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Radon transform of hyperfunctions and support theorem

Takashi TAKIGUCHI and Akira KANEKO

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Abstract. We define the Radon transform for a class of hyperfunctions which are not necessarily with bounded support. We give characterization of the image space for some basic spaces. Then we give a variant of support theorem by Helgason-Boman.

In this article we define the Radon transform for a class of hyperfunctions and discuss its properties. Especially, we prove a variant of support theorem by Helgason-Boman. Although the significance of extending the Radon transform to hyperfunctions is not so clear from the viewpoint of applications to industrial tomography, it will be interesting from purely mathematical viewpoint. Here we only treat the codimension one case i.e. the case of hyperplane integrals.

We should remark that in the theory of hyperfunctions there already exists another kind of Radon transformation theory (see e.g. [Kt]). Its viewpoint lies in the microlocalization of the classical Radon transformation and is different from ours laying stress on the global behavior of the transformation.

1. Introduction. Hyperfunctions

In this section we first give a short review on (Fourier) hyperfunctions, and then discuss the possibility of their Radon transform. For further details on (Fourier) hyperfunctions see [Kn2] and references therein.

A hyperfunction $f(x)$ on $\boldsymbol{R}^n$ is the equivalence class of formal expressions of the form

$$f(x)=\sum_{j=1}^{N} F_j(x+i\Gamma_j 0), \tag{1.1}$$

where Γ_j denotes an open convex cone with vertex at the origin and $F_j(z)$ a function holomorphic in a wedge-like domain with asymptotic form $\boldsymbol{R}^n+i\Gamma_j$ at the real axis. The equivalence means the natural rewriting among the defining functions F_j, and its precise expression is given by Martineau's edge of the wedge theorem which is a concrete expression of the zero cohomology class. There is a theory of localization based on the cohomology theory, and we can legally speak of the local vanishing, local analyticity, support or singular support of a hyperfunction. Distributions and ultra-distributions are included in hyperfunctions in the sense that they are realized as the limit to the real axis of $F_j(x+iy)$ as $y \in \Gamma_j$ tends to 0 in respective sense of topology instead of the formal one.

A Fourier hyperfunction $f(x)$ is a similar expression as above, but the defining functions are defined on a wedge conserving a fixed breadth up to infinity and there satisfy the *infra-exponential* growth estimate locally uniformly in Imz: More precisely, for any $\Gamma_j' \subset\subset \Gamma_j$ there exists $c(\Gamma_j')>0$ such that $F_j(z)$ is holomorphic on $(\boldsymbol{R}^n+i\Gamma_j') \cap \{|\operatorname{Im} z|<c(\Gamma_j')\}$, and for any $\delta>0$ it satisfies the following estimate uniformly on $(\boldsymbol{R}^n+i\Gamma_j') \cap \{\delta<|\operatorname{Im} z|<c(\Gamma_j')\}$: For any $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$|F_j(z)| \leq C_\varepsilon e^{\varepsilon|\operatorname{Re} z|}. \tag{1.2}$$

Here and in the sequel $\Gamma' \subset\subset \Gamma$ means that $\overline{\Gamma'} \cap \boldsymbol{S}^{n-1} \subset \Gamma \cap \boldsymbol{S}^{n-1}$. We then call that Γ' is a proper subcone of Γ. For Fourier hyperfunctions equivalence is understood under this growth condition and domain of definition. In this sense, the space of Fourier hyperfunctions $\mathcal{Q}(\boldsymbol{D}^n)$ is not a subspace of the space of hyperfunctions $\mathscr{B}(\boldsymbol{R}^n)$, but is a kind of extension of the latter to the directional compactification $\boldsymbol{D}^n=\boldsymbol{R}^n \sqcup \boldsymbol{S}_\infty^{n-1}$ of $\boldsymbol{R}^n$ by adding the points at infinity: There is a natural restriction mapping defined by "forgetting the growth condition"

$$\mathcal{Q}(\boldsymbol{D}^n) \longrightarrow \mathscr{B}(\boldsymbol{R}^n),$$

which is surjective but not injective, contrary to the case of distributions $\mathscr{S}' \hookrightarrow \mathscr{D}'(\boldsymbol{R}^n)$. Fourier hyperfunctions are also characterized as the dual of the space $\mathscr{P}_*(\boldsymbol{D}^n)$ of exponentially decaying holomorphic functions defined on a strip-neighborhood of the real axis. In the sequel we shall simply call them exponentially decaying real analytic functions. $\mathscr{P}_*(\boldsymbol{D}^n)$ becomes a DFS type space by the natural topology of inductive limit of Banach spaces with weighted supremum norm of those holomorphic functions with fixed strip breadth of domain of definition and fixed exponential decay. Since $\mathscr{P}_*(\boldsymbol{D}^n) \hookrightarrow \mathscr{S}$ is a continuous inclusion with dense range, we have $\mathscr{S}' \hookrightarrow \mathcal{Q}(\boldsymbol{D}^n)$. Hyperfunctions with compact supports $\mathscr{B}_c(\boldsymbol{R}^n)$, which are characterized as the dual of the space $\mathscr{A}(\boldsymbol{R}^n)$ of real analytic functions, are also contained in $\mathcal{Q}(\boldsymbol{D}^n)$. Exponentially decaying Fourier hyperfunctions, simply defined as

$$\mathcal{Q}_*(\boldsymbol{D}^n) := \bigcup_{\varepsilon>0} e^{-\varepsilon\sqrt{1+x^2}} \mathcal{Q}(\boldsymbol{D}^n) \tag{1.3}$$

also form a subclass of $\mathcal{O}(\boldsymbol{D}^n)$.

Fourier transform is easily seen to act isomorphically to the space $\mathcal{P}_*(\boldsymbol{D}^n)$, hence it can be defined on $\mathcal{O}(\boldsymbol{D}^n)$ by the duality, in consistency with the one for $\mathcal{S}'$. If the defining functions $F_j(z)$ are exponentially decaying on $(\boldsymbol{R}^n \backslash \mathring{\Delta}_k)+iy$ locally uniformly in $y \in \Gamma_j$ with a fixed decay type $-\delta$, then for any small choice of $y_j \in \Gamma_j$ the Fourier transform can be directly calculated by

$$\hat{f}(\xi) := \int_{\boldsymbol{R}^n} f(x) e^{-ix\xi} dx = G_k(\xi - i\Delta_k 0),$$

where

$$G_k(\zeta) = \sum_{j=1}^{N} \int_{\boldsymbol{R}^n} F_j(x+iy_j) e^{-i(x+iy_j)\zeta} dx \quad \text{for} \quad \zeta \in (\boldsymbol{D}^n - i\Delta_k) \cap \{|\mathrm{Im}\,\zeta| < \delta\}. \tag{1.4}$$

The integral converges even on $\mathring{\Delta}_k$ because of the exponential decay factor appearing by the shift $\xi \mapsto \zeta = \xi + i\eta$ with $\eta \in -\Delta_k$. $G_k(\zeta)$ is seen to be of infra-exponential growth by letting $|y_j| \to 0$. For a general Fourier hyperfunction we can decompose it to such ones with various Δ_k's. Especially, the Fourier transform of an exponentially decaying Fourier hyperfunction can be directly calculated by means of the given defining functions and becomes an analytic function which grows infra-exponentially on a strip-neighborhood of the real axis. The Fourier image of $\mathcal{O}_*(\boldsymbol{D}^n)$ agrees with the space $\mathcal{P}(\boldsymbol{D}^n)$ of functions each holomorphic and infra-exponential on some strip neighborhood of $\boldsymbol{R}^n$.

There is another type of Fourier hyperfunctions: $f(x)$ is said to be a modified Fourier hyperfunction if it is represented by the boundary values of defining functions $F_j(z)$ which are now defined and of infra-exponential growth on wedges whose size in the imaginary direction increases linearly with $\mathrm{Re}\,z$: More precisely, for any $\Gamma_j' \subset\subset \Gamma_j$, there exists $c(\Gamma_j')>0$ such that $F_j(z)$ is holomorphic in $(\boldsymbol{R}^n + i\Gamma_j') \cap \{|\mathrm{Im}\,z| < c(\Gamma_j')(|\mathrm{Re}\,z|+1)\}$ and for any $\delta>0$ it satisfies the infra-exponential estimate (1.2) uniformly on $(\boldsymbol{R}^n + i\Gamma_j') \cap \{\delta(|\mathrm{Re}\,z|+1) < |\mathrm{Im}\,z| < c(\Gamma_j')(|\mathrm{Re}\,z|+1)\}$. The equivalence relation is adjusted correspondingly. We denote by $\tilde{\mathcal{O}}(\boldsymbol{D}^n)$ the space of modified Fourier hyperfunctions.

$\tilde{\mathcal{O}}(\boldsymbol{D}^n)$ is the dual of the space of modified type exponentially decaying real analytic functions $\tilde{\mathcal{P}}_*(\boldsymbol{D}^n)$. This latter consists of functions holomorphic and exponentially decaying on a conical neighborhood of the real axis $|\mathrm{Im}\,z| < \delta(|\mathrm{Re}\,z|+1)$, and again becomes a DFS space. Since the Fourier transform acts isomorphically on $\tilde{\mathcal{P}}_*(\boldsymbol{D}^n)$, so does it on $\tilde{\mathcal{O}}(\boldsymbol{D}^n)$. There is a natural continuous inclusion $\tilde{\mathcal{P}}_*(\boldsymbol{D}^n) \hookrightarrow \mathcal{P}_*(\boldsymbol{D}^n)$ with dense range. Hence $\tilde{\mathcal{O}}(\boldsymbol{D}^n)$ is a generalization of functions for the Fourier transformation larger than $\mathcal{Q}(\boldsymbol{D}^n)$, but containing the latter densely.

In order to calculate the Fourier transform of an element $f(x) \in \tilde{\mathcal{Q}}(\boldsymbol{D}^n)$ we need a better decomposition: We decompose the support of f via convex closed cones $\Delta_k^\circ \subset \boldsymbol{D}^n$ with vertex at the origin, and set

$$\hat{f}(\xi) = \sum_k \hat{f}_k(\xi) = \sum_k G_k(\xi - i\Delta_k 0),$$

where

$$G_k(\zeta) = \langle f_k, e^{-ix\zeta} \rangle, \quad \text{for} \quad \mathrm{Im}\zeta \in -\Delta_k.$$

The inner product has sense because on $\Delta_k^\circ \supset \mathrm{supp} f_k$, $e^{-ix\zeta}$ serves as a test function depending holomophically on ζ in the indicated region. Actually, to obtain a set of defining functions of the above mentioned type for the Fourier transform of a hyperfunction even with bounded support, we have to employ just the same decomposition as for a general element of $\tilde{\mathcal{Q}}(\boldsymbol{D}^n)$. Alternatively, we can choose a boundary value representation such that each defining function behaves like a modified type exponentially decaying real analytic function outside a closed convex cone Δ_k°. For components with this fixed Δ_k, we calculate the integral (1.4) where now the integral path is of the form $\{x + iy(x)\,;\, x \in \boldsymbol{R}^n\}$, with $|y(x)| \sim \delta(|x|+1)$. Choosing $y(x)$ suitably in such a way that outside Δ_k°, $y(x)\xi$ produces a decay factor, we see that the resulting function $G_k(\zeta)$ becomes such that $G_k(\xi - i\Delta_k 0)$ is a modified Fourier hyperfunction. The full image of $f(x)$ is obtained by further adding these with respect to k. This way may be more practical than the one based on the decomposition by supports. But the theory assuring the possibility of obtaining such a representation is not yet explicitly developped in the literature.

Let us introduce the following modified type analogue of (1.3):

$$\tilde{\mathcal{Q}}_*(\boldsymbol{D}^n) := \bigcup_{\varepsilon > 0} e^{-\varepsilon\sqrt{1+x^2}}\, \tilde{\mathcal{Q}}(\boldsymbol{D}^n). \tag{1.5}$$

We call it the space of exponentially decaying modified Fourier hyperfunctions.

Remark. The notation for various classes of Fourier hyperfunctions is not well fixed. Here we shall use the upper $\sim$ to indicate the modified type, the suffix $*$ to indicate the exponential decay, and the superfix $*$ the infra-exponential growth if necessary, throughout. This will produce in some case a notation much different from Kawai's original one, but will finally be accepted to be more consistent. (Saburi [S] uses a similar notation but with a longer suffix *inc* and *dec*.)

The above class did not appear in the literature, because its Fourier image is not so nice. But it is important for the theory of Radon transformation.

The following assertions can be proved by a standard argument. Therefore we only sketch their proofs. (A full development of these materials are planned in our forthcoming paper.)

