Hokkaido Mathematical Journal Vol. 23 (1994) p. 21-34

Brauer functors and Fisher's inequality for t-designs over a finite field with group action

Toyoharu ITOH

(Received March 7, 1992, Revised July 7, 1993)

Abstract. In [I-Y], we proved a Fisher type inequality for a t-design with group action. In this paper, we prove a q-analogue version of a Fisher type inequality in [I-Y]. Let G be a finite group with a normal p-subgroup P and $\mathfrak{B}\subset\begin{bmatrix}V\\k\end{bmatrix}$ a t-design over $\boldsymbol{F}_q$ on which G acts, where V is a finite dimensional vector space over $\boldsymbol{F}_q$ Assume that $2s\leqq t$ and the prime p does not divide $\lambda_s^0\lambda_s^1\cdots\lambda_s^s$. Then $\left|\begin{bmatrix}V\\s\end{bmatrix}^P/G\right|\leqq|\mathfrak{B}^P/G|$, where $\mathfrak{B}^P$ denotes the set of blocks fixed by all elements of P and $\mathfrak{B}^P/G$ denotes the set of G-orbits in $\mathfrak{B}^P$.

1. Introduction

In [I-Y], we proved a Fisher type inequality for a t-design with group action. Let G be a finite group with a normal p-subgroup P and (X, $\mathfrak{B}$) an ordinary (classical) t-design on which G acts. Assume that $2s\leqq t$ and the prime p does not divide $\lambda_s^0\cdots\lambda_s^s$, where λ_i^j is the number of blocks which contain a fixed i-element subset I of X but disjoint from a j-element subset of $X-I$ for $i+j\leqq t$. Then

$$\left|\binom{X}{s}^P/G\right|\leqq|\mathfrak{B}^P/G|, \tag{1}$$

$$\left|\left(\binom{X}{s}^P\times\binom{X}{s}^P\right)/G\right|\leqq\left|\left(\binom{X}{s}^P\times\mathfrak{B}^P\right)/G\right|\leqq|(\mathfrak{B}^P\times\mathfrak{B}^P)/G|, \tag{2}$$

where $\mathfrak{B}^P$ (resp. $\binom{X}{s}^P$) denotes the set of blocks (resp. s-element subsets) fixed by all elements of P and $\mathfrak{B}^P/G$ (resp. $\binom{X}{s}^P/G$) denotes the set of G-orbits in $\mathfrak{B}^P$ (resp. $\binom{X}{s}^P$). In this paper, we consider a q-analogue version of (1) and (2).

A t-$(v, k, \lambda; q)$ *design*, or t-design over $\boldsymbol{F}_q$, is a nonempty collection $\mathfrak{B}$ of k-dimensional subspaces of a v-dimensional vector space over $\boldsymbol{F}_q$

with the property that any t-dimensional subspace is contained in exactly λ members of $\mathfrak{B}$. Here $0<t\leqq k\leqq v$, $\lambda\geqq 1$. It is also known as a q-analogue of the ordinary (classical) t-design. If $\mathfrak{B}$ is the collection of all k-dimensional subspaces, clearly, $\mathfrak{B}$ becomes a t-design over $\boldsymbol{F}_q$ for any $t\leqq k$. This design is called *trivial*. Nontrivial examples for $t\geqq 2$ were given by S. Thomas [T] and H. Suzuki [Su 1] [Su 2].

P. Delsarte [D] introduced the concept of an algebraic (Delsarte) t-design $\mathfrak{B}\subset X$ for a Q-polynomial association scheme $\mathfrak{X}=(X, \{R_i\}_{0\leqq i\leqq d})$. P. Delsarte [D], A. Munemasa [M] and D. Stanton [St] gave a geometrical interpretation of each of this algebraic t-design for a Q-polynomial association scheme. For example, an ordinary t-(v, k, λ) design corresponds to an algebraic t-design for the Johnson scheme $J(v, k)$, and an orthogonal array of strength t corresponds to an algebraic t-design for the Hamming scheme $H(n, q)$. Also a t-design over $\boldsymbol{F}_q$ corresponds to an algebraic t-design for the q-Johnson scheme $J_q(v, k)$. Since $J_q(v, k)$ is a q-analogue of $J(v, k)$, a t-design over $\boldsymbol{F}_q$ is a q-analogue of an ordinary t-design.

Throughout this paper, let G denote a finite group and R a commutative ring with identity. For the rest of this section, we define a Brauer functor (see [I-Y]).

DEFINITION 1.1. An *action* of G on a set X is described by a mapping

$$G\times X\to X\,;\ (g, x)\mapsto gx$$

such that

$$(gh)x=g(hx),\ 1_G x=x \quad \text{for } g, h\in G,\ x\in X,$$

and we shall say that G acts on X or simply that X is a G-*set*. A *permutation RG-module* is a finitely generated R-free (left) RG-module RX (or $R[X]$) with a specified basis X on which G acts. The *Hecke category* $\mathbf{Hec}(G, R)$ is the category of permutation RG-modules and RG-homomorphisms. The Hecke category $\mathbf{Hec}(G, R)$ is a full R-additive subcategory of $\mathbf{Mod}_{RG}$, the category of RG-modules.

