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# Nonlinear oscillations in the wave of a string

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## Introduction

In this paper we investigate the existence of solutions $u(x, t)$ for a piecewise linear perturbation $bu^{+}-au^{-}$ of the 1-dimensional wave operator $u_{tt}-u_{xx}$ under Dirichlet boundary condition on the interval $-\frac{\pi}{2}<x<\frac{\pi}{2}$ and $\pi$-periodic condition on the variable $t$,

$$u_{tt}-u_{xx}+bu^{+}-au^{-}=f(x, t) \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\times \mathbf{R} \tag{0.1}$$
$$u\left(\pm\frac{\pi}{2}, t\right)=0$$
$$u(x, t+\pi)=u(x, t).$$

Here $bu^{+}$ is an upward restoring force and $au^{-}$ a downward restoring force. We shall assume that $f$ is even in $x$ and $t$, periodic in $t$ with period $\pi$, and we shall look for $\pi$-periodic solutions of (0. 1).

Let $L$ be the 1-dimensional wave operator, in $\mathbf{R}^2$

$$Lu=u_{tt}-u_{xx}.$$

Then the eigenvalue problem

$$Lu=\lambda u \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\times \mathbf{R} \tag{0.2}$$
$$u\left(\pm\frac{\pi}{2}, t\right)=0 \tag{0.3}$$
$$u(x, t)=u(-x, t)=u(x, -t)=u(x, t+\pi) \tag{0.4}$$

has infinitely many eigenvalues

$$\lambda_{mn}=(2n+1)^2-4m^2 \quad (m, n=0, 1, 2, \cdots).$$

Hence the eigenvalues in the interval $(-15, 9)$ are given by

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$$\lambda_{32}=-11<\lambda_{21}=-7<\lambda_{10}=-3<\lambda_{00}=1<\lambda_{11}=5.$$

The eigenfunction $\phi_{mn}$ corresponding to $\lambda_{mn}$ is given by

$$\phi_{mn}=\cos 2mt\cos(2n+1)x \quad (m,n=0,1,2,\cdots).$$

Let $Q$ be the square $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $H$ the Hilbert space defined by

$$H=\{u\in L^2(Q)|u \text{ is even in } x \text{ and } t\}.$$

Then the set $\{\phi_{mn}|m, n=0,1,2,\cdots\}$ is an orthogonal set in $H$.

The existence of multiple solutions of equation (0. 1), when the forcing term $f$ is supposed to be a multiple $s\phi_{00}(s\neq 0, s\in\mathbf{R})$ of the first positive eigenfunction and the nonlinearity $-(bu^+-au^-)$ crosses the first negative eigenvalue, was shown by a variational reduction method in [2].

In Section 1, we shall investigate the existence of multiple solutions of equation (0. 1) when the nonlinearity crosses the first positive eigenvalue.

In Section 2, we shall investigate the existence of multiple solutions of equation (0, 1) when the nonlinearity crosses the first positive eigenvalue and the first negative eigenvalue.

In proving theorems in Section 1 and Section 2 we applied the topological method developed in [5].

## 1 The nonlinearity crosses the first positive eigenvalue

In this section we study equation (0. 1) when the nonlinearity $-(bu^+-au^-)$ crosses the first positive eigenvalue. To get some feeling for this situation we study a simple equation

$$Lu+bu^+-au^-=s\phi_{00} \text{ in } H. \tag{1. 1}$$

If $\mu_1$ and $\mu_2$ are successive eigenvalues and $-\mu_2<a, b<-\mu_1$, then equation (1. 1) bas exactly one solution for all real $s$ (see[3]).

We now examine the equation when $a<-1<b$.

THEOREM 1.1. *Assume $a<-\lambda_{00}=-1<b$. Then we have:*

(i) *If $s<0$, (1. 1) has no solution.*
(ii) *If $s=0$, (1. 1) has only the trivial solution.*

PROOF. We rewrite (1. 1) as

$$(L-\lambda_{00})u+(b+\lambda_{00})u^+-(a+\lambda_{00})u^-=s\phi_{00}.$$

Multiply across by $\phi_{00}(x)$ and integrate over $Q$. Since $((L-\lambda_{00})u, \phi_{00})=0$, we have

$$\int_Q \{(b+\lambda_{00})u^+ - (a+\lambda_{00})u^-\}\phi_{00} = s\int_Q \phi_{00}^2 = \frac{\pi^2}{2}s. \tag{1.2}$$

But $(b+\lambda_{00})u^+ - (a+\lambda_{00})u^- \geq 0$ for all real valued function $u$. Also $\phi_{00}(x) > 0$ in $Q$. Therefore the left hand side of (1.2) is always greater than or equal to zero, hence there is no solution of (1.1) if $s<0$. Also, if $s=0$, then the only possibility is that $u\equiv 0$. ■

If $b<-1<a$ and $s>0$, then the left hand side of (1.2) is less than or equal to zero and the right hand side of it is positive. Therefore we have the following theorem.