Proposition 1.1 *The space $\tilde{\mathscr{Q}}_*(\boldsymbol{D}^n)$ endowed with the natural inductive limit topology is in duality with the following space of asymptotically infra-exponential modified type real analytic functions endowed with the natural topology*:

$$\begin{aligned}\tilde{\mathscr{P}} &= \{\varphi(z)\,;\ \forall\varepsilon>0\ \exists\delta>0\ \varphi \text{ is holomorphic in } \\ &\qquad |\mathrm{Im}z|\le\delta(|\mathrm{Re}z|+1) \text{ and } |\varphi(z)|\le Ce^{\varepsilon|z|}\} \\ &= \varprojlim_{\varepsilon>0}\varinjlim_{\delta>0}\{\varphi\,;\ \varphi \text{ is holomorphic in } |\mathrm{Im}z|\le\delta(|\mathrm{Re}z|+1) \text{ and} \\ &\qquad |\varphi(z)|\le Ce^{\varepsilon|z|}\}.\end{aligned} \tag{1.6}$$

In fact, a continuous linear functional Φ on $\tilde{\mathscr{Q}}_*(\boldsymbol{D}^n)$ is by definition continuous on every $e^{-\varepsilon\sqrt{1+x^2}}\tilde{\mathscr{Q}}(\boldsymbol{D}^n)$. Thus for any $\varepsilon>0$ there exists $\delta>0$ such that Φ is represented by a function φ holomorphic on $|\mathrm{Im}\,z|\le\delta(|\mathrm{Re}z|+1)$ and of $O(e^{\varepsilon|\mathrm{Re}z|})$ there. Since there is a uniqueness for the representative of an element of the dual of $\tilde{\mathscr{Q}}(\boldsymbol{D}^n)$, we can finally find a representative φ as above. Conversely, let f be a continuous linear functional on $\tilde{\mathscr{P}}$. It is continuous for some fixed $\varepsilon>0$ on the space in the right-hand side of the above. Then $e^{-2\varepsilon\sqrt{x^2+1}}f$ becomes a continuous linear functional on $\tilde{\mathscr{P}}_*(\boldsymbol{D}^n)$ in an obvious manner. Thus $f\in e^{-2\varepsilon\sqrt{x^2+1}}\tilde{\mathscr{Q}}(\boldsymbol{D}^n)$.

Remark that in view of the Phragrén-Lindelöf principle, the estimate in (1.6) is equivalent with the following:

$$\begin{aligned}&\exists A>0,\quad \delta>0 \quad \text{such that for} \quad \forall\varepsilon>0 \\ &|\varphi(z)|\le C_\varepsilon e^{\varepsilon|\mathrm{Re}z|+A|\mathrm{Im}z|} \quad \text{on} \quad |\mathrm{Im}z|\le\delta(|\mathrm{Re}z|+1).\end{aligned} \tag{1.7}$$

Proposition 1.2 *The Fourier image of $\tilde{\mathscr{P}}$ is the subspace of $f\in\tilde{\mathscr{Q}}_*(\boldsymbol{D}^n)$ consisting of those with bounded singular supports, i.e. analytic and exponentially decaying on a conical neighborhood of the complement of some compact set in $\boldsymbol{R}^n$.*

This is essentially contained in [Kw], Lemma 5.1.2, and can be proved similarly. (Note, however, that there is a small confusion in the statement of that Lemma. Cf. the reformulation given in (1.7).) To prove the direct implication, a use of partitions of unity employing factors $\prod_{j=1}^n 1/(e^{\sigma_j x_j}+1)$, $\sigma_j=\pm1$ can also be efficiently used as in [Kn2] to calculate the defining functions of the Fourier image.

Proposition 1.3 *The Fourier image of $\tilde{\mathcal{O}}_*(\boldsymbol{D}^n)$ is the subspace of $\tilde{\mathcal{Q}}(\boldsymbol{D}^n)$ consisting of those of which the defining functions are analytically continued to a strip neighborhood of $\boldsymbol{R}^n$.*

In fact, applying the decomposition of support to $f(x)e^{\delta\sqrt{x^2+1}}\in\tilde{\mathcal{Q}}(\boldsymbol{D}^n)$ as is employed in [Kw], and then by multiplication by the exponential decay factor, we obtain a representation

$$f(x)=\sum_{j,k}f_k e^{-\delta\sqrt{x^2+1}},\qquad f_k\in\tilde{\mathcal{Q}}(\boldsymbol{D}^n),\quad \operatorname{supp} f_k\subset\Delta_k^\circ.$$

Then

$$f(\xi)=\sum_k G_k(\xi-i\Delta_k 0),$$

where

$$G_k(\zeta)=\langle f_k e^{-\delta\sqrt{x^2+1}},\, e^{-ix\zeta}\rangle_x=\langle f_k,\, e^{-\delta\sqrt{x^2+1}-ix\zeta}\rangle_x.$$

Seeking for the condition that $e^{-\delta\sqrt{x^2+1}-ix\zeta}\in\tilde{\mathcal{P}}(\Delta_k^\circ)$, we can see that the result becomes analytic on a strip neighborhood of $\boldsymbol{R}^n$. (But we cannot manipulate the growth order in general.) The verification of the converse is similar.

Corollary 1.4 *The image spaces described in Propositions* 1.2, 1.3 *are dual to each other by their natural topologies.*

Modified Fourier hyperfunctions do not behave well in respect with convolutions:

Proposition 1.5 *The convolution of $f\in\tilde{\mathcal{Q}}(\boldsymbol{D}^n)$ and $\varphi\in\tilde{\mathcal{P}}_*(\boldsymbol{D}^n)$, which may be defined by the inner product*

$$f*\varphi=\langle f(\xi),\,\varphi(x-\xi)\rangle_\xi$$

realized via integration of appropriately chosen defining functions through appropriately chosen paths, gives an element of the space $\mathcal{P}(\boldsymbol{D}^n)$ of infra-exponential analytic functions on a strip neighborhood of $\boldsymbol{R}^n$. Hence so is the case for the convolution between $\tilde{\mathcal{O}}_*(\boldsymbol{D}^n)$ and $\tilde{\mathcal{P}}(\boldsymbol{D}^n)$.

Note that the result of the convolution does not necessarily belong to $\tilde{\mathcal{P}}(\boldsymbol{D}^n)$. This inconvenience comes from the inconsistency with the infinite translation of the notion of modified type spaces.

The Radon transform has no converging factor available as $e^{-ix\xi}$ in the case of Fourier transform. Thus we must restrict the growth order in some sense to calculate it. For example, the Fourier transform of 1 is a well defined hyperfunction $(2\pi)^n\delta(\xi)$, but the Radon transform of 1 is meaningless. As definition of Radon transform, we can adopt several ways: The first is to use the formula

$$Rf(\omega, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\rho\omega)e^{it\rho}d\rho.$$

In this case, the validity of the substitution $\rho\omega$ to $\hat{f}$ requires some regularity of $\hat{f}$ at the origin, and this is equivalent to some decay condition of f. For example, in relation with the above example remark that the substitution $\delta(\rho\omega)$ is meaningless. The second is to define the Radon transform by duality. (Here the duality is understood to be the Plancherel type formula for the Radon transform.) For this we have to examine the image of the adjoint Radon transform on the space of real analytic functions exponentially decaying to the radial direction. The third is to calculate the Radon transform by interpreting the formula

$$Rf(\omega, t) = \int_{\boldsymbol{R}^n} \delta(t - x\omega)f(x)dx.$$

We may introduce the hyperfunctional defining function of $\delta(t-x\omega)$ and realize the integral as a kind of inner product or integration on a path deformed to the complex domain. This is a hyperfunctional interpretation of the hyperplane integrals. As a matter of fact, for a fairly wide subclass of hyperfunctions, it is directly realized as the integral of the defining functions on a suitabe shift of the hyperplane $x\omega = t$ to the complex domain.

It is well known that for elements of $\mathscr{S}$ these definitions of Radon transform all agree. In the sequel we practice these ideas.

2. Radon transform of subspaces of Fourier hyperfunctions

In this section we collect Paley-Wiener type theorems for the Radon transform of some subspaces of Fourier hyperfunctions with enough decay for which Radon transform is rather easily extended. As the fundamental space of test functions on the image side we adopt the space $\mathscr{P}_*(\boldsymbol{S}^{n-1}\times\boldsymbol{D}^1)$ consisting of functions $g(\omega, t)$ satisfying the following conditions: g is analytic and exponentially decaying in t when ω runs a complex neighborhood of $\boldsymbol{S}^{n-1}$ and t a strip neighborhood of $\boldsymbol{R}$. We define the modified version $\tilde{\mathscr{P}}_*(\boldsymbol{S}^{n-1}\times\boldsymbol{D}^1)$ in a similar way. In our framework, the result of Radon transform shold be defined at least as a continuous linear functional on this space.

Now recall the homogeneity condition of Helgason [Hl] for functions $g(\omega, t)$ on $\boldsymbol{S}^{n-1}\times\boldsymbol{D}^1$:

$$\text{for each } k=0, 1, 2,\ldots\ p_g^k(\omega)=\int_{-\infty}^{\infty} g(\omega, t)t^k\,dt \quad \text{becomes a homogeneous polynomial of order } k. \tag{2.1}$$

First we give a characterization of the image by Radon transform of the space of test functions of Fourier hyperfunctions, imitating Helgason's characterization for the Radon image of $\mathscr{S}$. The result is neater for modified type case:

Theorem 2.1 *Let $\widetilde{\mathscr{P}}_{*H}(\boldsymbol{S}^{n-1}\times\boldsymbol{D}^1)$ denote the subspace of $\widetilde{\mathscr{P}}_*(\boldsymbol{S}^{n-1}\times\boldsymbol{D}^1)$ consisting of even functions $g(\omega, t)$ satisfying the homogeneity condition* (2.1). *Then the Radon transform induces an isomorphism of $\widetilde{\mathscr{P}}_*(\boldsymbol{D}^n)$ onto $\widetilde{\mathscr{P}}_{*H}(\boldsymbol{S}^{n-1}\times\boldsymbol{D}^1)$. Incidentally the polynomials thus arising in* (2.1) *as a whole satisfy the estimate*

$$\overline{\lim_{k\to\infty}}\sqrt[k]{\sup_{\omega\in S^{n-1}}|p_g^k(\omega)|/k!}<\infty, \tag{2.2}$$

or equivalently,

$$\overline{\lim_{k\to\infty}}\sqrt[k]{\sup_{|\zeta|\le 1}|p_g^k(\zeta)|/k!}<\infty. \tag{2.2$'$}$$

Proof. We have

$$g(\omega, t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\rho}\hat{f}(\rho\omega)d\rho.$$

Since $f\in\widetilde{\mathscr{P}}_*(\boldsymbol{D}^n)$, it follows that $\hat{f}\in\widetilde{\mathscr{P}}_*(\boldsymbol{D}^n)$. Note that we can also consider $g(\omega, t)$ to be a function positively homogeneous of order -1 in $(\omega, t)\in(\boldsymbol{R}^n\setminus\{0\})\times\boldsymbol{R}$. If we let t complexify to $t+is$ with $|s|<\delta$ for suitable $\delta>0$ and ω to $\omega+i\eta$ running in a suitable complex neighborhood U of $\boldsymbol{S}^{n-1}$ in $\boldsymbol{C}^n$(without posing the condition of $(\omega+i\eta)^2=1$), it is obvious that the above integral still remains absolutely convergent. Hence, $g(\omega, t)$ is jointly holomorphic in ω, t on $U\times\{|s|<\delta\}$. Moreover, shifting the integral path to $\rho+i\sigma$ with $\mathrm{sgn}\sigma=\mathrm{sgn}t$, we see that it is exponentially decaying in t there.

Consider finally the Taylor expansion of $\hat{f}(\xi)=\sum a_\alpha\xi^\alpha$. Since

$$\begin{aligned} p_g^k(\omega) &= \int_{-\infty}^{\infty} g(\omega, t)t^k e^{-it\rho}dt\Big|_{\rho=0} = \left(i\frac{d}{d\rho}\right)^k\int_{-\infty}^{\infty} g(\omega, t)e^{-it\rho}dt\Big|_{\rho=0} \\ &= \left(i\frac{d}{d\rho}\right)^k \hat{f}(\rho\omega)\Big|_{\rho=0}, \end{aligned}$$

comparing with

$$\hat{f}(\rho\omega)=\sum_k\Bigl(\sum_{|\alpha|=k}a_\alpha\omega^\alpha\Bigr)\rho^k$$

we obtain

$$p_g^k(\omega)=i^k k!\sum_{|\alpha|=k}a_\alpha\omega^\alpha.$$

The estimate (2.2) or (2.2′) is easily obtained from the positivity of the radius of convergence of the Taylor series.

Conversely, let $g(\omega,t)\in\widetilde{\mathscr{F}}_{*H}(\boldsymbol{S}^{n-1}\times\boldsymbol{D}^1)$. Set

$$\hat{f}(\rho\omega)=\int_{-\infty}^{\infty}e^{-it\rho}g(\omega,t)dt.$$

As in Helgason[H1], this is legitimate because the right-hand side is even in ω,ρ by the assumption of evenness of g and for $\rho=0$ it reduces to a constant by the homogeneity condition (2.1). The analyticity of $\hat{f}(\xi)$ outside the origin on some conical neighborhood of the real axis is obvions, because, for any $\zeta\in\boldsymbol{C}^n$ in such a neighborhood, we can find $\omega+i\eta$ with $|\omega|=1$ in the domain of g and $\rho>0$ such that $\rho(\omega+i\eta)=\zeta$. Note, however, that in order to assure that $\hat{f}(\xi)$ is well defined and holomorphic, we need the homogeneity of $g(\omega,t)$ in a complex neighborhood, whence the conical domain of definition in t is required. Also we can see its exponential decay from the same property of g by shifting t to the strip neighborhood in the above integral. What is not so obvious is its analyticity at the origin. Rewrite

$$\int_{-\infty}^{\infty}e^{it\rho}g(\omega,t)dt=\int_{-\infty}^{\infty}\sum_{k=0}^{N}\frac{(-it\rho)^k}{k!}g(\omega,t)dt$$
$$+\int_{-\infty}^{\infty}\sum_{k=N+1}^{\infty}\frac{(-it\rho)^k}{k!}g(\omega,t)dt.$$

Here the first term in the right-hand side is equal to

$$\sum_{k=0}^{N}(-i)^k\frac{p_g^k(\xi)}{k!},\quad \xi=\rho\omega.$$

This converges, as $N\to\infty$, to a germ of analytic function at $\xi=0$ provided (2.2′) holds, which we assume for the moment. On the other hand, the second term is estimated as

$$\left|\int_{-\infty}^{\infty}\sum_{k=N+1}^{\infty}\frac{(-it\rho)^k}{k!}g(\omega,t)dt\right|\le C\int_{-\infty}^{\infty}\sum_{k=N+1}^{\infty}\frac{|t\rho|^k}{k!}e^{-\delta|t|}dt$$
$$\le\int_{-\infty}^{\infty}e^{|\rho||t|}e^{-\delta|t|}dt,$$

for some $\delta>0$. Therefore, if $|\rho|<\delta$, this converges to 0 as $N\to\infty$ by Lebesgue's theorem. Thus $\hat{f}(\xi)$ is equal to the above analytic function near $\xi=0$. Thus $\hat{f}(\xi)\in\tilde{\mathscr{P}}_*(\boldsymbol{D}^n)$, hence it is actually the Fourier transform of a function $f\in\tilde{\mathscr{P}}_*(\boldsymbol{D}^n)$, and we conclude that $g(\omega, t)=Rf$.