REMARK. The Hecke category $\mathbf{Hec}(G, R)$ is equivalent to the *category of G-matrices* in which an object is a finite G-set and a morphism from Y to X is a $X\times Y$-matrix $(a_{x,y})_{x\in X, y\in Y}$ over R such that $a_{gx,gy}=a_{x,y}$ for all $x\in X$, $y\in Y$, $g\in G$, together with composition defined by matrix multiplication.

DEFINITION 1.2. Let F be a field of characteristic $p>0$, and P a p-subgroup of a finite group G with the normalizer $N_G(P)$. Then the *Brauer functor* Br_P : $\mathbf{Hec}(G, F) \rightarrow \mathbf{Hec}(N_G(P), F)$ is defined by $Br_P(FX)$ $:=F[X^P]$ and

$$Br_P(f: FX \rightarrow FY) := (F[X^P] \xrightarrow{\text{incl.}} FX \xrightarrow{f} FY$$
$$= F[Y^P] \oplus F[Y - Y^P] \xrightarrow{\text{proj.}} F[Y^P]).$$

Here, $X^P=\{x \in X | ux=x \text{ for all } u \in P\}$. If we use the matrix form for the Hecke categories (see the remark above), the Brauer functor is given by $Br_P(X)=X^P$ and $Br_P((a_{y,x})_{y\in Y, x\in X})=(a_{y,x})_{y\in Y^P, x\in X^P}$.

LEMMA 1.3. [I-Y]. *If F is a field of characteristic $p>0$, and P is a p-subgroup of G, then the Brauer functor Br_P :* $\mathbf{Hec}(G, F) \rightarrow \mathbf{Hec}$ *$(N_G(P), F)$ is really an F-additive functor.* □

This paper is organized as follows. In section 2, we consider fundamental properties of t-designs over $\boldsymbol{F}_q$. In section 3, we give an invertible condition of det $N_sN_s^T$ in R where N_s in an s^{th} incidence matrix of a t-design over $\boldsymbol{F}_q$ with $2s \leqq t$ and R is a commutative ring. In section 4, we prove a q-analogue version of (1) and (2).

2. Notation and fundamental properties of t-designs over $\boldsymbol{F}_q$.

In this section, we fix notation and prove fundamental properties for t-designs over $\boldsymbol{F}_q$.

Let $0<t\leqq k \leqq v$ be positive integers, and q a prime power. Let V be a v-dimensional vector space over $\boldsymbol{F}_q$. For each non-negative integer i, $\begin{bmatrix} V \\ i \end{bmatrix}$ denotes the set of all i-dimensional subspaces of V. Hence

$$\# \begin{bmatrix} V \\ i \end{bmatrix} := \begin{bmatrix} v \\ i \end{bmatrix}_q = \prod_{l=0}^{i-1} \frac{q^{v-l}-1}{q^{i-l}-1}.$$

DEFINITION 2.1. A nonempty subset $\mathfrak{B}$ of $\begin{bmatrix} V \\ k \end{bmatrix}$ is called a t-$(v, k, \lambda;$ $q)$ *design* or a *t-design over $\boldsymbol{F}_q$* if for any element $\alpha \in \begin{bmatrix} V \\ t \end{bmatrix}$, the number

$$\lambda_t(\alpha) := \#\{b \in \mathfrak{B} | \alpha \subset b\}$$

is a constant λ (independent of the choice of $\alpha \in \begin{bmatrix} V \\ t \end{bmatrix}$).

LEMMA 2.2. *Let* $\mathfrak{B}\subset\begin{bmatrix} V \\ k \end{bmatrix}$ *be a* t-$(v, k, \lambda ; q)$ *design. Then the following hold*:

(1) $\mathfrak{B}$ *is an* i-$(v, k, \lambda_i ; q)$ *design, where*

$$\lambda_i=\lambda\frac{\begin{bmatrix} v-i \\ t-i \end{bmatrix}_q}{\begin{bmatrix} k-i \\ t-i \end{bmatrix}_q}=\lambda\frac{\begin{bmatrix} v-i \\ k-i \end{bmatrix}_q}{\begin{bmatrix} v-t \\ k-t \end{bmatrix}_q}$$

for any $i\leqq t$.

(2) *Let* i *and* j *be non-negative integers with* $i+j\leqq t$. *Let* $\alpha\in\begin{bmatrix} V \\ i \end{bmatrix}$, $\beta\in\begin{bmatrix} V \\ j \end{bmatrix}$ *with* $\alpha\cap\beta=\{0\}$. *Then*

$$\#\{b\in\mathfrak{B}|\alpha\subset b,\ b\cap\beta=\{0\}\}=\lambda\frac{\begin{bmatrix} v-i-j \\ k-i \end{bmatrix}_q}{\begin{bmatrix} v-t \\ k-t \end{bmatrix}_q}q^{j(k-i)}.$$

Thus this number is independent of the choices of α *and* β. *This number is denoted by* λ_i^j.

