin{cases} \partial c_2 = 0 \\ \partial c_3 = D c_2. \end{cases}$

The last formula implies, in particular, that if c_2 is constant and if $\dim M \geq 4$ then $c_3 = 0$, since, as easily verified, $\partial : \mathrm{Hom}(\wedge^2 \mathfrak{l}_{-1}, \mathfrak{l}_1) \to \mathrm{Hom}(\wedge^3 \mathfrak{l}_{-1}, \mathfrak{l}_0)$ is injective if $\dim \mathfrak{l}_{-1} \geq 4$.

This example offers a prototype to construct Cartan connections.

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Department of Mathematics
Kyoto University of Education
Kyoto 612 Japan