Now we show (2. 2′). First note that

$$|p_g^k(\omega)|\leq\left|\int_{-\infty}^{\infty}g(\omega, t)t^k dt\right|\leq 2C\int_0^{\infty}e^{-\delta t}t^k dt$$
$$=2C\delta^{-k-1}\int_0^{\infty}e^{-s}s^k ds=2C\delta^{-k-1}k!.$$

Hence $\{p_g^k(\omega)\}$ satisfies (2. 2). We cannot let ω vary in the complex ball in the above estimation to deduce (2. 2′) directly, because we can only let ω run in a small complex neighborhood of the real unit ball. But we have the following general fact in view of which (2. 2′) follows from (2. 2). □

Lemma 2. 2 *Let $p(\zeta)$ be a polynomial of degree $\leq m$ in n variables $\zeta=(\zeta_1\ldots, \zeta_n)$. Let Δ^n denote the polydisc $\{\zeta\in\boldsymbol{C}^n; |\zeta_j|\leq 1, j=1, \ldots, n\}$ and I^n the rectangle $\{\xi\in\boldsymbol{R}^n; |\xi_j|\leq 1, j=1, \ldots, n\}$. Then we have*

$$\sup_{\zeta\in\Delta^n}|p(\zeta)|\leq c_n^m\sup_{\xi\in I^n}|p(\xi)| \tag{2. 3}$$

with a constant $c_n\leq 5^n$ independent of m.

Proof. We show it by induction on n. First notice that for a polynomial of one variable

$$p(\tau)=c\prod_{j=1}^{m}(\tau-\lambda_j)$$

we have

$$\sup_{\tau\in\Delta}|p(\tau)|\leq 5^m\sup_{t\in I}|p(t)|. \tag{2. 4}$$

For λ_j satisfying $|\lambda_j|\geq 3/2$ we have

$$\frac{|\tau-\lambda_j|}{|t-\lambda_j|}\leq\frac{|\lambda_j|+1}{|\lambda_j|-1}\leq 5$$

whatever $\tau\in\Delta$ and $t\in I$ may be. If $|\lambda_j|\leq 3/2$ we have

$$|\tau-\lambda_j|\leq 1+|\lambda_j|\leq\frac{5}{2}.$$

We need an estimate from below of

$$\sup_{t \in I} \Pi' |t - \lambda_j|,$$

where Π' denotes the product of factors corresponding to such λ_j. Let m' denote their total number. By a minimax theorem we have

$$\sup_{t \in I} \Pi' |t - \lambda_j| \geq \sup_{t \in I} \Pi' |t - \mathrm{Re}\lambda_j| \geq \sup_{t \in I} \frac{1}{2^{m'-1}} T_{m'}(t) \geq \frac{1}{2^{m'-1}}.$$

where $T_{m'}$ denotes the Chebyshev polynomial of degree m'. See e.g. Henrici [Hn], Theorem 9. 4. Thus in any case we have (2. 4).

Now consider a polynomial $p(\zeta)$ of n variables ζ. Regarding $p(\zeta)$ as a polynomial of one variable ζ_n, with parameters $\zeta' = (\zeta_1, \ldots, \zeta_{n-1})$, we can apply what we have shown above and obtain that for any $\xi' \in I^{n-1}$ fixed

$$|p(\xi', \zeta_n)| \leq \sup_{\xi_n \in \Delta} |p(\xi', \zeta_n)| \leq 5^m \sup_{\xi_n \in I} |p(\xi', \zeta_n)|.$$

Now, $p(\xi', \zeta_n)$, with $\zeta_n \in \Delta$ fixed, is a polynomial of $n-1$ variables ξ' of degree $\leqq m$. Hence by the induction hypothesis we have

$$\sup_{\zeta' \in \Delta^{n-1}} |p(\zeta', \zeta_n)| \leq 5^{(n-1)m} \sup_{\xi' \in I^{n-1}} |p(\xi', \zeta_n)|.$$

Combining these we obtain (2. 3). $\square$

We still do not have a complete characterization of the Radon image of the fundamental space $\mathscr{P}_*(\boldsymbol{D}^n)$. What we know at present is the following:

Proposition 2. 3 *The Radon transform of $f \in \mathscr{P}_*(\boldsymbol{D}^n)$ becomes an even function $g(\omega, t)$ of (ω, t) on $\boldsymbol{S}^{n-1} \times \boldsymbol{D}^1$, satisfying the homogeneity condition* (2. 1) *and the essimate* (2. 2) *or equivalently* (2. 2′). *Further, it can be considered as a $\mathscr{P}_*(\boldsymbol{D}^1_t)$-valued Gevrey function of index* 2 *in $\omega \in \boldsymbol{S}^{n-1}$. Namely, it satisfies the following estimate: There exist $\delta > 0$, $B > 0$ such that*

$$|D^\alpha_\omega g(\omega, \tau)| \leq C\delta^{-|\alpha|}(|\alpha|\,!\,)^2 e^{-\delta|\tau|} \qquad on \quad |\mathrm{Im}\tau| < \delta. \tag{2. 5}$$

Proof. It is obvious that for each fixed ω, $g(\omega, t)$ belongs to $\mathscr{P}_*(\boldsymbol{D}^1_t)$. We have $\hat{f}(\xi) \in \mathscr{P}_*(\boldsymbol{D}^n)$, hence

$$\begin{aligned}
|D^\alpha_\omega g(\omega, t+is)| &\leq \left| D^\alpha_\omega \int_{-\infty}^{\infty} e^{i(\rho+i\sigma)(t+is)} \hat{f}((\rho+i\sigma)\omega) d\rho \right| \\
&\leq \int_{-\infty}^{\infty} e^{-\rho s - \sigma t} |\rho + i\sigma|^{|\alpha|} |(D^\alpha_\xi \hat{f})((\rho+i\sigma)\omega)| d\rho \\
&\leq Ce^{-\sigma t} \int_{-\infty}^{\infty} e^{-\rho s} |\rho|^{|\alpha|} \frac{|\alpha|!}{\delta^{|\alpha|}} e^{-\delta|\rho|} d\rho.
\end{aligned}$$

Here we have used, with $\zeta=\xi+i\eta$, $z=x+iy$, $y=\delta\xi/|\xi|$,

$$|(D_\xi^\alpha \hat{f})(\zeta)|\leq \left|\int_{\boldsymbol{R}^n} e^{-iz\zeta}(-iz)^\alpha f(z)dx\right|$$
$$\leq C\int_{\boldsymbol{R}^n} e^{|\eta||x|-\delta|\xi|}|x^\alpha|e^{-2\delta|x|}dx\leq \frac{C}{\delta^{|\alpha|}}|\alpha|!e^{-\delta|\xi|},$$
$$\text{for } |\eta|<\delta,$$

for sufficiently small $\delta>0$. Choosing sgnσ suitably, we obtain (2.5) with a smaller $\delta>0$. The remaining assertion can be shown similarly. □

Remark. (2.1) and (2.5) are not enough to assure $\hat{f}(\xi)\in\mathscr{P}_*(\boldsymbol{D}^n)$, although we have no counter-example. Note that the Gevrey 2 regularity of $g(\omega, t)$ in ω is the best possible as will be shown in Example 3.3, 4).

Now we pass to the study of the Radon image of generalized functions. First we consider the space $\mathscr{B}_c(\boldsymbol{R}^n)$ of hyperfunctions with compact supports. For $f(x)\in\mathscr{B}_c(\boldsymbol{R}^n)$, the Radon transform can be defined either by the following hyperfunctional integration along fiber as in [GGV] for the case of distributions:

$$Rf(\omega, t) = \int_{\boldsymbol{R}^n}\delta(t-x\omega)f(x)dx, \tag{2.6}$$

or by the formula via the Fourier transform:

$$Rf(\omega, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\rho}\hat{f}(\rho\omega)d\rho.$$

In the first method, the result is interpreted as a hyperfunction by

$$g(\omega, t) = G(\omega, t+i0)-G(\omega, t-i0), \tag{2.7}$$

where

$$G(\omega, \tau) = \left\langle f(x), -\frac{1}{2\pi i}\frac{1}{\tau-x\omega}\right\rangle_x. \tag{2.8}$$

In the second method, it is interpreted by the same formula (2.6), where now

$$G(\omega, \tau) = \frac{1}{2\pi}\int_0^{\pm\infty} e^{i\tau\rho}\hat{f}(\rho\omega)d\rho, \quad \text{for} \quad \pm\mathrm{Im}\tau>0. \tag{2.8$'$}$$

The integral converges because by the Paley-Wiener-Ehrenpreis theorem $\hat{f}(\rho\omega)$ grows infra-exponentially (i.e. more slowly than any exponential type). (2.8) and (2.8$'$) give the same function, as is easily seen by substitution of $\hat{f}(\rho\omega)=\langle f(x), e^{-i\rho x\omega}\rangle_x$. The equivalence of these two definitions follows from this. Actually the above $G(\omega, \tau)$ can be patched to one analytic function outside a bounded interval on $\mathrm{Re}\,\tau$, corresponding to the fact that $g(\omega, t)$ has bounded support in t.

Theorem 2.4 *Let $\mathscr{B}_{cH}(S^{n-1}\times \boldsymbol{R})$ denote the space of even hyperfunctions $g(\omega, t)$ with compact support, containing ω as real analytic parameters and satisfying the homogeneity condition* (2.1). *Then the Radon transform induces an isomorphism of $\mathscr{B}_c(\boldsymbol{R}^n)$ onto $\mathscr{B}_{cH}(S^{n-1}\times \boldsymbol{R})$. Incidentally, the arising polynomials are subordinate to the estimate*

$$\overline{\lim_{k\to\infty}}\sqrt[k]{\sup_{\omega\in S^{n-1}}|p_g^k(\omega)|} < \infty, \tag{2.9}$$

or equivalently

$$\overline{\lim_{k\to\infty}}\sqrt[k]{\sup_{|\zeta|\le 1}|p_g^k(\zeta)|} < \infty. \tag{2.9$'$}$$

Proof. As is well known (see e.g. [Kn2] Theorem 8.1.1) $\hat{f}$ is an entire function of exponential type. Although this time $g(\omega, t)$ is not necessarily a usual function, we can apply almost the same discussion as in Theorem 2.1. Namely,

$$p_g^k(\zeta) = \int_{-\infty}^{\infty} g(\zeta, t)t^k e^{-it\rho}dt\Big|_{r=0} = \Big(i\frac{d}{d\rho}\Big)^k \hat{f}(\rho, \zeta)\Big|_{\rho=0} = i^k k!\,\Big(\sum_{|\alpha|=k} a_\alpha \zeta^\alpha\Big).$$

The estimate (2.9) or (2.9$'$) follows from the fact that $\hat{f}(\zeta)$ is an entire function of exponential type.

Conversely, given $g(\omega, t)\in\mathscr{B}_{cH}(S^{n-1}\times \boldsymbol{R})$, consider its partial Fourier transform

$$\hat{f}(\omega, \rho) = \int_{-\infty}^{\infty} e^{-it\rho} g(\omega, t)dt = -\oint_\gamma e^{-i\tau\rho}G(\omega, \tau)d\tau, \tag{2.10}$$

where $\gamma\subset \boldsymbol{C}$ is a simple closed curve surrounding $\mathrm{supp}\,g(\omega, t)\subset\boldsymbol{R}$, the support of g as a hyperfunction of one variable t, and G is a defining function. Estimating

$$p_g^k(\omega) = -\oint_\gamma \tau^k G(\omega, \tau)d\tau$$

we obtain $|p_g^k(\omega)|\le CB^{k+1}$, provided $\gamma\subset\{|\tau|\le B\}$, whence (2.9) and (2.9$'$). As in the proof of Theorem 2.1 we can show that it becomes an analytic function of $\xi=\rho\omega$ on $\boldsymbol{R}^n$ including the origin. (No estimate is needed because now the chain of integration is compact.) The estimate (2.9$'$) then implies that its Taylor expansion at the origin defines an entire function of exponential growth. Since it is of infra-exponential growth on the real axis as is seen from (2.10) by choosing γ in the ε-neighborhood of the real axis, it is actually the Fourier transform of an element $f\in\mathscr{B}_c(\boldsymbol{R}^n)$. We have $g=Rf$ as before. $\square$

The following result may be known from long ago. But it is not explicitely written anywhere, including [GGV], [H1].