For the rest of this paper, we regard λ_i and λ_i^j as parameters of the t-design over $\boldsymbol{F}_q$.

PROOF. (1) Let $\alpha\in\begin{bmatrix} V \\ i \end{bmatrix}$ and $\lambda(\alpha)=\#\{b\in\mathfrak{B}|\alpha\subset b\}$. Counting in two ways the number

$$\#\{(\beta, b)\in\begin{bmatrix} V \\ t-i \end{bmatrix}\times\mathfrak{B}|\alpha\cap\beta=\{0\}, \alpha, \beta\subset b\},$$

we obtain

$$\lambda(\alpha)=\lambda\frac{\begin{bmatrix} v-i \\ t-i \end{bmatrix}_q}{\begin{bmatrix} k-i \\ t-i \end{bmatrix}_q}=\lambda\frac{\begin{bmatrix} v-i \\ k-i \end{bmatrix}_q}{\begin{bmatrix} v-t \\ k-t \end{bmatrix}_q}.$$

Thus $\lambda(\alpha)$ is independent of the choice of $\alpha\in\begin{bmatrix} V \\ i \end{bmatrix}$.

(2) Let i and j be non-negative integers with $i+j\leqq t$, and $\alpha\in\begin{bmatrix} V \\ i \end{bmatrix}$

and $\beta\in\begin{bmatrix}V\\ j\end{bmatrix}$ with $\alpha\cap\beta=\{0\}$. The number $\#\{b\in\mathfrak{B}|\alpha\subset b, \beta\cap b=\{0\}\}$ is independent of the choices of α and β, moreover, we have $\lambda_i^j=\lambda_i^{j-1}-q^{j-1}\lambda_{j+1}^{j-1}$ by Lemma 2. 1 in [Su 3]. Thus, we have that by induction on j

$$\lambda_i^j=\sum_{u=0}^{j}(-1)^u q^{\binom{u}{2}}\begin{bmatrix}j\\ u\end{bmatrix}_q\lambda_{i+u}. \tag{a}$$

Let $B_i^j=\#\{y\in\begin{bmatrix}V\\ k\end{bmatrix}|\alpha\subset y, \beta\cap y=\{0\}\}=q^{j(k-i)}\begin{bmatrix}v-i-j\\ k-i\end{bmatrix}_q$, so that

$$B_i^j=\sum_{u=0}^{j}(-1)^u q^{\binom{u}{2}}\begin{bmatrix}j\\ u\end{bmatrix}_q B_{i+u}^0$$

$$\lambda\frac{B_i^0}{\begin{bmatrix}v-t\\ k-t\end{bmatrix}_q}=\lambda\frac{\begin{bmatrix}v-i\\ k-i\end{bmatrix}_q}{\begin{bmatrix}v-t\\ k-t\end{bmatrix}_q}=\lambda_i$$

by (a) and (1). Thus

$$\begin{aligned}B_i^j&=q^{j(k-i)}\begin{bmatrix}v-i-j\\ k-i\end{bmatrix}_q\\ &=\sum_{u=0}^{j}(-1)^u q^{\binom{u}{2}}\begin{bmatrix}j\\ u\end{bmatrix}_q B_{i+u}^0\\ &=\sum_{u=0}^{j}(-1)^u q^{\binom{u}{2}}\begin{bmatrix}j\\ u\end{bmatrix}_q\lambda_{i+u}\frac{\begin{bmatrix}v-t\\ k-t\end{bmatrix}_q}{\lambda}=\lambda_i^j\frac{\begin{bmatrix}v-t\\ k-t\end{bmatrix}_q}{\lambda}.\end{aligned}$$

Hence

$$\lambda_i^j=\lambda\frac{\begin{bmatrix}v-i-j\\ k-i\end{bmatrix}_q}{\begin{bmatrix}v-t\\ k-t\end{bmatrix}_q}q^{j(k-i)}.$$

□

Clearly, $\lambda_i^0=\lambda_i$ and $\lambda_0=\#\mathfrak{B}$.

Let $\mathfrak{B}\subset\begin{bmatrix}V\\ k\end{bmatrix}$ be a t-$(v, k, \lambda; q)$ design. The i^{th} incidence matrix of $\mathfrak{B}$ is a matrix N_i whose rows are indexed by $\begin{bmatrix}V\\ i\end{bmatrix}$, whose columns are indexed by $\mathfrak{B}$, and where the (a, b)-entry is 1 if $a\subset b$ and *0* otherwise for $i=0, 1, 2, \cdots$. Also the incidence matrix of the incidence structure is a matrix

W_{ij} whose rows are indexed by $\begin{bmatrix} V \\ i \end{bmatrix}$, whose columns are indexed by $\begin{bmatrix} V \\ j \end{bmatrix}$, and where the (α, β)-entry is 1 if $\alpha \subset \beta$ and 0 otherwise for $0 \leqq i \leqq j \leqq v$. The following lemma and proposition are q-analogues of R. Wilson's results [W].