THEOREM 1.2. *Assume* $b<-1<a$. *Then we have*:
(i) *If* $s>0$, (1.1) *has no solution.*
(ii) *If* $s=0$, (1.1) *has only the trivial solution.*

THEOREM 1.3. *Assume that* $-5<a<-1<b<3$ *and* $s>0$. *Then equation* (1.1) *has exactly two solutions.*

PROOF. Let $V$ be the subspace of $H$ spanned by $\phi_{00}$ and $W=V^\perp$. Let $P$ be the orthogonal projection in $H$ onto $V$. Then for all $u\in H$, $u=v+w$, where $v=Pu$, $w=(I-P)u$. Since the operators $P$ and $I-P$ commute with $L$, we have that equation (1.1) is equivalent to the pair of equations

$$\begin{aligned} &\text{(a)} \quad Lw+(I-P)(b(v+w)^+ - a(v+w)^-)=0, \\ &\text{(b)} \quad Lv+P(b(v+w)^+ - a(v+w)^-)=s\phi_{00}. \end{aligned} \tag{1.3}$$

First we show that for fixed $v\in V$, equation (1.3.a) has a unique solution $w(v)$. Let $\delta=-1$ and $g(\xi)=b\xi^+ - a\xi^- - \delta\xi$. Then equation (1.3.a) is equivalent to

$$w=(L+\delta)^{-1}(I-P)(-g(v+w)). \tag{1.4}$$

Since $(L+\delta)^{-1}(I-P)$ is a self-adjoint, compact linear map from $(I-P)H$ into itself, the eigenvalues of $(L+\delta)^{-1}(I-P)$ in $W$ are $(\lambda_{mn}+\delta)^{-1}$, where $\lambda_{mn}\geq 5$ or $\lambda_{mn}\leq -3$. Therefore its $L^2$ norm is $\frac{1}{4}$. Since

$$|g(\xi_2)-g(\xi_1)| \leq \max\{|b-\delta|, |\delta-a|\}|\xi_2-\xi_1| < 4|\xi_2-\xi_1|,$$

it follows that the right hand side of (1.4) defines, for fixed $v\in V$, a Lipschitz mapping $(I-P)H$ into itself with Lipschitz constant $\gamma<1$. Therefore, by the contraction mapping principle, for given $v\in V$, there exists a unique $w\in(I-P)H$ which satisfies (1.4). We note that $w\equiv 0$ is a solution of (1. 3. a) for any $v \in V = PH$. Thus $bv^+ - av^- = bv$ for all $v > 0$ and $bv^+ - av^- = av$ for all $v < 0$.

In either case we have $(I-P)(bv^+ - av^-) = 0$ and hence $w \equiv 0$ satisfies

$$L0 + (I-P)(bv^+ - av^-) = 0.$$

Thus equation (1. 1) is reduced to

$$Lv + P(bv^+ - av^-) = s\phi_{00},$$

where $v = c\phi_{00}$, $c \in \mathbf{R}$.

Case 1. $c > 0$. In this case, we have

$$-\lambda_{00}c + bc = s, \qquad c = \frac{s}{b - \lambda_{00}},$$

Case 2. $c < 0$. In this case, we have

$$-\lambda_{00}c + ac = s. \qquad c = \frac{s}{a - \lambda_{00}},$$

and therefore (1. 1) has exactly two solutions. This concludes the proof of this theorem. ■

## 2 The nonliniarity crosses the first positive eigenvalue and the first negative eigenvalue

In this section we study equation (0. 1) in the case where the nonlinearity $-(bu^+ - au^-)$ crosses the first positive eigenvalue and the first negative one. We first consider a simple equation, $-5 < a < -1 = -\lambda_{00}$, $-\lambda_{10} = 3 < b < 7$,

$$Lu + bu^+ - au^- = s\phi_{00} \quad \text{in } H. \tag{2. 1}$$

In section 1, we proved that equation (2. 1) has no solution for $s < 0$.

THEOREM 2.1. *Let* $-5 < a < -1, 3 < b < 7$, *and* $s > 0$. *Then equation* (2. 1) *has at least four solutions.*

PROOF. We use the contraction mapping theorem to reduce the problem from an infinite dimensional one in $L^2(Q)$ to a finite dimensional one.