Corollary 2.5 *Let $\mathscr{E}'_H(\boldsymbol{S}^{n-1}\times\boldsymbol{R})$ denote the space of even distributions with compact support, containing ω as real analytic parameters and satisfying the homogeneity condition* (2.1). *Then the Radon transform induces an isomorphism of $\mathscr{E}'(\boldsymbol{R}^n)$ onto $\mathscr{E}'_H(\boldsymbol{S}^{n-1}\times\boldsymbol{R})$. The arising polynomials are subordinate to the estimate* (2.9) *or* (2.9′).

In fact, the only difference is the singularity of $g(\omega, t)$ which is now in distributions reflecting the same singularity of $f(x)$.

The following theorem can be proved in a similar way:

Theorem 2.6 *The Radon transform of a modified Fourier hyperfunction with bounded singular support, i.e. which is a section of $\mathscr{P}_*(\boldsymbol{D}^n)$ outside a compact subset of $\boldsymbol{R}^n$, is defined by the same formula as* (2.6)-(2.8). *The image is characterized as an even function $g(\omega, t)$ satisfying the homogeneity condition* (2.1), *which can be regarded as a real analytic function of $\omega\in\boldsymbol{S}^{n-1}$ with values in the space of modified Fourier hyperfunctions with bounded singular support in the variable t. Incidentally* (2.2) *or* (2.2′) *holds.*

Finally we consider the space of exponentially decaying Fourier hyperfunctions. To our astonishment, the result is not necessarily analytic in $\omega\in\boldsymbol{S}^{n-1}$ even in the weakest sense for the modified case, too:

$$\tilde{\mathscr{O}}_*(\boldsymbol{D}^n)=\bigcup_{\delta>0}e^{-\delta\sqrt{x^2+1}}\tilde{\mathscr{O}}(\boldsymbol{D}^n). \tag{1.4bis}$$

For $f\in\tilde{\mathscr{O}}_*(\boldsymbol{D}^n)$, its Radon transform is defined by

$$Rf(\omega, t)=G_+(\omega, t+i0)-G_-(\omega, t-i0),$$

where

$$\begin{aligned} G_\pm(\omega, \tau)&=-\frac{1}{2\pi i}\left\langle f(x), \frac{1}{\tau-x\omega}\right\rangle\\ &=-\frac{1}{2\pi i}\left\langle f(x)e^{\delta\sqrt{x^2+1}}, \frac{e^{-\delta\sqrt{x^2+1}}}{\tau-x\omega}\right\rangle, \text{ for } \pm\mathrm{Im}\,\tau>0, \end{aligned}$$

with some $\delta>0$ such that $f(x)e^{\delta\sqrt{x^2+1}}\in\tilde{\mathscr{O}}(\boldsymbol{D}^n)$. This is apparently the same as (2.7)-(2.8) but now $1/(\tau-x\omega)$ resp. $e^{-\delta\sqrt{x^2+1}}/(\tau-x\omega)$ is considered as a test function for exponentially decaying Fourier hyperfunctions resp. infra-exponential Fourier hyperfunctions. A paraphrase employing Fourier transform is similar. This time the analogy of (2.6) is ambiguous.

Proposition 2.7 *The Radon transform of an element of* $\tilde{\mathcal{O}}_*(\boldsymbol{D}^n)$ *becomes an even hyperfunction in* (ω, t) *satisfying* (2.1), *which can be regarded as a* $\tilde{\mathcal{O}}_*(\boldsymbol{D}_t^1)$*-valued Gevrey function of order* 2. *It satisfies* (2.2) *or equivalently* (2.2′).

Proof. Let $f \in \tilde{\mathcal{O}}_*(\boldsymbol{D}^n)$. In view of Proposition 1.3 $\hat{f}(\xi)$ becomes analytic on a neighborhood of the orign. Thus we obtain (2.1) just in the same way as in Theorem 2.1. We verify the Gevrey estimate for the derivatives of $g(\omega, t) = Rf(\omega, t)$ in ω. Let $\varphi(t) \in \tilde{\mathscr{P}}(\boldsymbol{D}_t^1)$. Denoting by $\tilde{\varphi} \in \tilde{\mathcal{O}}_*(\boldsymbol{D}_\rho^1)$ the inverse Fourier transform, we have

$$\begin{aligned} G(\omega) := \int_{-\infty}^{\infty} g(\omega, t)\varphi(t)dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\rho t}\hat{f}(\rho\omega)\varphi(t)dtd\rho \\ &= \int_{-\infty}^{\infty} \hat{f}(\rho\omega)\tilde{\varphi}(\rho)d\rho \\ &= \int_{-\infty}^{\infty}\int_{\boldsymbol{R}^n} e^{-i\rho\omega x} f(x)dx\tilde{\varphi}(\rho)d\rho \\ &= \int_{\boldsymbol{R}^n} f(x)\varphi(x\omega)dx. \end{aligned}$$

Let ω run in a neighborhood of $\boldsymbol{S}^{n-1}$ in $\boldsymbol{R}^n$. We have

$$D_\omega^\alpha G(\omega) = \int_{\boldsymbol{R}^n} f(x)x^\alpha(D_t^{|\alpha|}\varphi)(x\omega)dx.$$

Since $e^{\delta\sqrt{x^2+1}}f(x) \in \tilde{\mathcal{O}}(\boldsymbol{D}^n)$ for some fixed $\delta > 0$, we have, with a slightly smaller δ and with any $\lambda > 0$,

$$|D_\omega^\alpha G(\omega)| \leq \|e^{\delta\sqrt{x^2+1}}f(x)\| \sup_{|\mathrm{Im}z| \leq \lambda(|\mathrm{Re}z|+1)} |e^{-\delta\sqrt{z^2+1}}z^\alpha(D_t^{|\alpha|}\varphi)(z\omega)|,$$

$\|\cdot\|$ denoting the seminorm dual to the one employed for $e^{-\delta\sqrt{x^2+1}}x^\alpha\varphi^{(|\alpha|)}(x\omega)$. Since by Cauchy's inequality we have, for any fixed $\varepsilon > 0$, with some $\lambda = \lambda(\varepsilon)$,

$$|(D_\tau^m\varphi)(\tau)| \leq C\frac{m!}{(\lambda(|\mathrm{Re}\tau|+1))^m}e^{\varepsilon|\mathrm{Re}\tau|},$$

we obtain, again with a smaller $\delta > 0$,

$$|D_\omega^\alpha G(\omega)| \leq C\frac{(|\alpha|!)^2}{\delta^{|\alpha|}}.$$

Since the estimate is uniform in φ, we conclude that $g(\omega, t)$ is in Gevrey class of order 2 in $\omega\in \boldsymbol{S}^{n-1}$ in the sense of the strong topology of $\widetilde{\mathcal{Q}}_*(\boldsymbol{D}^1_t)$. $\square$

Next we consider the case of exponentially decaying Fourier hyperfunctions of ordinary type:

$$\mathcal{Q}_*(\boldsymbol{D}^n):=\bigcup_{\varepsilon>0} e^{-\varepsilon\sqrt{1+x^2}}\,\mathcal{Q}(\boldsymbol{D}^n). \tag{1.3bis}$$

The definition of Radon transform described above for the modified type case works equally well. Since now $\hat{f}(\xi)\in \mathscr{P}(\boldsymbol{D}^n)$, i.e. analytic and infra-exponential on a strip neighborhood of the real axis, the following formula via the Fourier transform is also convenient:

$$G_\pm(\omega, \tau) = \pm\int_0^{\pm\infty} e^{i\rho\tau}\hat{f}(\rho\omega)d\rho, \quad \text{for} \quad \pm\mathrm{Im}\,\tau>0.$$

Proposition 2.8 *The Radon transform of $f\in \mathcal{Q}_*(\boldsymbol{D}^n)$ becomes a function $g(\omega, t)$ of $\omega\in \boldsymbol{S}^{n-1}$ with values in $\widetilde{\mathcal{Q}}_*(\boldsymbol{D}^1_t)$ which is in Gevrey class of order 2 by the strong topology. It is even in ω, t and it satisfies the homogeneity condition* (2.1). *The estimate* (2.2) *or equivalently,* (2.2′) *is satisfied.*

Proof. $\hat{f}$ becomes a holomorphic function of infra-exponential growth on a strip neighborhood of the real axis. Arguing similarly as in Proposition 2, 3, we have

$$\begin{aligned}|D^\alpha_\omega G_+(\omega, t+is)| &= \left|D^\alpha_\omega\int_0^\infty e^{i(\rho+i\sigma)(t+is)}\hat{f}((\rho+i\sigma)\omega)d\rho\right| \\ &\le \left|\int_0^\infty e^{-\rho s-\sigma t}\rho^{|\alpha|}|(D^\alpha_\xi \hat{f})((\rho+i\sigma)\omega)|d\rho\right| \\ &\le C_\varepsilon\int_0^\infty \rho^{|\alpha|}|\alpha|!e^{-s\rho-\sigma t}d\rho \\ &\le C_\varepsilon\frac{(|\alpha|!)^2}{s^{|\alpha|}}e^{-\sigma t}.\end{aligned}$$

Similar estimate holds for $G_-(\omega, \tau)$. The remaining assertion can be shown similarly as before. $\square$

For these two theorems we do not know if the converse holds. We shall show in Example 3.3, 4) that the Gevrey 2 is the best possible regularity we can hope.

We give an identification result deduced from the fact that $g(\omega, t)$ is analytic in ω in some sense. Similar assertion is given in [Z] for usual functions.

Corollary 2.9 *If the original $f(x)$ is an exponenitally decaying Fourier hyperfunction (hence especially a hyperfunction with compact support), then it is uniquely determined from the restriction data $g(\omega^{(k)}, t)$, for a countably many points $\omega^{(k)} \in \boldsymbol{S}^{n-1}$, $k=1, 2, \ldots$ constituting a uniqueness set for real analytic functions on $\boldsymbol{S}^{n-1}$.*

Proof. By the assumption on $f(x)$, $\hat{f}(\xi) \in \mathscr{P}(\boldsymbol{D}^n)$ is analytic on a strip neighborhood of $\boldsymbol{R}^n$. We have, for each fixed real ω.

$$\hat{f}(\rho\omega) = \int_{\boldsymbol{R}} e^{-i\rho t} g(\omega, t) dt.$$

By the assumption, it vanishes on every line $\{\rho\omega^{(k)}; \rho \in \boldsymbol{R}\}$. Thus by the condition on $\omega^{(k)}$ its restriction to every sphere centered at the origin vanishes. Thus we conclude $\hat{f} \equiv 0$ as an element of $\mathscr{P}(\boldsymbol{D}^n)$, hence $f \equiv 0$. □

Note that when $f(x)$ is a hyperfunction with compact support, this is a variant of usual uniqueness theorem with respect to real analytic parameters, and may be discussed even locally with respect to ω. In case $f \in \tilde{\mathscr{Q}}(\boldsymbol{D}^n)$, the above argument does not work, because $\hat{f}(\xi)=0$ on $\boldsymbol{R}^n$ does not exclude the possibility of supp $\hat{f}$ remaining at infinity. Similar assertion in this case is still open.

3. Radon hyperfunctios and their Radon transform

In order to define the Radon transform for more general hyperfunctions we rather need the characterization of the image by the adjoint Radon transform of $\mathscr{P}_*(\boldsymbol{S}^{n-1} \times \boldsymbol{D}^1)$:

$$R^{\#}\varphi(x) = \int_{S^{n-1}} \varphi(\omega, x\omega) d\omega.$$

Lemma 3.1 *The adjoint Radon transform of a function $\varphi(\omega, t) \in \mathscr{P}_*(\boldsymbol{S}^{n-1} \times \boldsymbol{D}^1)$ (resp. $\varphi(\omega, t) \in \tilde{\mathscr{P}}_*(\boldsymbol{S}^{n-1} \times \boldsymbol{D}^1)$) becomes real analytic in a strip neighborhood (resp. conical neighborhood) of the real axis and satisfies the decay condition*:

$$|\mathrm{Re}z| R^{\#}\varphi(z) \textit{ is bounded.} \tag{3.1}$$

Proof. Let $x \neq 0$ be fixed and write $\omega = \eta + sx/|x| \in \boldsymbol{S}^{n-1}$, where η runs in the $(n-1)$-dimensional disc perpendicular to the vector x. We have $|\eta|^2 + s^2 = 1$. Denoting by c_n various constants depending only on n, we find

$$\begin{aligned}|R^{\#}\varphi(x+iy)| &= \left|\int_{S^{n-1}}\varphi(\omega,(x+iy)\omega)d\omega\right| \\ &= \left|\int_{-1}^{1}\frac{1}{\sqrt{1-s^2}}ds\int_{|\eta|=\sqrt{1-s^2}}\varphi(\omega,s|x|+iy\omega)d\eta\right| \\ &\leq c_n\int_0^1(1-s^2)^{(n-3)/2}e^{-\delta s|x|}ds \\ &\leq\frac{c_n}{|x|}\int_0^\infty e^{-\delta t}dt\leq\frac{c_n}{|x|},\end{aligned}$$

where in the last line we used $(1-s^2)^{(n-3)/2}\leq 1$. Hence we need a more careful calculation for $n=2$: Instead of the last line we have

$$\begin{aligned}&\leq c_n\left\{\int_0^{1/2}(1-s^2)^{(n-3)/2}e^{-\delta s|x|}ds+\int_{1/2}^1(1-s^2)^{(n-3)/2}e^{-\delta s|x|}ds\right\} \\ &\leq c_n\left\{\sup_{0\leq s\leq 1/2}(1-s^2)^{(n-3)/2}\int_0^\infty e^{-\delta s|x|}dx+\int_{1/2}^1(1-s^2)^{(n-3)/2}\frac{ds}{s}\sup_s se^{-\delta s|x|}\right\} \\ &\leq\frac{c_n}{|x|}.\end{aligned}$$

The analyticity of $R^{\#}\varphi(x)$ is obvious. □

Taking the above in mind, we can introduce the Radon transform by the duality. The heuristic background is the following Plancherel type formula:

Proposition 3.2 *Let $f(x)\in\mathcal{O}_*(\boldsymbol{D}^n)$ and $\varphi(\omega,t)\in\mathcal{P}_*(\boldsymbol{S}^{n-1}\times\boldsymbol{D}^1)$ (resp. $f\in\widetilde{\mathcal{O}}_*(\boldsymbol{D}^n)$ and $\varphi(\omega,t)\in\widetilde{\mathcal{P}}_*(\boldsymbol{S}^{n-1}\times\boldsymbol{D}^1)$). Then we have*

$$\int_{S^{n-1}}\int_R(Rf)(\omega,t)\varphi(\omega,t)d\omega dt=\int_{R^n}f(x)\,R^{\#}\varphi(x)dx.$$

These integrals are to be understood as the respective inner products.