LEMMA 2.3. *Let* $\mathfrak{B}$ *be a* t-$(v, k, \lambda; q)$ *design. Then*

$$N_i N_j^T \sum_{l=0}^{\min\{i,j\}} \lambda_{i+j-l}^l W_{li}^T W_{lj}$$

where $i+j \leqq t$.

PROOF. Let $\alpha \in \begin{bmatrix} V \\ i \end{bmatrix}$ and $\beta \in \begin{bmatrix} V \\ j \end{bmatrix}$ with $\alpha \cap \beta \in \begin{bmatrix} V \\ u \end{bmatrix}$. Then

$$N_i N_j^T(\alpha, \beta) = \#\{b \in \mathfrak{B} | \alpha, \beta \subset b\} = \lambda_{i+j-u},$$

and

$$W_{li}^T W_{lj}(\alpha, \beta) = \#\{\gamma \in \begin{bmatrix} V \\ l \end{bmatrix} | \gamma \subset \alpha \cap \beta\} = \begin{bmatrix} u \\ l \end{bmatrix}_q.$$

Now, we fix $\rho \in \begin{bmatrix} V \\ u \end{bmatrix}$ such that $(\alpha+\beta) \cap \rho = \{0\}$, and we put

$$\Lambda_l(\eta) := \{b \in \mathfrak{B} | \alpha, \beta \subset b, \rho \cap b = \eta\}$$

for $\eta \in \begin{bmatrix} \rho \\ u-l \end{bmatrix}$, then $\#\Lambda_l(\eta) = \lambda_{i+j-l}^l$ and

$$\lambda_{i+j-u} = \sum_{l=0}^{u} \sum_{\eta \in \begin{bmatrix} \rho \\ u-l \end{bmatrix}} \#\Lambda_l(\eta) = \sum_{l=0}^{u} \lambda_{i+j-l}^l \begin{bmatrix} u \\ u-1 \end{bmatrix}_q = \sum_{l=0}^{u} \lambda_{i+j-l}^l \begin{bmatrix} u \\ l \end{bmatrix}_q.$$

Hence

$$N_i N_j^T = \sum_{l=0}^{\min\{i,j\}} \lambda_{i+j-l}^l W_{li}^T W_{lj}.$$

□

REMARK. By Lemma 2.3, the following inequality holds

$$\begin{bmatrix} v \\ s \end{bmatrix}_q \leqq \#\mathfrak{B},$$

where $\mathfrak{B}$ is a t-$(v, k, \lambda; q)$ design with $2s \leqq t$. Indeed $N_s N_s^T$ is positive definite, and in particular, it is nonsingular. But there is no tight

t-design over $\boldsymbol{F}_q$ by L. Chihara [C] and H. Suzuki [Su 3].

PROPOSITION 2.4. *Suppose that $\mathfrak{B}$ is a t-$(v, k, \lambda; q)$ design with $t \geqq 2s$ and $v \geqq k+s$. Then*

$$\det(N_s N_s^T) = \prod_{i=0}^{s} \left(\begin{bmatrix} k-i \\ s-i \end{bmatrix}_q \lambda_s^i \right)^{\left[v \atop i \right]_q - \left[v \atop i-1 \right]_q}.$$

PROOF. Since $N_s N_s^T$ and $N_s^T N_s$ have the same nonzero eigenvalues and multiplicities, we consider the eigenvalues of $N_s^T N_s$.

We now put $M_i := N_i^T N_i$. Let U_i denote the row vector space over $\boldsymbol{Q}$ of the matrix N_i. Since $W_{is} N_s = \begin{bmatrix} k-i \\ s-i \end{bmatrix}_q N_i$,

$$U_0 \subset U_1 \subset \cdots \subset U_k,$$

and U_i is also the row space of M_i. By Lemma 2.3,

$$M_e M_f = \sum_{i=0}^{\min\{e,f\}} \lambda_{e+f-i}^i \begin{bmatrix} k-i \\ e-i \end{bmatrix}_q \begin{bmatrix} k-i \\ f-i \end{bmatrix}_q M_i, \tag{b}$$

where $e+f \leqq t$. Let $W_0 := U_0$ and $W_i := U_i \cap U_{i-1}^\perp$ for $i = 1, 2, \cdots s$. If W denotes the $\#\mathfrak{B}$-dimensional vector space whose coordinates are indexed by the set $\mathfrak{B}$, then

$$W = W_0 \oplus W_1 \oplus \cdots \oplus W_s \oplus U_s^\perp,$$
$$\dim W_i = \begin{bmatrix} v \\ i \end{bmatrix}_q - \begin{bmatrix} v \\ i-1 \end{bmatrix}_q.$$