Let $V$ be the two dimensional subspace of $H$ spanned by $\{\phi_{00}, \phi_{10}\}$ and $W$ the orthogonal complement of $V$ in $H$. Let $P$ be an orthogonal projection $H$ onto $V$. Then for all $u \in H$, $u = v + w$, where $v = Pu$, $w = (I - P)u$. Therefore equation (2. 1) is equivalent to

$$\begin{aligned}&\text{(a)}\quad Lw+(I-P)(b(v+w)^{+}-a(v+w)^{-})=0,\\&\text{(b)}\qquad\quad Lv+P(b(v+w)^{+}-a(v+w)^{-})=s\phi_{00}.\end{aligned}\tag{2.2}$$

We look on this as a system of two equations in the two unknowns $v$ and $w$. Let us show that for fixed $v$, (2.2.a) has a unique solution $w=\vartheta(v)$, and that, furthermore, $\vartheta(v)$ is Lipschitz continuous in terms of $v$. This step is similar to the proof of Theorem 1.3.

Let $\delta=\frac{1}{2}(\lambda_{00}+\lambda_{10})=-1$. Write (2.2.a) as

$$(L-\delta)w=-(I-P)(b(v+w)^{+}-a(v+w)^{-}+\delta(v+w))$$

or equivalently

$$w=-(L-\delta)^{-1}(I-P)g_v(w),\tag{2.3}$$

where

$$g_v(w)=b(v+w)^{+}-a(v+w)^{-}+\delta(v+w).$$

Since

$$\begin{aligned}&|g_v(w_1)-g_v(w_2)|\leq\max\{|b+\delta|,\ |a+\delta|\}|w_1-w_2|,\\&\|g_v(w_1)-g_v(w_2)\|\leq\max\{(|b+\delta|,\ |a+\delta|\}\|w_1-w_2\|,\end{aligned}$$

where $\|\ \|$ is the norm in $L^2(Q)$. The operator $(L-\delta)^{-1}(I-P)$ is a self-adjoint, compact linear map from $(I-P)H$ into itself, the eigenvalues of $(L-\delta)^{-1}(I-P)$ in $W$ are $(\lambda_{mn}-\delta)^{-1}$, where $\lambda_{mn}\geq 5$ or $\lambda_{mn}\leq -7$. Therefore its $L^2$ norm is $\frac{1}{6}$. Since max $\{|b+\delta|,\ |a+\delta|\}<6$, it follows that for fixed $v\in V$, the right hand side of (2.3) defines a Lipschitz mapping $(I-P)H$ into itself with Lipschitz constant $\gamma<1$. Therefore, by the contration mapping principle, for given $v\in V$, there exists a unique $w\in W$ which satisfies (2.3). Also it follows, by the standard argument principle that $\vartheta(v)$ is Lipschitz continuous in terms of $v$.

Therefore we have reduced equation (2.1) to the study of an equivalent problem

$$Lv+P(b(v+\theta(v))^{+}-a(v+\theta(v))^{-})=s\phi_{00}\tag{2.4}$$

definded on the two dimensional subspace $PH$ spanned by $\{\phi_{00},\ \phi_{10}\}$.

While one feels instinctively that (2.4) ought be easier to solve, there is the disadvantage of an implicitly defined term $\theta(v)$ in the equation. However, in our case, it turns out that we know $\theta(v)$ for some very important $v$'s.

If $v\geq 0$ or $v\leq 0$, then, $\theta(v)\equiv 0$. For example, let us take $v\geq 0$ and $\theta(v)=0$. Then equation (2. 2. a) reduces to

$$L0+(I-P)(bv^{+}-av^{-})=0$$

which is satisfied because $v^{+}=v$, $v^{-}=0$ and $(I-P)v=0$, since $v\in PH$.

Since $v=c_1\phi_{00}+c_2\phi_{10}$, there exists a cone $C_1$ defined by $c_1\geq 0$, $|c_2|\leq \varepsilon_0 c_1$ so that $v\geq 0$ for all $v\in C_1$ and a cone $C_2$, $c_1\leq 0$, $|c_2|\leq\varepsilon_0|c_1|$ so that $v\leq 0$ for all $v\in C_2$.

Thus, we do not know $\theta(v)$ for all $v\in PH$, but we know $w\equiv 0$ for $v\in C_1\cup C_2$, and we need to study the map

$$v\rightarrow\Phi(v)=Lv+P(b(v+\theta(v))^{+}-a(v+\theta(v))^{-}).$$

First we consider the image of the cone $C_1$. If $v=c_1\phi_{00}+c_2\phi_{10}\geq 0$ we have

$$\begin{aligned}\Phi(v)&=-\lambda_1c_1\phi_{00}-\lambda_2c_2\phi_{10}+b(c_1\phi_{00}+c_2\phi_{10})\\&=(b-\lambda_1)c_1\phi_{00}+(b-\lambda_2)c_2\phi_{10}.\end{aligned}$$