Notice that $R^{\#}\varphi\in\mathcal{P}(\boldsymbol{D}^n)$ (rest. $R^{\#}\varphi\in\widetilde{\mathcal{P}}(\boldsymbol{D}^n)$) in view of the above lemma, hence the right-hand side has sense. Also, the left-hand side has sense in view of Propositions 2.7, 2.8. The proof is otherwise straightforward. (Cf. the calculation done in the proof of Proposition 2.7).

Let us denote by $\mathcal{P}_{(-1)}$ (resp. $\widetilde{\mathcal{P}}_{(-1)}$) the space of functions holomorphic in a strip neighborhood of the real axis and satisfying there (3.1). It allows a natural structure of DF type topological linear space as the inductive limit of Banach spaces defined by the supremum norms (3.1) taken on $|\mathrm{Im}z|<1/k$, $k=1,2,\ldots$. Because the topology is defined via supremum norms with a fixed weight, this space is unfortunately not reflexive. Its dual will be called the space of Radon hyperfunctions (resp.

modifed Radon hyperfunctions). As a matter of fact, elements of $R^{\#}\mathscr{P}_{*}(\boldsymbol{S}^{n-1}\times\boldsymbol{D}^{1})$ or of $R^{\#}\widetilde{\mathscr{P}}_{*}(\boldsymbol{S}^{n-1}\times\boldsymbol{D}^{1})$ should have stronger restrictions than the mere estimate (3. 1), and if we take them into account, we might have a wider space of Fourier hyperfunctions admitting the Radon transform. But here as a first attempt we adopt these handy spaces.

$\mathscr{P}_{*}(\boldsymbol{D}^{u})$ is continuously imbedded into $\mathscr{P}_{(-1)}$, but not dense therein. The closure of the image of the former consists of those elements such that $|z|\varphi(z)\rightarrow 0$ as $\mathrm{Re}z\rightarrow 0$. Such is the case for $\widetilde{\mathscr{P}}_{*}(\boldsymbol{D}^{n})\hookrightarrow\widetilde{\mathscr{P}}_{(-1)}$, too. Consequently, the space of Radon hyperfunctions cannot be considered as a subspace of Fourier hyperfunctions; we only have a canonical mapping from the former to the latter which is not injective. Thus to a general Radon hyperfunction the Fourier transform does not act faithfully. Also, we have to prepare the sheaf theory for the Radon hyperfunctions independently of the one for Fourier hyperfunctions. Because we only treat global sections of Radon hyperfunctions in this article, we only define the notion of supports: For a compact subset K of the compactification $\boldsymbol{D}^{n}$, let $\mathscr{P}_{(-1)}(K)$ denote the space of real analytic functions defined on a neighborhood of K in $\boldsymbol{D}^{n}+i\boldsymbol{R}^{n}$ (hence possessing a fixed breadth near points at infinity of K), and satisfying (3. 1) there. It has a natural structure of topological vector space similar to $\mathscr{P}_{(-1)}$. A Radon hyperfunction f is said to have support in K if it is extended to a continuous linear functional up to $\mathscr{P}_{(-1)}(K)$. Malgrange's theorem with bound given in Lemma A. 1 in the Appendix assures that

$$\mathrm{supp}f\subset K_{1},\quad \mathrm{supp}f\subset K_{2}\quad\Longrightarrow\quad \mathrm{supp}f\subset K_{1}\cap K_{2}.$$

(See Corollary A. 2.) Thus the notion is legitimate. We still do not know if similar assertion is valid for modified type case, too. Thus we will not speak of support of a modified Radon hyperfunction in the sequel. We shall call the natural image of (modified) Radon hyperfunctions in Fourier hyperfunction as *(modified) Fourier Radon hyperfunctions*. As we see later, there is a good subclass common to both spaces for which the interpretation as Fourier hyperfunction is faithful.

For a (modified) Radon hyperfunction $f(x)$ its Radon transform is defined via the duality;

$$\langle Rf(\omega, t),\ \varphi(\omega, t)\rangle=\langle f(x),\ R^{\#}\varphi(x)\rangle.$$

For an ordinary type Radon hyperfunction, we can also calculate Rf directly, interpreting the formula

$$Rf(\omega, t)=\int_{\boldsymbol{R}^{n}}\delta(t-x\omega)f(x)dx. \tag{3. 2}$$

Here the integral is to be understood in the sense of hyperfunctions as follows: Decompose f into the sum of f_k, $k=1, 2, \ldots, N$ such that each f_k has support in a closed proper convex cone Δ_k°, which is the dual of an open convex cone $\Delta_k \subset \boldsymbol{R}^n$. (For the possibility of such decomposition see Theorem A. 7 in Appendix.) Then

$$Rf(\omega, t) = \sum_{k=1}^{N}(G_k(\omega - i\Delta_k 0,\ t+i0) - G_k(\omega + i\Delta_k 0,\ t-i0)),$$

where

$$G_k(\zeta, \tau) = -\frac{1}{2\pi i}\int_{\boldsymbol{R}^n}\frac{1}{\tau - x\zeta}f_k(x)dx \quad \text{for } \pm\mathrm{Re}\,\tau>0,\ \eta\in\mp\Delta_k. \qquad (3.3)$$

Here $1/(\tau - x\zeta)$ can be considered as a test function for the Radon hyperfunctions, because it is of $O(|x|^{-1})$ thanks to the term $x\eta \neq 0$ in the denominator. Thus the integral (3. 3) is meaningful in this sense of inner product. Remark that the limit $t\pm i0$ is necessary for this, because $\omega \pm i\Delta_k 0$ is not an efficient regularization when $x=0$. In this way we obtain a well defined hyperfunction of (ω, t) on $(\boldsymbol{R}^n\setminus\{0\})\times\boldsymbol{R}$ which is homogeneous of degree -1. Hence it satisfies Euler's differential equation.

$$\omega\nabla_\omega f + t\partial_t f = 0,$$

which is non-characteristic to the radial direction in ω. Thus the restriction to $|\omega|^2=1$ is legitimate, and finally we obtain the desired Radon transform.

If we consider ω to be points on $\boldsymbol{S}^{n-1}$ from the beginning in the above formulas, they must be modified a little: Denoting by $\boldsymbol{CS}^{n-1}$ the complex sphere

$$\boldsymbol{CS}^{n-1} = \{\zeta\in\boldsymbol{C}^n\ ;\ \zeta_1^2+\cdots+\zeta_n^2=1\},$$

and by Δ_k^ω the local profile of the $(n-1)$-dimensional twisted wedge $(\boldsymbol{R}^n + i\Delta_k)\cap\boldsymbol{CS}^{n-1}$, we have

$$Rf(\omega, t) = \sum_{k=1}^{N}(G_k(\omega - i\Delta_k^\omega 0,\ t+i0) - G_k(\omega + i\Delta_k^\omega 0,\ t-i0)),$$

with the same G_k as above. Let us examine the range where η runs. $\zeta=\omega+i\eta\in\boldsymbol{CS}^{n-1}$ implies

$$\omega^2 - \eta^2 = 1, \qquad \omega\eta = 0.$$

Hence, in the region $\omega\in\boldsymbol{S}^{n-1}\cap(\pm\Delta_k^\circ)$ it follows $\Delta_k\cap\{\omega\eta=0\}=\phi$. But if Δ_k° are small enough so that $\Delta_k^\circ\subset\Delta_k$ holds for each k, then $x\omega\neq0$ and (3. 3) has sense even with $\mathrm{Im}\eta=0$. This implies that the result is analytic in ω on such a region. Outside that region, it has S. S. (analytic wavefront set) confined in the directions $(\omega^{\perp}\cap\pm\Delta_k)^{\circ}$, where the dual cone is in the sense of $(n-1)$-dimensional inner product in any local cooridnate system. Usually it is not convenient to choose Δ_k satisfying $\Delta_k^{\circ}\subset\Delta_k$. For example, for the orthant cones, which are the most practical choices, this condition is not satisfied. Thus in practical calculations it is preferable to find first the homogeneous hyperfunction $f(\omega, t)$ on $\boldsymbol{R}^n\setminus\{0\}\times\boldsymbol{R}$ and then restrict it to $|\omega|=1$.

As is easily seen by the duality argument, the space of Radon hyperfunctions injectively contains the space $\mathscr{B}_c(\boldsymbol{R}^n)$ of hyperfunctions with compact supports and the space $\mathscr{O}_*(\boldsymbol{D}^n)$ of exponentially decaying Fourier hyperfunctions. So it is for modified case. In these situations, specialization with respect to ω is legitimate as we saw in the preceding section. More generally, it contains Fourier hyperfunctions which admit representations by difining functions $F_j(z)$ such that each $F_j(z)/(|\mathrm{Re}z|+1)$ is absolutely integrable on $\boldsymbol{R}^n+iy, y\in\Gamma_j$. This is a subclass common to both spaces of Radon hyperfunctions and Fourier hyperfunctions. Hence for such a subclass, the Radon transform agrees with the one calculated via the Fourier transform as in § 2. This situation includes most important examples. We expect that the property stated above completely characterizes the Fourier Radon hyperfunctions, i. e. the natural image of Radon hyperfunctions in Fourier hyperfunctions. If it is true, we will have a canonical right inverse of the natural mapping from the Radon hyperfunctions to Fourier hyperfunctions.

Usual mesurable functions f such that $f(x)/(|x|+1)$ is absolutely integrable are contained therein as a particular case. For such a function f, the Radon transform becomes a distribution, but the above formula calculating it may be interesting even in such a case.

There is a good subclass of Radon hyperfunctions, which may be called absolutely integrable hyperfunctions. It constitutes of Fourier hyperfunctions possessing a set of defining functions which are absolutely integrable in $\mathrm{Re}z$ locally uniformly in $\mathrm{Im}z$, constitute a nice subclass of Radon hyperfunctions. Such is the case in particular if the defining functions $F_j(z)$ are of order $O(|\mathrm{Re}z|^{-n-\varepsilon})$ at infinity for some $\varepsilon>0$. For such class, the integral (3. 3) converges absolutely, irrespective of the contribution of $1/(\tau-x\zeta)$ as damping factor. Hence its Radon transform becomes continuous in $\omega\in\boldsymbol{S}^{n-1}$ in an appropriate sense. In this case a formula like

$$Rf(\xi) = \sum_{j=1}^{N} \int_{\xi} F_j(x+iy_j)dS$$

works for the Radon transform of a hyperfunction under a suitable interpretation:

$$Rf(\omega, t) = \sum_{j=1}^{N} G_j(\omega, t+i\sigma_j 0),$$

$$G_j(\omega, t+is) = \int_{(x+iy)\omega=t+is} F_j(x+iy)dS.$$

If we rotate $\boldsymbol{S}^{n-1}$ in such a way that $\omega=(1, 0, \dots, 0)$, then setting $x'=(x_2, \dots, x_n)$ etc. the last formula leads to:

$$G_j(\omega, t+is) = \int_{\boldsymbol{R}^{n-1}} F_j(t+is, x'+iy')dx',$$

where $(s, y')\in\Gamma_j$, and s can run in the projection image of Γ_j to the y_1-axis. Note that the shift to the complex domain $t+is$ is necessary unless f is micro-analytic to the conormal direction of the hyperplane ξ (i.e. unless $y_1=0$ intersects Γ_j in the last situation).

The Radon transform thus defined is compatible with the usual one or with the one already defined for $\mathscr{B}_c(\boldsymbol{R}^n)$, as is easily seen via duality argument employing test functions from $\mathscr{P}_*(\boldsymbol{S}^{n-1}\times\boldsymbol{D}^1)$. The equivalence of this definition with the one by duality is also shown in the same way. Especially, given any Radon hyperfunction we can always decompose it to two parts, one with bounded support and the other with support confined near the points at infinity, and calculate the Radon transform separately.

In the Appendix we give justification to these "delicate" subspaces of Fourier hyperfunctions which were never treated in the literature. In this article we do not discuss their analogy for modified Fourier hyperfunctions, because it requires a basically profounder provision.

Example 3.3. 1) Consider a hyperfunction with compact support $f(x)=J(D)\delta(x-a)$, where $a\in\boldsymbol{R}^n$ is a fixed point and $J(D)$ is a local operator. Then we have

$$\begin{aligned} Rf(\omega, t) &= \int_{\boldsymbol{R}^n} \delta(t-x\omega)J(D)\delta(x-a)dx \\ &= \int_{\boldsymbol{R}^n} J(-D_x)\delta(t-x\omega)\cdot\delta(x-a)dx \\ &= \int_{\boldsymbol{R}^n} J(\omega\frac{d}{dt})\delta(t-x\omega)\cdot\delta(x-a)dx \end{aligned}$$

$$=J\left(\omega\frac{d}{dt}\right)\int_{R^n}\delta(t-x\omega)\cdot\delta(x-a)dx$$

$$=J\left(\omega\frac{d}{dt}\right)\delta(t-a\omega).$$

This result is micro-analytic in ω in concordance with Theorem 2.3. It is also micro-analytic in t in the region $a\omega\neq 0$, although not in the whole. In this sense, the result is more regular for $a\neq 0$ than for $a=0$.