Clearly, vectors of $U_s^\perp$ are eigenvectors of 0 for M_s. Given $x \in W_e$, we have $x \in U_e$, so we can write $x = yM_e$ for some $y \in W$. Since $x \in U_i^\perp$ for $i < e$,

$$0 = xM_i = yM_e M_i = yM_i Me,$$

so $yM_i \in U_e^\perp$. Thus

$$xM_s = yM_e M_s = \sum_{i=0}^{e} \begin{bmatrix} k-i \\ s-i \end{bmatrix}_q \begin{bmatrix} k-i \\ e-i \end{bmatrix}_q \lambda_{e+s-i}^i y M_i$$
$$= \begin{bmatrix} k-e \\ s-e \end{bmatrix}_q \lambda_s^e x.$$

Hence vectors of W_e are eigenvectors of an eigenvalue $\begin{bmatrix} k-e \\ s-e \end{bmatrix} \lambda_s^e$ for M_s, so we have

$$\det(N_sN_s^T)=\prod_{i=0}^{s}\left(\begin{bmatrix}k-i\\s-i\end{bmatrix}_q\lambda_s^i\right)^{\begin{bmatrix}v\\i\end{bmatrix}_q-\begin{bmatrix}v\\i-1\end{bmatrix}_q}.$$

□

3. An invertibility of det $(N_sN_s^T)$ in a commutative ring

Let R be a commutative ring. In this section, we give an invertible condition of $\det(N_sN_s^T)$ in R.

LEMMA 3.1. *Let $\mathfrak{B}\subset\begin{bmatrix}V\\k\end{bmatrix}$ be a t-$(v, k, \lambda; q)$ design with $t\geqq 2s$ and $v\geqq k+s$. Then the following holds:*

$$\prod_{i=0}^{s}\frac{\lambda_s^i}{\begin{bmatrix}k-i\\s-i\end{bmatrix}_q}=\det\begin{pmatrix}\lambda_s & \lambda_{s-1} & \cdots & \cdots & \lambda_0\\ \lambda_{s+1} & \lambda_s & \ddots & & \vdots\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \lambda_s & \lambda_{s-1}\\ \lambda_{2s} & \cdots & \cdots & \lambda_{s+1} & \lambda_s\end{pmatrix}.$$

PROOF. By Lemma 2. 2,

$$\lambda_i=\lambda\frac{\begin{bmatrix}v-i\\k-i\end{bmatrix}_q}{\begin{bmatrix}v-t\\k-t\end{bmatrix}_q},$$

$$\lambda_s^i=\lambda\frac{\begin{bmatrix}v-s-i\\k-s\end{bmatrix}_q}{\begin{bmatrix}v-t\\k-t\end{bmatrix}_q}q^{i(k-s)}.$$

Hence, the left hand side is equal to

$$\prod_{i=0}^{s}\frac{\lambda_s^i}{\begin{bmatrix}k-i\\s-i\end{bmatrix}_q}=\prod_{i=0}^{s}\frac{\lambda\begin{bmatrix}v-s-i\\k-s\end{bmatrix}_q}{\begin{bmatrix}k-i\\s-i\end{bmatrix}_q\begin{bmatrix}v-t\\k-t\end{bmatrix}_q}q^{i(k-s)}$$

$$=\left(\frac{\lambda}{\begin{bmatrix}v-t\\k-t\end{bmatrix}_q}\right)^{s+1}\prod_{i=0}^{s}\frac{\begin{bmatrix}v-s-i\\k-s\end{bmatrix}_q}{\begin{bmatrix}k-s+i\\i\end{bmatrix}_q}q^{i(k-s)},$$

and the right hand side is equal to

$$\det((\lambda_{s+i-j})_{0\leqq i,j\leqq s})=\left(\frac{\lambda}{\begin{bmatrix} v-t \\ k-t \end{bmatrix}_q}\right)^{s+1}\det\left(\begin{bmatrix} v-s-i+j \\ v-k \end{bmatrix}_q\right)_{0\leqq i,j\leqq s}.$$

Thus, it is sufficient to show that

$$\prod_{i=0}^{s}\frac{\begin{bmatrix} v-s-i \\ k-s \end{bmatrix}_q}{\begin{bmatrix} k-s+i \\ i \end{bmatrix}_q}q^{i(k-s)}=\det\left(\begin{bmatrix} v-s-i+j \\ v-k \end{bmatrix}_q\right)_{0\leqq i,j\leqq s}.$$