Thus the images of the rays $c_1\phi_{00}\pm\varepsilon_0c_1\phi_{10}$ can be explicitly calculated and they are

$$(b-\lambda_1)c_1\phi_{00}\pm(b-\lambda_2)\varepsilon_0c_1\phi_{10}$$

or in other words the rays

$$d_1\phi_{00}\pm\varepsilon_0\left(\frac{b-\lambda_2}{b-\lambda_1}\right)\ d_1\phi_{00}.$$

Thus $\Phi$ maps $C_1$ into the cone

$$D_1=\left\{d_1\phi_{00}+d_2\phi_{10}\,\middle|\,d_1\geq 0,\ |d_2|\leq\varepsilon_0\left(\frac{b-\lambda_2}{b-\lambda_1}\right)\right\}.$$

Similary for $C_1$ we can explicitly calculate the image under $\Phi$. If $c\leq 0$,

$$\Phi(c_1\phi_{00}\pm\varepsilon_0c_1\phi_{10})=(a-\lambda_1)c_1\phi_{00}\pm\varepsilon_0(a-\lambda_2)c_1\phi_{10}.$$

Thus, $\Phi(v)=t\phi_{00}$ has one solution in each of the cones $C_1$, $C_2$, namely $\frac{t\phi_{00}}{b-\lambda_{00}}$, $\frac{t\phi_{00}}{a-\lambda_{00}}$. At this stage we need a lemma.

LEMMA 2. 1. *There exists $d>0$ so that*

$$(\Phi(c_1\phi_{00}+c_2\phi_{10}),\ \phi_{00})\geq d|c_2|. \tag{2. 5}$$

PROOF. Let us write $f(u)=bu^{+}-au^{-}$ for brevity. Then

$$\Phi(c_1\phi_{00}+c_2\phi_{10})=A(c_1\phi_{00}+c_2\phi_{10})+P(f(c_1\phi_{00}+c_2\phi_{10}+\theta(c_1, c_2))).$$

So, if $u=c_1\phi_{00}+c_2\phi_{10}+\theta(c_1, c_2)$, then

$$(\Phi(c_1\phi_{00}+c_2\phi_{10}),\ \phi_{00})=((A+\lambda_1)(c_1\phi_{00}+c_2\phi_{10}),\ \phi_{00})+(f(u)-\lambda_1 u,\ \phi_{00}).$$

The first term is zero because $(A+\lambda_1)\phi_{00}=0$ and $A$ is self-adjoint. The second term satisfies $f(u)-\lambda_1 u\geq\gamma|u|$, where $\gamma=\min\{b-\lambda_1, \lambda_1-a\}>0$. Therefore $<\Phi(c_1\phi_{00}+c_2\phi_{10}),\ \phi_{00}>\geq\gamma\int|u|\phi_{00}$. Now there exists $d>0$ so that $\gamma\phi_{00}\geq d|\phi_{10}|$ and therefore

$$\gamma\int|u|\phi_{00}\geq d\int|u||\phi_{10}|\geq d\left|\int u\phi_{10}\right|=d|(u,\ \phi_{10})|$$

which concludes the proof of the lemma.

We are now in a position to describe the behaviour of $\Phi$ in the complement of the two cases $C_1$ and $C_2$. Let us consider the image under $\Phi$ of $c_1\phi_{00}+c_2\phi_{10}$ with $c_2\geq\varepsilon|c_1|$, $c_2=k$ for some $k>0$.

The lemma tells us that the image $\Phi(L)$ of $c_2=k$, $|c_1|\leq\frac{1}{\varepsilon}k$ must lie to the right of the line $c_1=dk$, and must therefore cross the positive $\phi_{00}$ axis in the image space.

Thus we have shown that if $u=c_1\phi_{00}+k\phi_{10}+\theta(c_1, k)$, $k>0$, $|c_1|\leq\left(\frac{k}{\varepsilon}\right)$. Then $u$ satisfies, for some $c_1$, $Lu+bu^+-au^-=s\phi_{00}$ for some $s>dk$ and $k$ positive. Letting $\tilde{u}=\left(\frac{t}{s}\right)u$, we see that $\tilde{u}$ satisfies

$$L\tilde{u}+b\tilde{u}-a\tilde{u}=t\phi_{00}.$$

Similarly one shows the existence of another solution $\underset{\sim}{u}$ satisfying $A\underset{\sim}{u}+b\underset{\sim}{u}^+-a\underset{\sim}{u}^-=t\phi_{00}$, with $(u, \phi_{10})<0$. Thus we have four solutions, one in each of the four cones, which $C_1$, $C_2$ divide the $\phi_{00}$, $\phi_{10}$ plane into. This concludes the proof of Theorem 2. 1. ■

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