2) Take $f(x)=1/(x_1+ix_2)^{m+1}$ in $\boldsymbol{R}^2$. The interpretation at $x=0$ is given by

$$\frac{1}{(x_1+ix_2)^{m+1}}=\frac{(-1)^m}{m!}\partial^m\frac{1}{x_1+ix_2},$$

with

$$\partial=\frac{1}{2}\left(\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial x_2}\right),\ \bar{\partial}=\frac{1}{2}\left(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}\right),$$

and $1/(x_1+ix_2)$ is considered to be a hyperfunction coming from the locally integrable function by the canonical imbedding. This example allows a Radon hyperfunction for $m\geq 1$. Calculating its Radon transform, we obtain

$$Rf(\omega,t)=2\pi\frac{(-1)^m}{m!2^m}\frac{(\omega_1-i\omega_2)^m}{\omega_1+i\omega_2}\delta^{(m-1)}(t). \tag{3.4}$$

On the unit sphere $\boldsymbol{S}^{n-1}$ the factor $\omega_1+i\omega_2$ in the denominator may be transformed to $\omega_1-i\omega_2$ and joined to the numerator. But this causes the change in the homogeneity degree. In this sense, (3.4) does not satisfy the homogeneity condition (2.1), hence does not come from a hyperfunction with compact support, although after this rewriting it becomes a polynomial in ω and has support bounded in t.

For $m=0$ we would have

$$R\frac{1}{x_1+ix_2}=\pi\frac{1}{\omega_1+i\omega_2}\mathrm{sgn}\,t.$$

But this is not covered by our present theory. It requres a kind of principal value at infinity for the line integral of Radon transform.

3) Consider another Radon hyperfunction $f(x)=\prod_{j=1}^n(x_j+i\varepsilon)^{-1}$ with $\varepsilon\geq 0$. By a similar calculus, we obtain

$$Rf(\omega,t)=-(-2\pi i)^{n-1}\left(\frac{\chi(\omega)}{t+i\varepsilon|\omega_1+\cdots+\omega_n|}-\frac{\chi(-\omega)}{t-i\varepsilon|\omega_1+\cdots+\omega_n|}\right),$$

where $\chi(\omega)$ denotes the characteristic function of the part of S^{n-1} lying in the first orthant. This is analytic in t for $\varepsilon>0$ but discontinuous in ω whatever ε may be. By a similar calculus we obtain the Radon transform of $f(x)=\prod_{j=1}^{n}(x_j+i\varepsilon)^{-m}$:

$$\begin{aligned}Rf(\omega, t) &= -(-2\pi i)^{n-1}\frac{(2m-2)!}{(m-1)!^2}(\omega_1\cdots\omega_n)^{m-1}\\ &\quad\times\left(\frac{\chi(\omega)}{t+i\varepsilon|\omega_1+\cdots+\omega_n|}-\frac{\chi(-\omega)}{t-i\varepsilon|\omega_1+\cdots+\omega_n|}\right).\end{aligned}$$

This becomes more and more regular in ω as m grows.

4) Let $f(x)=e^{-|x_1|-|x_2|}$ on $\boldsymbol{R}^2$. This is in $\mathscr{O}_*(\boldsymbol{D}^2)$. A simple calculation gives

$$Rf(\omega, t)=\frac{2}{\omega_1^2-\omega_2^2}(|\omega_1|e^{-|t|/|\omega_1|}-|\omega_2|e^{-|t|/|\omega_2|}).$$

This is not analytic in ω, e.g., near $\omega_1=0$, even if considered to be $\widetilde{\mathscr{O}}_*(\boldsymbol{D}_t^1)$-valued. To see this more clearly, let $\varphi(t)\in\widetilde{\mathscr{P}}_*(\boldsymbol{D}_t^1)$ and consider the inner product

$$\begin{aligned}G(\omega) := \langle|\omega_1|e^{-|t|/|\omega_1|},\ \varphi(t)\rangle_t &= |\omega_1|\int_{-\infty}^{\infty}e^{-|t|/|\omega_1|}\varphi(t)dt\\ &=|\omega_1|^2\int_0^{\infty}e^{-s}\{\varphi(|\omega_1|s)+\varphi(-|\omega_1|s)\}ds\\ &=|\omega_1|^2\int_0^{\infty}e^{-s}\left\{\sum_{k=0}^{2N}a_k t^k+\varphi_{2N+2}(|\omega_1|s)\right\}ds\\ &=|\omega_1|^2\sum_{k=0}^{N}a_{2k}(2k)!|\omega_1|^{2k}+O(|\omega_1|^{2N+4}).\end{aligned}$$

Here a_k are the Taylor coefficients of φ at the origin and φ_{2N+2} is the composed remainder. Note that generally what we can expect is the estimate $|a_k|\leq cb^k k!$. Hence this shows the Gevrey 2 regularity of $G(\omega)$ at $\omega_1=0$ and no more in general. Thus the regularity asserted in Proposition 2.7 or 2.8 is the best possible.

Next choose a radially symmetric element $\varphi(x)\in\widetilde{\mathscr{P}}_*(\boldsymbol{D}^n)$ (e.g. $\varphi(x)=e^{-x^2}$) and consider $h=f*\varphi$, which lies in $\mathscr{P}_*(\boldsymbol{D}^n)$ in view of Proposition 1.5. Then $\hat{\varphi}(\xi)$ is again radially symmetric. Denoting by $\psi(t)$ the inverse Fourier transform of $\hat{\varphi}$ as a function of one variable ρ, we will have

$$\begin{aligned}Rh(\omega, t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\rho t}\hat{f}(\rho\omega)\hat{\varphi}(\rho)\,d\rho = Rf(\omega, t)*_t\psi(t)\\ &= \frac{2}{\omega_1^2-\omega_2^2}\int_{-\infty}^{\infty}(|\omega_1|e^{-|u|/|\omega_1|}-|\omega_2|e^{-|u|/|\omega_2|})\psi(t-u)du\end{aligned}$$

$$=\frac{2}{\omega_1^2-\omega_2^2}\int_0^\infty (|\omega_1|^2e^{-v}\{\psi(t-|\omega_1|v)+\psi(t+|\omega_1|v)\}$$
$$-|\omega_2|^2e^{-v}\{\psi(t-|\omega_2|v)+\psi(t+|\omega_2|v)\}\,dv.$$

By the same way as above we can see that this is in Gevrey 2, and not analytic in ω in general. This shows that the regularity asserted in Proposition 2.3 is also best possible. In view of Theorem 2.1 we can infer that $h(x)\notin \widetilde{\mathscr{F}}_*(\boldsymbol{D}^n)$, hence the assertion of Proposition 1.5 is also just what we can expect.

5) The Radon tranform of $f(x)=|x|^{-\mu}$, $\mu>n-1$, which is regularized by the finite part at the origin, becomes as follows when $\mu-n$ is not an even integer:

$$Rf(\omega, t)=\pi^{(n-1)/2}\frac{\Gamma\left(\frac{\mu-n+1}{2}\right)}{\Gamma\left(\frac{\mu}{2}\right)}\mathrm{f.p.}|t|^{n-\mu-1}.$$

When $\mu-n$ is an even integer, it should be replaced by

$$Rf(\omega, t)=\pi^{(n-1)/2}\frac{\Gamma\left(\frac{\mu-n+1}{2}\right)}{\Gamma\left(\frac{\mu}{2}\right)}\left\{c_{\mu,n}\delta^{(\mu-n)}(t)+t_+^{-\mu+n-1}\frac{i}{\pi}[\tau^{-\mu+n-1}\log\tau]\right\},$$

$$c_{\mu,n}=\frac{1}{\pi}\mathrm{f.p.}\Gamma(n-\mu)-\frac{1}{2\pi}\frac{1}{(\mu-n)!}\left(\frac{\Gamma'\left(\frac{\mu-n+1}{2}\right)}{\Gamma\left(\frac{\mu-n+1}{2}\right)}-\frac{\Gamma'\left(\frac{\mu}{2}\right)}{\Gamma\left(\frac{\mu}{2}\right)}\right.$$
$$\left.-2\frac{\Gamma'(\mu-n+1)}{(\mu-n)!^2}\right).$$

Here we used the standard notation for hyperfunctions of one variable:

$$[F(\tau)]:=F(t+i0)-F(t-i0).$$

This suggests the possibility of generalization of Radon transforms by means of analytic continuation for some special hyperfunctions which do not necessarily decay at all. This type of calculation is elaborately made in [GGV]. But our present objective is to introduce a general theory, instead of treating special functions.

Some people use Radon transform defined on the projective space $\boldsymbol{P}^n$ by means of a standard hyperplane measure element on it. In that sense, every hyperfunction on $\boldsymbol{P}^n$ admits the Radon transform, because it reduces to an integration with respect to compact fibers. We therefore examine here relation between that one and our Radon transform which agrees with the classical definition. First note that Radon hyperfunction can be canonically extended to hyperfunctions on the compact manifold $\boldsymbol{P}^n$. In fact, fix a volume element $d\mu$ on $\boldsymbol{P}^n$. For example, we can use the one whose restriction to $\boldsymbol{R}^n$ reads as $(1+x^2)^{-(n+1)/2}\,dx$. This measure takes the same form after the coordinate transformation on $\boldsymbol{P}^n$, e.g.,

$$\xi_1=1/x_1,\quad \xi_j=x_j/x_1,\quad j=2,\ \dots,\ n. \tag{3.5}$$

As a measure on a hypersurface $\boldsymbol{P}^n$ it is natural to adopt the one induced from the above. In the local coordinates of $\boldsymbol{R}^n$, this reads as $ds/(1+x^2)^{n/2}$, ds being usual Euclidean hyperplane element. See [GGV].

Now let φ be an element of $\mathscr{A}(\boldsymbol{P}^n)$. Its restriction to $\boldsymbol{R}^n$ is a bounded real analytic function extendable up to a strip neighborhood. Hence, the inner product

$$\langle \tilde{f}, \varphi\rangle_{\boldsymbol{P}^n} := \int_{\boldsymbol{R}^n} f(x)\varphi(x)d\mu = \left\langle f(x), \frac{\varphi(x)}{(1+x^2)^{(n+1)/2}}\right\rangle_{\boldsymbol{R}^n} \tag{3.6}$$

is meaningful. Notice that the duality between $\mathscr{A}(\boldsymbol{P}^n)$ and $\mathscr{B}(\boldsymbol{P}^n)$ is meanigful although $\boldsymbol{P}^n$ is non-orientable.

Lemma 3.4 *The canonical map defined by* (3.6) *is injective. Near the hyperplane* $t=0$ *at infinity, the extended hyperfunction has the regularity such that it is canonically divisible by* t^n.

In fact, notice that the Cauchy kernel $(-2\pi i)^{-n}\prod_{j=1}^{n}1/(\zeta_j-x_j)$ as a function of x becomes an element of $\mathscr{A}(\boldsymbol{P}^n)$ for $\operatorname{Im}\zeta_j\neq 0$, $j=1, \dots, n$. Thus if the extended elements $\tilde{f}$ is zero, we will have

$$\begin{aligned} G(\zeta) &= \left\langle f(x), \frac{1}{(-2\pi i)^n}\prod_{j=1}^{n}\frac{1}{\zeta_j-x_j}\frac{1}{(1+x^2)^{(n+1)/2}}\right\rangle_x \\ &= \left\langle \frac{f(x)}{(1+x^2)^{(n+1)/2}}, \frac{1}{(-2\pi i)^n}\prod_{j=1}^{n}\frac{1}{\zeta_j-x_j}\right\rangle_x = 0, \\ &\qquad \text{for}\quad \forall\zeta\in(\boldsymbol{C}\setminus\boldsymbol{R})^n. \end{aligned}$$

Since in view of Corollary A.4 in Appendix $f/(1+x^2)^{(n+1)/2}$ as a Fourier hyperfunction admits a set of defining functions which are of class L^1 in $\operatorname{Re} z$, we can easily see by a standard argument (cf. e.g. [K2], proof of Theorem 4.1.5) that $\boldsymbol{G}(\zeta)$ is again a defining function of $f/(1+x^2)^{(n+1)/2}$. Thus $f/(1+x^2)^{(n+1)/2}=0$, hence $f=0$ in view of the fact that $(1+x^2)^{(n+1)/2}\mathscr{P}_*(\boldsymbol{D}^n)=\mathscr{P}_*(\boldsymbol{D}^n)$.

Concerning the latter assertion, notice that for a Radon hyperfunction f the last term of (3.6) is meaningful even if $\varphi(x)/(1+x^2)^{(n+1)/2}$ is replaced by $\varphi(x)/(1+x^2)^{1/2}$. This means that not only a Radon hyperfunction but also its multiple by $(1+x^2)^{n/2}$ can be canonically extended to $\boldsymbol{P}^n$ by the above method. This gives a canonical solution of the division by t^n.