Now, we put

$$L_{ij}=q^{(k-s-i+j)j}\begin{bmatrix} i \\ j \end{bmatrix}_q\frac{\begin{bmatrix} v-s-i \\ v-k-j \end{bmatrix}_q}{\begin{bmatrix} v-s-j \\ k-s \end{bmatrix}_q},\quad D_{ij}=\delta_{ij}\frac{\begin{bmatrix} v-s-i \\ k-s \end{bmatrix}_q}{\begin{bmatrix} k-s+i \\ i \end{bmatrix}_q},$$

$$U_{ij}=\begin{bmatrix} j \\ i \end{bmatrix}_q\frac{\begin{bmatrix} v-s+j-i \\ k-s+j \end{bmatrix}_q}{\begin{bmatrix} v-s \\ k-s+i \end{bmatrix}_q}$$

for $0\leqq i, j\leqq s$. Then for $0\leqq i, j\leqq s$,

$$\sum_{l=0}^{s}L_{il}D_{ll}U_{lj}=\sum_{l=0}^{s}\begin{bmatrix} i \\ l \end{bmatrix}_q\begin{bmatrix} j \\ l \end{bmatrix}_q\frac{\begin{bmatrix} v-s-i \\ v-k-l \end{bmatrix}_q\begin{bmatrix} v-s+j-l \\ v-k-l \end{bmatrix}_q}{\begin{bmatrix} k-s+l \\ l \end{bmatrix}_q\begin{bmatrix} v-s \\ k-s+l \end{bmatrix}_q}q^{(k-s-i+l)l}$$

$$=\frac{\begin{bmatrix} v-s-i \\ v-k \end{bmatrix}_q\begin{bmatrix} v-s+j \\ v-k \end{bmatrix}_q}{\begin{bmatrix} v-s \\ v-k \end{bmatrix}_q}$$

$$\times\left\{1+\sum_{l=1}^{s}\frac{(q\,;q)_i(q\,;q)_j(q\,;q)_{v-k}(q\,;q)_{v-s+j-l}(q\,;q)_{k-s-i}}{(q\,;q)_{i-l}(q\,;q)_{j-l}(q\,;q)_{v-k-l}(q\,;q)_{v-s+j}(q\,;q)_{k-s-i+l}}\frac{q^{(k-s-i+l)l}}{(q\,;q)_l}\right\}$$

$$=\frac{\begin{bmatrix} v-s-i \\ v-k \end{bmatrix}_q\begin{bmatrix} v-s+j \\ v-k \end{bmatrix}_q}{\begin{bmatrix} v-s \\ v-k \end{bmatrix}_q}\sum_{l=0}^{s}\frac{(q^{-i}\,;q)_l(q^{-j}\,;q)_l(q^{-v+k}\,;q)_l}{(q^{-v+s-j}\,;q)_l(q^{k-s-i+1}\,;q)_l(q\,;q)_l}q^l.$$

In above, the notation $(a\,;q)_n$ is defined by the following

$$(a\,;q)_n:=\begin{cases}1, & n=0,\\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & n=1,2,\cdots.\end{cases}$$

Note that we have that for any positive integers a, b with $a\geqq b>l\geqq 0$,

$$\begin{bmatrix}a\\ b\end{bmatrix}_q=\frac{(q\,;q)_a}{(q\,;q)_b(q\,;q)_{a-b}},$$

$$\frac{(q\,;q)_a}{(q\,;q)_{a-l}}=q^{al-\binom{l}{2}}(q^{-a}\,;q)_l=(q^{a-l+1}\,;q)_l.$$

Now we can use a q-analogue of Saalschütz summation formula ([G-R]p. 13):

$$\sum_{l=0}^{\infty}\frac{(a\,;q)_l(b\,;q)_l(q^{-n}\,;q)_l}{(c\,;q)_l(abc^{-1}q^{-n+1}\,;q)_l(q\,;q)_l}q^l=\frac{(ca^{-1}\,;a)_n(cb^{-1}\,;q)_n}{(c\,;q)_n(c(ab)^{-1}\,;q)_n}.$$

By putting $a:=q^{-i}$, $b:=q^{-v+k}$, $c:=q^{k-s-i+1}$, $n:=j$ in the above formula, we have that

$$\sum_{l=0}^{s}\frac{(q^{-i}\,;q)_l(q^{-j}\,;q)_l(q^{-v+k}\,;q)_l}{(q^{-v+s-j}\,;q)_l(q^{k-s-i+1}\,;q)_l(q\,;q)_l}q^l=\frac{(q^{k-s+1}\,;q)_j(q^{v-s-i+1}\,;q)_j}{(q^{k-s-i+1}\,;q)_j(q^{v-s+1}\,;q)_j}.$$

Thus

$$\begin{aligned}\sum_{l=0}^{s}L_{il}D_{ll}U_{lj}&=\frac{\begin{bmatrix}v-s-i\\ v-k\end{bmatrix}_q\begin{bmatrix}v-s+j\\ v-k\end{bmatrix}_q}{\begin{bmatrix}v-s\\ v-k\end{bmatrix}_q}\\ &\quad\frac{(q\,;q)_{k-s+j}(q\,;q)_{v-s-i+j}(q\,;q)_{k-s-i}(q\,;q)_{v-s}}{(q\,;q)_{k-s}(q\,;q)_{v-s-i}(q\,;q)_{k-s-i+j}(q\,;q)_{v-s+j}}\\ &=\frac{(q\,;q)_{v-s-i}}{(q\,;q)_{v-k}(q\,;q)_{k-s-i}}\,\frac{(q\,;q)_{v-s+j}}{(q\,;q)_{v-k}(q\,;q)_{k-s+j}}\\ &\quad\frac{(q\,;q)_{v-k}(q\,;q)_{k-s}}{(q\,;q)_{v-s}}\\ &\quad\times\frac{(q\,;q)_{k-s+j}(q\,;q)_{v-s-i+j}(q\,;q)_{k-s-i}(q\,;q)_{v-s}}{(q\,;q)_{k-s}(q\,;q)_{v-s-i}(q\,;q)_{k-s-i+j}(q\,;q)_{v-s+j}}\\ &=\frac{(q\,;q)_{v-s-i+j}}{(q\,;q)_{v-k}(q\,;q)_{k-s-i+j}}=\begin{bmatrix}v-s-i+j\\ v-k\end{bmatrix}_q.\end{aligned}$$

Therefore

$$LDU=\left(\begin{bmatrix}v-s-i+j\\ v-k\end{bmatrix}_q\right)_{0\leqq i,j\leqq s}$$

for $L=((L_{ij})_{0\leqq i,j\leqq s})$, $D=((D_{ij})_{0\leqq i,j\leqq s})$, $U=((U_{ij})_{0\leqq i,j\leqq s})$. Clearly, L (resp. U) is a lower (resp. upper) matrix. Hence

$$\det\left(\begin{bmatrix} v-s-i+j \\ v-k \end{bmatrix}_q\right)_{0\leqq i,j\leqq s}=\prod_{i=0}^{s}\frac{\begin{bmatrix} v-s-i \\ k-s \end{bmatrix}_q}{\begin{bmatrix} k-s+i \\ i \end{bmatrix}_q}q^{(k-s)i}.$$

The lemma is proved. □

Since

$$\det(N_sN_s^T)=\prod_{i=0}^{s}\left(\begin{bmatrix} k-s \\ s-i \end{bmatrix}_q\lambda_s^i\right)^{\begin{bmatrix} v \\ i \end{bmatrix}_q-\begin{bmatrix} v \\ i-1 \end{bmatrix}_q},$$

we obtain the follwoing corollary easily.

COROLLARY 3.2. *The following conditions on a commutative ring R are equivalent*:

(i) $\det(N_sN_s^T)$ *is invertible,*

(ii) $\prod_{i=0}^{s}\lambda_s^i$ *is invertible.*

□

4. An equivariant Fisher's inequality for t-designs over F_q

Let V be a v-dimensional vector space over F_q and let $\mathfrak{B}\subset\begin{bmatrix} V \\ k \end{bmatrix}$ a t-$(v, k, \lambda; q)$ design with $2s\leqq t$ on which a finite group G acts.

PROPOSITION 4.1. *Let R be a commutative ring in which $\prod_{i=0}^{s}\lambda_s^i$ is invertible. Let $\mathfrak{F}$: $\mathbf{Hec}(G, R)\to\mathfrak{C}$ be an additive functor to an additive category $\mathfrak{C}$. Then $R\begin{bmatrix} V \\ s \end{bmatrix}$ (resp. $\mathfrak{F}(R\begin{bmatrix} V \\ s \end{bmatrix})$) is isomorphic to a direct summand of $R\mathfrak{B}$ (resp. $\mathfrak{F}(R\mathfrak{B})$) in $\mathbf{Hec}(G, R)$ (resp. $\mathfrak{C}$).*

PROOF. We define linear maps φ and φ' as follows:

$$\varphi: R\mathfrak{B}\to R\begin{bmatrix} V \\ s \end{bmatrix};\ b\mapsto\sum_{a\in\begin{bmatrix} X \\ s \end{bmatrix}, a\subset b}a$$

$$\varphi': R\begin{bmatrix} V \\ s \end{bmatrix}\to R\mathfrak{B};\ a\mapsto\sum_{b\in\mathfrak{B}, a\subset b}b,$$

so that φ, φ' correspond to the s^{th} incidence matrices N_s, N_s^T respectively. By Corollary 3. 2 and the assumption of this proposition, we have that det $(N_sN_s^T)$ is invertible in R, and so the linear map $\varphi\circ\varphi'$ is an R-isomorphism in $\overline{\mathbf{Hec}}(G, R)$. Furthermore since $\mathfrak{F}$ is an additive functor, $\mathfrak{F}\left(R\begin{bmatrix} V \\ s \end{bmatrix}\right)$ is isomorphic to a direct summand of $\mathfrak{F}(R\mathfrak{B})$ in additive category $\mathfrak{C}$. □

Now, we define a functor $\mathfrak{F}_Y$ for any finite G-set Y as follows:

$$\mathfrak{F}_Y: \mathbf{Hec}(G, F) \xrightarrow{\otimes FY} \mathbf{Hec}(G, F) \xrightarrow{\mathrm{Br}_P} \mathbf{Hec}(N_G(P), F)$$
$$; FA \mapsto FA \otimes_F FY = F[A \times Y] \mapsto F[(A \times Y)^P] = F[A^P \times Y^P],$$

where F is a field of characteristic $p>0$, and P is a p-subgroup of G. Then $\mathfrak{F}_Y$ is an F-additive functor (see [I-Y]).