Rewriting our Radon tranform to the one on $\boldsymbol{P}^n$ for thus defined extension of f, we obtain an integral with respect to a measure having singularity of the type dS/t^n at the hyperplane $t=0$ at infinity. For example, the integral $\int_{x_n=\lambda x_1} f(x)dS=\int f(x_1, \ldots, x_{n-1}, \lambda x_1)\sqrt{1+\lambda^2}\, dx_1 \cdots dx_{n-1}$ on $x_n=\lambda x_1$ after the transformation (3. 5) leads to

$$\int f(1/\xi_1, \xi_2, \ldots, \xi_{n-1}, \lambda/\xi_1)\sqrt{1+\lambda^2}\frac{1}{\xi_1^n}d\xi_1 \cdots d\xi_{n-1}.$$

By the above lemma, the integrand has a canonical meaning as a hyperfunction near $\xi_1=0$. Thus the line integral is meaningful just in the same sense as the Radon transform for $\mathscr{B}_c$ on $\boldsymbol{R}^n$. Thus we have a well defined paraphrase between these two kinds of Radon transforms. For this subject in a more general setting, see [B1].

Remark. General Fourier hyperfunctions cannot be considered canonically to be a hyperfunction on $\boldsymbol{P}^n$. But for some good (hyper)functions, such interpretation may be possible. Some of the examples of calculations given in [GGV], such as the Radon transform of the characteristic function of an orthant, are made in that sense. In our next paper, we shall try to include such a situation from a new point of view. Note however, that the neglection of things at infinity is sometimes very dangerous. In this respect we refer to a counter-example to the uniqueness of the Radon transform given by Zalcman [Z]: It is an entire function $f(x)$ on $\boldsymbol{C}\simeq\boldsymbol{R}^2$ such that $f(z)=O(|z|^{-3})$ and $f'(z)=O(|z|^{-2})$ on every line, hence its line integral converges absolutely and the Radon transform is equal to 0. (In order to be consistent with Fubini's theorem, e.g., $\int|f(x+iy)|dy$ can no more be an integrable function.) This function should have strong singularity at infinity, so that any of its interpretation as a Fourier hyperfunction should possess a non-zero Fourier image in spite of the vanishing of the line integral itself. We cannot ignore what is going on at infinity. As a matter of fact, any hyperfunction can be extended to an exponentially decaying Fourier hyperfunction, which possesses a well defined Radon tranform image. But the result is meaningless as the Radon transform of the original hyperfunction, because the ambiguity of extension at infinity remains in the result. We already know such a situation in the case of

Fourier transformation.

We leave the discussion of inversion formula to our forthcoming paper. Remark that for elements of $\mathscr{P}_*(\boldsymbol{D}^n)\subset\mathscr{S}$ the usual definition of Radon transform applies. Hence the inverse formula is the same as given in Helgason [H1] or Natterer [N].

4. Support theorem

In this article we use the term "rapidly decreasing Fourier hyperfunctions" to indicate those hyperfunctions which admit representation by defining functions of order $O(|x|^{-N})$ for any $N>0$. (The class of Fourier hyperfunctions usually called by this name (cf [Kn2]) is called here as of exponential decay for distinction.) This subclass of Radon hyperfunctions has a special meaning in the support theorem for the Radon transform:

Theorem 4.1 *Let $f(x)$ be a rapidly decreasing Fourier hyperfunction. Assume that $Rf(\omega, t)$ vanishes for $|t|\geq A$. Then $f(x)$ vanishes on $|x|\geq A$.*

We can prove this either by Helgason's method [H1] or by Boman's method [B1]. For the former we cannot employ the regularization by convolution and have to prove the sphere theorem directly for hyperfunctions. For the latter, we should notice that near the hyperplane at infinity, we can consider $Rf(\omega, 1/t)$ as a hyperfunction defined on $\boldsymbol{S}^{n-1}\times\{|t|<\varepsilon\}$ and even in ω, t, containing t as real analytic parameter and with support compact in ω. The compactness of support in ω, allows us to use the usual uniqueness theorem for hyperfunctions with analytic parameters. Since both proofs are interesting to clarify the characteristic feature of hyperfunctions, we explain below the essential points of them.

First we prepare;

Lemma 4.2 *Let f be a rapidly decreasing Fourier hyperfunction defined on $\boldsymbol{R}^n$. Suppose f has surface integral 0 over every sphere which encloses the unit ball in the sence (4.1) below). Then $f(x)\equiv 0$ for $|x|>1$.*

Proof. Denote the sphere of radius R with center x by $S(x\,;\,R)$ and the corresponding ball by $B(x\,;\,R)$. The assumption means that

$$\int_{S(0;R)} f(x+s)ds=\int_{S^{n-1}} f(x+Rs)R^{n-1}d\omega(s)=0, \tag{4.1}$$

for any $(x, R)\in\boldsymbol{R}^n\times(0,\infty]$ such that $S(x\,;\,R)$ encloses the unit ball, where ds resp. $d\omega(s)$ denotes the standard surface measure on $S(0\,;R)$ resp. $S^{n-1}=S(0\,;1)$. Note that the integral is legitimate as the one along compact fibers of a hyperfunction of the variables (x, R, s) defined on

$$\Sigma \equiv \{(x, R, s) | s \in \boldsymbol{S}^{n-1},\ B(0; 1) \subset B(x; R)\}.$$

Since f decays rapidly enough as $|x| \to \infty$ we obtain for $|x| < \varepsilon < 1$, $R \geq 1 + \varepsilon$

$$\int_{\boldsymbol{R}^n} f(y) dy = \int_{\boldsymbol{R}^n} f(x+y) dy = \int_0^\infty r^{n-1} dr \int_{\boldsymbol{S}^{n-1}} f(x+Rs) d\omega(s)$$
$$= \int_{B(0;R)} f(x+y) dy.$$

Thus the last integral is constant for such x, R. Differentiating with respect to x_i and then employing the divergence theorem just as in [H1], we obtain

$$\int_{S(0;R)} f(x+s) s_i \, d\omega(s) = 0. \tag{4.2}$$

Since this holds for any such x, R, we can say that (4.1) holds for $f(x+s)$ replaced by $f(x+s)s_i$. Repeating this, we conclude that

$$\int_{S(0;R)} f(x+s) P(s) \, d\omega(s) = 0, \tag{4.3}$$

for any polynomial $P(s)$.
Hence

$$\begin{aligned} \int_{\boldsymbol{R}^n} f(x+y) P(y) dy &= \int_{B(0;R)} f(x+y) P(y) dy \\ &= \int_{\boldsymbol{R}^n} f(x+y) \chi_R(y) P(y) dy, \end{aligned} \tag{4.4}$$

where χ_R denotes the characteristic function of $B(0; R)$. Viewing this as an identity in $\mathcal{O}(\boldsymbol{D}^n)$, we see that it can be extended by continuity to any $P(y)$ which is analytic and of polynomial growth on a strip neighborhood of $\boldsymbol{R}^n$. Thus we see that (4.3) holds for any $P \in \mathscr{A}(S^{n-1})$. By a theorem of [Kn 4] we first conclude that the hyperfunction $F(x, R, s) \equiv f(x+Rs)$ of (x, R, s) defined on Σ contains x, R as real analytic parameters and support compact in s. Then by Theorem 4.4.7 of [Kn2] we conclude that $F(x, R, s) = f(x+Rs) \equiv 0$ for such x, R, s, hence finally on Σ. Therefore the lemma follows. $\square$

From this Lemma we can easily obtain a proof of Theorem 4.1 just as in the same way as in the proof of Theorem 2.6 in [H1]. Note that the explicit inversion formula for radial functions is still valid by continuity for rapidly decreasing radial Fourier hyperfunctions (at least if the origin is apart from their supports, which case is enough for us).

For the second proof of Theorem 4.1, first remark that a rapidly decreasing Fourier hyperfunction $f(x)$ can be naturally considered to be a hyperfunction $\tilde{f}(x)$ on the projective space $\boldsymbol{P}^n$ via the duality, as is explained at the end of §2. The assumption on the support of $g(\omega, t)$ implies that this hyperfunction $\tilde{f}(x)$ becomes micro-analytic to the conormal direction at the hyperplane at infinity. The proof of this fact is just similar to the one in Boman [B1]. In fact, the pseudo-differential operators with analytic coefficients used there equally act to hyperfunctions. (Note that the Radon transform on $\boldsymbol{P}^n$ used in [B1] is not the natural one on $\boldsymbol{P}^n$ but the one induced from the one on $\boldsymbol{R}^n$, hence singular at infinity. Thus we do not need any paraphrase in this respect.)

Now consider $\tilde{f}$ to be an even hyperfunction $\tilde{f}$ on the 2-fold covering $\boldsymbol{S}^n$ of $\boldsymbol{P}^n$. Then by introducing the local coordinates near the great circle corresponding to the hypersurface at infinity in a canonical way, we can consider $\tilde{f}$ to be a hyperfunction $\tilde{f}(\omega, t)$ on the product manifold $\boldsymbol{S}^{n-1}\times \{|t|<\delta\}$ with the real analytic parameter t, such that $\tilde{f}(\omega, t)=f((1/t)\omega)$ for $t\neq 0$. We shall show that

$$\left(\frac{\partial}{\partial t}\right)^k \tilde{f}\Big|_{t=0}=0, \qquad k=0, 1, 2,\dots . \tag{4.5}$$

Let $\varphi(\omega)\in\mathscr{A}(\boldsymbol{S}^{n-1})$ be arbitrary. Noticing that

$$\begin{aligned}\langle \tilde{f}, \varphi(\omega)\rangle_\omega dt &= \tilde{f}\varphi(\omega)\, d\mu = f(x)\varphi\left(\frac{x}{|x|}\right)\frac{dx}{(1+x^2)^{(n+1)/2}} \\ &= f(r\omega)\varphi(\omega)\frac{r^{n-1}}{(1+r^2)^{(n+1)/2}}\, d\omega\, dr \\ &= f((1/t)\omega)\varphi(\omega)d\omega\frac{dt}{(1+t^2)^{(n+1)/2}},\end{aligned}$$

we have

$$\begin{aligned}\left\langle\left(\frac{\partial}{\partial t}\right)^k \tilde{f}\Big|_{t=0}, \varphi(\omega)\right\rangle_{S^{n-1}} &= \left(\frac{\partial}{\partial t}\right)^k \langle \tilde{f}, \varphi(\omega)\rangle_\omega\Big|_{t=0} \\ &= \lim_{t\to+0}\left(\frac{\partial}{\partial t}\right)^k\left[\frac{1}{(1+t^2)^{(n+1)/2}}\langle f((1/t)\omega), \varphi(\omega)\rangle_\omega\right].\end{aligned}$$

By the assumption, the Fourier hyperfunction f allows a set of rapidly decreasing defining functions. Their derivatives also decrease rapidly by Cauchy's inequality. Thus the last term in the above, when calculated via such defining functions, turns out to be equal to zero. Then by the duality we conclude (4.5). This calculas of limit may formally seem trivial, but the fact is not so much. It had better be verified carefully:

It suffices to consider $g(r)=\langle f(r\omega), \varphi(\omega)\rangle_\omega$ as $r\to\infty$. From the lemma below, we can see that $g(r)$ then becomes a rapidly decreasing hyperfunction of one variable. Since we know that it is in fact analytic and bounded in $r>>1$ on the real axis, we can apply the three-line theorem to a function bounded on a strip and rapidly dereasing on one of the sides to conclude that $g(r)$ is rapidly decreasing on the real axis, too. Hence the limit is zero.

Since $\tilde{f}(\omega, t)$ has support compact in ω, the local vanishing theorem holds. (See e.g. Theorem 4.4.5 in [Kn2]. Although it is stated for the case when ω is limited in a compact subset of a Euclidean space, it holds also for compact manifolds. To see this, we can simply imbedd the manifold into a Euclidean space, and extend f correspondingly, using the single layer.) Thus we conclude $\tilde{f}\equiv 0$, hence $f\equiv 0$ near the hyperplane at infinity. The remaining precise estimation of support is the same as [B1].