THEOREM 4.2. *Suppose that* $\mathfrak{B} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ *is a* t-$(v, k, \lambda; q)$ *design with* $2s \leqq t$ *on which a finite group* G *acts. Let* P *be a normal* p-*subgroup of* G. *Assume that the prime* p *does not divide* $\prod_{i=0}^{s} \lambda_s^i$. *Then the following hold*:

(1) $\left| \begin{bmatrix} V \\ s \end{bmatrix}^P / G \right| \leqq |\mathfrak{B}^P / G|$

In particular, if G *is a* p-*group, then* $\left| \begin{bmatrix} V \\ s \end{bmatrix}^G \right| \leqq |\mathfrak{B}^G|$.

(2) $\left| \left(\begin{bmatrix} V \\ s \end{bmatrix}^P \times \begin{bmatrix} V \\ s \end{bmatrix}^P \right) / G \right| \leqq \left| \left(\begin{bmatrix} V \\ s \end{bmatrix}^P \times \mathfrak{B}^P \right) / G \right| \leqq |(\mathfrak{B}^P \times \mathfrak{B}^P)/G|$.

PROOF. Let F be a field of characteristic $p>0$.

(1) By Lemma 1.3, the Brauer functor

$$Br_P: \mathbf{Hec}(G, F) \to \mathbf{Hec}(N_G(P), F)$$
$$; F[A] \mapsto F[A^P],$$

is an F-additive functor. Since

$$Br_P\left(F\begin{bmatrix} V \\ s \end{bmatrix}\right) = F\left[\begin{bmatrix} V \\ s \end{bmatrix}^P\right], \quad Br_P(F\mathfrak{B}) = F[\mathfrak{B}^P],$$

Proposition 4.1 implies that $F\left[\begin{bmatrix} V \\ s \end{bmatrix}^P\right]$ is isomorphic to a direct summand of $F[\mathfrak{B}^P]$ in $\mathbf{Hec}(G, F)$. Moreover, G-fixed-point submodules $\left(F\left[\begin{bmatrix} V \\ s \end{bmatrix}^P\right]\right)^G$ and $(F[\mathfrak{B}^P])^G$ are isomorphic to $F\left[\begin{bmatrix} V \\ s \end{bmatrix}^P / G\right]$ and $F[\mathfrak{B}^P/G]$ respectively. Therefore

$$\left| \begin{bmatrix} V \\ s \end{bmatrix}^P / G \right| \leqq |\mathfrak{B}^P/G|.$$

(2) Since $\mathfrak{F}_Y$ is an F-additive functor, $\mathfrak{F}_Y\left(F\begin{bmatrix} V \\ s \end{bmatrix}\right)$ is isomorphic to direct summand of $\mathfrak{F}_Y(F\mathfrak{B})$ by Proposition 4.1. Applying this to $Y=\begin{bmatrix} V \\ s \end{bmatrix}$, $\mathfrak{B}$ and using a similar way as in the proof of (1), we can easily prove (2). □

REMARK. Let V be a v-dimensional vector space over $\boldsymbol{F}_q$ and $\mathfrak{B}\subset\begin{bmatrix} V \\ k \end{bmatrix}$ a t-$(v, k, \lambda; q)$ design with $2s\leqq t$, on which a finite group G acts.

Assume that q is a power of the prime number p. Since $\lambda_s^i=\lambda_s^{i-1}-q^{i-1}\lambda_{s+1}^{i-1}$ by Lemma 2.1 in [Su 3],

$$\begin{aligned}\prod_{i=0}^{s}\lambda_s^i&=\lambda_s\prod_{i=1}^{s}(\lambda_s^{i-1}-q^{i-1}\lambda_{s+1}^{i-1})\\&=\lambda_s(\lambda_s^1)^s \quad \text{in } \boldsymbol{F}_q.\end{aligned}$$

Since

$$\lambda_s^1=\lambda\frac{\begin{bmatrix} v-s-1 \\ k-s \end{bmatrix}_q}{\begin{bmatrix} v-t \\ k-t \end{bmatrix}_q}q^{k-s}$$

by Lemma 2.2 (2), p divides $\prod_{i=0}^{s}\lambda_s^i$.

Hence, if q is a power of p, $\prod_{i=0}^{s}\lambda_s^i$ is not invertible in $\boldsymbol{F}_q$. Therefore, the results in Corollary 3.2 cannot be applicalbe if q is power of p.

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Department of Mathematics
Hokkaido University
Kita 10 Nishi 8, Sapporo 060, JAPAN