Lemma 4.3 *Let f be a rapidly decreasing Fourier hyperfunction. Then $g(\rho)=\langle f(\rho\omega), \varphi(\omega)\rangle_{S^{n-1}}$, calculated in the sense of integration along compact fiber for $\rho>0$, becomes a rapidly decreasing Fourier hyperfunction.*

Proof. By decomposing the support of f employing Theorem A.7, we can assume that supp f is contained in a truncated proper cone Δ, say $\Delta=\{x_n>|x'|, x_n>1\}$, where $x'=(x_1,\dots, x_{n-1})$ in this proof. Then the integral can be rewritten as

$$\begin{aligned} g(\rho)&=\int_{|\omega'|\le\omega_n} f(\rho\omega', \rho\omega_n)\varphi(\omega', \omega_n)\frac{d\omega'}{\sqrt{1-|\omega'|^2}} \\ &=\int_{|\omega'|\le 1/\sqrt{2}} f(\rho\omega', \rho\sqrt{1-|\omega'|^2})\varphi(\omega', \sqrt{1-|\omega'|^2})\frac{d\omega'}{\sqrt{1-|\omega'|^2}} \\ &=\int_{|\xi'|\le\rho/\sqrt{2}} f(\xi', \sqrt{\rho^2-|\xi'|^2})\varphi((1/\rho)\xi', \sqrt{1-|\xi'|^2/\rho^2})\frac{d\xi'}{\sqrt{\rho^2-\xi'^2}}. \end{aligned}$$

In the domain of integration, $\psi(\xi', r)=\varphi((1/\rho)\xi', \sqrt{1-|\xi'|^2/\rho^2})/\sqrt{\rho^2-\xi'^2}$ is bounded analytic. Here remark that we cas assume f given as

$$f(x)=\chi(x)\sum_{k=1}^{M} F_k(x+i\Gamma_k 0),$$

where F_k are holomorphic up to the real axis near the boundary of Δ, and χ denotes the charecteristic function of Δ. (This can be seen from the proof of Theorem A.7.) Then we have obviously

$$\begin{aligned} g(\rho\pm is)&=\int_{D(0,s)} f(\xi', \sqrt{(\rho\pm is)^2-|\xi'|^2})\psi(\xi', \rho\pm is)d\xi' \\ &=\sum_{k=1}^{M}\int_{D(\eta_k', s)} f(\xi'+i\eta_k', \sqrt{(\rho\pm is)^2-|\xi'|^2+|\eta_k'|^2\pm 2\rho is-2i\xi'\eta_k'}) \\ &\qquad\times\psi(\xi'+i\eta_k', \rho\pm is)d(\xi'+i\,\eta_k'), \end{aligned}$$

where $\xi' \mapsto \eta_k' = \eta_k'(\xi')$ is a mapping such that it is zero on the boundary of the original domain of integration D and that on the deformed domain of integral

$$D(\eta_k', s) := \{\zeta' ; \zeta' \in (\rho \pm is)D + i\eta'(\mathrm{Re}\,\zeta)\}$$

the running point $(\xi' + i\eta_k', \sqrt{\rho^2 - s^2 - |\xi'|^2 + |\eta_k'|^2 \pm 2\rho is - 2i\xi'\eta_k'})$ falls in the domain where F_k is defined. In view of

$$\sqrt{\rho^2 - s^2 - |\xi'|^2 + |\eta_k'|^2 \pm 2\rho is - 2i\xi'\eta_k'})$$
$$\sim \sqrt{\rho^2 - s^2 - |\xi'|^2 + |\eta_k'|^2} + \frac{\pm i\rho s - i\xi'\eta_k'}{\sqrt{\rho^2 - s^2 - |\xi'|^2 + |\eta_k'|^2}}$$

this is legitimate. Now it is clear that the rapid decrease follows from the same property of F_k. $\square$

Remark. It should be recalled that there are Fourier hyperfunctions, even exponentially decreasing, such that their supports are concentrated at infinity. In the case of one variable, the defining functions of such Fourier hyperfunctions patch holomorphically on the real axis but with growth like $e^{e^{x^2}}$, hence the three-line theorem does not apply. For such a Fourier hyperfunction, its canonical extension to $\boldsymbol{P}^n$ defined in §2 has support concentrated in the hyperplane at infinity, but is never a trivial extension by zero in view of Lemma 3. 4.

In the case of distributions Boman [B1] gave a local version of Theorem 4. 1. It holds also for non-quasi-analytic type ultradistributions without modification [TT]. As remarked by Boman, the agrument does not apply to hyperfunctions in view of M. Sato's counter-example to the local uniqueness. Nevertheless we expect a local type support theorem for hyperfunctions. The main difficulty for this seems to us not lying in the local vanishing theorem but in the possibility of interpretation of the local rapid decay as the local vanishing of the restriction data to the hyperplane at infinity. This is not at all trivial because local test functions with compact supports are not available in the analytic category.

Our discussion can be generalized to the X-ray transform or to the integral transform with respect to general linear subvarieties. We can treat as well the two point homogeneous spaces as in Helgason [H1]. These will be studied in our forthcoming papers.

The outline of these results were reported in [KnT]. Some conditions are improved after that report.

Appendix. Properties of Fourier hyperfunctions of slow decay

Here we gather and prove some properties of Radon hyperfunctions or polynomially decaying Fourier hyperfunctions cited above. These properties are more or less of cohomological nature. But instead of developing the corresponding cohomology theory, we preferred here to deduce the necessary properties by elementary arguments from the known facts on Fourier hyperfunctions given in [Kw], [Kn2], [Kn3]. In this article we do not intend to develop the local theory, or the sheaf theoretic treatment. But we use the word "support" for these, because it is meaningful as the one for a global section of the sheaf of usual Fourier hyperfunctions (except for the Radon hyperfunctions themselves for which we prepare this notion separately).

In the sequel we will denote by $\mathcal{O}^*(\boldsymbol{D}^n+i\Gamma_j 0)$ the space of holomorphic functions of infra-exponential growth defined on a tubular domain whose imaginary profile is asymptotically equal to the cone Γ_j near the real axis. ($\mathcal{O}^*$ is the complexified notion of $\mathscr{P}$. Here we add superfix $*$ to indicate the infra-exponential growth property. As remarked in §1, we avoid the formerly used notation "over $\sim$" because it is confusing with the symbol for the modified objects.)

Our main tool is an argument based on Kashiwara's twisted Radon decomposition of the delta function. We employ its exponential decay version:

$$\delta(x)=\int_{S^{n-1}} W_*(x,\omega)\,d\omega,$$

$$W_*(x,\omega)=\frac{(n-1)!}{(-2\pi i)^n}\frac{(1-ix\omega)^{n-1}-(1-ix\omega)^{n-2}(x^2-(x\omega)^2)}{(x\omega+i(x^2-(x\omega)^2)+i0)^n}e^{-x^2}.$$

When we treat modified version of these notions, we need a variant where the factor in the denominator is asymptotically linear in x as was employed in [Kn3] in order that the component becomes a section of $\widetilde{\mathscr{P}}_*(\boldsymbol{D}^n)$ outside the origin:

$$\widetilde{W}_*(x,\omega)=\frac{(n-1)!}{(-2\pi i)^n}\frac{J(x,\omega)}{(x\omega+i(x^2-(x\omega)^2)/\sqrt{x^2+1}+i0)^n}e^{-x^2},$$

where $J(x,\omega)$ denotes a function calculated by the same procedure as Kashiwara's from the denominator. However, the property of convolution for modified Fourier hyperfunctions is not so simple, as remarked after Proposition 1.5. In the sequel we do not treat the case of modified Fourier hyperfunctions.

Put

$$W_*(x, \Delta^\circ)=\int_{S^{n-1}\cap\Delta^\circ} W_*(x, \omega)d\omega.$$

This is the boundary value of $W_*(z, \Delta^\circ)\in \mathcal{O}_*(\boldsymbol{D}^n+i\Delta 0)$ which decays (more rapidly than) exponentially. Then for a Fourier hyperfunction $F(x+i\Gamma 0)$ the convolution $F(x+i\Gamma 0) * W_*(x, \Delta^\circ)$ is meaningful, and the result can be written as $G(x+i(\Gamma+\Delta)0)$, where

$$G(x+iy)=\int_{\boldsymbol{R}^n} F(\xi+i\eta) W_*(x+iy-\xi-i\eta, \Delta^\circ)d\xi, \quad y-\eta\in\Delta, \ \eta\in\Gamma.$$

Note that G satisfies the same decay estimate as F if there are any, such as $F(z)=O(|\mathrm{Re}\, z|^{-m})$ as $|\mathrm{Re}\, z|\to\infty$. Recall that $(\Gamma+\Delta)^\circ=\Gamma^\circ\cap\Delta^\circ$. Hence in particular, if $\Gamma^\circ\cap\Delta^\circ=\emptyset$, then G becomes holomorphic in a strip neighborhood of the real axis, conserving the estimate. For a general Fourier hyperfunction $f=\sum_{j=1}^N F_j(x+i\Gamma_j 0)$, the convolution

$$f * W_*(x, \Delta) = \sum_{j=1}^N F_j(x+i\Gamma_j 0) * W_*(x, \Delta), \tag{A.1}$$

which is calculated termwise as above, can be shown to be independent of the choice of defining functions, based on the Martineau type edge of the wedge theorem for $\mathcal{O}^*$. It can also be shown by noticing that (A. 1) agrees with the one given via the inner product:

$$f * W_*(x, \Delta)=G(x+i\Delta 0), \qquad G(z)=\langle f(\xi), W_*(z-\xi, \Delta)\rangle_\xi, \tag{A.2}$$

where $W_*(z-\xi, \Delta)$ is considered to be a test function in $\mathscr{P}_*(D^n)$ of the variable ξ with holomorphic parameter z. (A. 2) is also meaningful for Radon hyperfunctions because $\mathscr{P}_*(\boldsymbol{D}^n)$ is continuously imbedded into the space $\mathscr{P}_{(-1)}$ of test functions for the Radon hyperfunctions.

Proposition A. 1 (Weak form of Malgrange's theorem with bounds) *Let K_j, $j=1, 2$ be two compact subsets of $\boldsymbol{D}^n$. Let $\varphi(x)$ be holomorphic and bounded on a strip complex neighborhood of $K_1\cap K_2$ with fixed breadth at infinity. Then we can find similar functions $\varphi_j(x)$ on a neighborhood of K_j, $j=1, 2$ resp. such that*

$$\varphi(x)=\varphi_1(x)-\varphi_2(x) \qquad on \quad K_1\cap K_2.$$

Proof. Choose a real neighborhood D of $K_1\cap K_2$ with a smooth boundary. Set

$$F_\sigma(x)=\int_D W_*(x-\xi, \Gamma_\sigma)\varphi(\xi)\, d\xi,$$

where Γ_σ denotes the σ-th orthant. Obviously, each $F_\sigma(z)$ is holomorphic and bounded on a strip wedge of type $\boldsymbol{R}^n+i\Gamma_\sigma$. Further, by deformation of the integral path we can see that it is further holomorphic and bounded on a strip neighborhood of the interior of D. Choose a thus deformed path D_σ for each σ, and decompose it to two parts $D_{\sigma,1}$, $D_{\sigma,2}$ in such a way that $D_{\sigma,j}\cap K_j=\emptyset$, $j=1, 2$. Then

$$\varphi_j(x)=\int_{D_{\sigma,j}} W_*(x-\xi, \Gamma_\sigma)\varphi(\xi)\,d\xi, \qquad j=1,\ 2$$

will be a desired decomposition. $\square$

Actually, the above proof works when $\varphi(x)$ is a real analytic function given on an open set $D\subset\boldsymbol{D}^n$ which is the intersection of two open sets $D_j\subset\boldsymbol{D}^n$, $j=1, 2$, which is extendable as bounded holomorphic function to respective type of complex neighborhood of D. We only have to apply the same formula. This gives the cohomology vanishing theorem in degree 1 of the presheaf of bounded real analytic functions of respective type. That's why we referred to Malgrange.

Corollary A. 2 *If* supp $f\subset K_j$, $j=1,\ 2$, *then we have* supp $f\subset K_1\cap K_2$.

Proof. In view of Banach's open mapping theorem we can choose φ_j in the above Proposition in such a way that their supremum norms are bounded by constant times the bound of the original φ. Applying this observation after multiplication by $(1+x^2)^{-1/2}$, we see that a continuous linear functional on both $\mathscr{P}_*(K_j)$, $j=1, 2$ can be extended continuously on $\mathscr{P}_*(K_1\cap K_2)$. $\square$

Lemma A. 3 1) *The convolution* $\varphi\in\mathscr{P}_{(-1)}\mapsto\varphi * W_*(x, \Delta)$ *defines a continuous linear mapping from* $\mathscr{P}_{(-1)}$ *into itself.*

2) *For a Radon hyperfunction f, $G(z)$ in* (A. 2) *satisfies* $G(z)=O(|\mathrm{Re}\,z|^{-1})$ *and* $G(x+i\Delta 0)$ *again becomes a Radon hyperfunction* (*of which the interpretation is given in the proof*).

Proof. 1) Choosing $\mathrm{Im}\,z-\mathrm{Im}\,\zeta\in\Delta$ and $N>n+1$, we have

$$\begin{aligned}
&|\mathrm{Re}\,z|\left|\int\varphi(\zeta)\,W_*(z-\zeta, \Delta)d(\mathrm{Re}\,\zeta)\right|\\
&\qquad\le c\int(|\mathrm{Re}\,\zeta|+1)\varphi(\zeta)\frac{|\mathrm{Re}\,z|}{|\mathrm{Re}\,\zeta|+1}\frac{1}{(|z-\zeta|+1)^N}d(\mathrm{Re}\,\zeta)\\
&\qquad\le C\sup_{|\mathrm{Im}\,\zeta|<\varepsilon}(|\mathrm{Re}\,\zeta|+1)\varphi(\zeta)\int\frac{1}{(|z-\zeta|+1)^{N-1}}d(\mathrm{Re}\,\zeta)\\
&\qquad\le C,
\end{aligned}$$

where we have used Peetre's inequality $(\mathrm{Re}z|+1)/(|\mathrm{Re}\zeta|+1)\leq|\mathrm{Re}z-\mathrm{Re}\zeta|+1$. The continuity is obvious.

2) Estimating (A. 2) in a similar way by choosing $N\geq 1$ and $\varepsilon>0$ such that $\mathrm{Im}z-B(0;\varepsilon)\subset\varDelta$, we find

$$\begin{aligned}|\langle f(\xi), W_*(z-\xi,\varDelta)\rangle_\xi &\leq C\sup_{|\mathrm{Im}\zeta|<\varepsilon}|\mathrm{Re}\zeta||W_*(z-\zeta,\varDelta)|\\ &\leq C(|\mathrm{Re}z|+1)\sup_{|\mathrm{Im}\zeta|<\varepsilon}\frac{|\mathrm{Re}\zeta|}{|\mathrm{Re}z|+1}\frac{1}{(|z-\zeta|+1)^N}\\ &\leq C(|\mathrm{Re}z|+1),\end{aligned}$$

C depending on $\mathrm{Im}z$ in a locally uniform way. The result of the convolution is again a Radon hyperfunction. In fact, first choose $\varphi\in\mathscr{P}_{(-1)}(\boldsymbol{D}^n)$. Then, in the sense of calculus for usual Fourier hyperfunctions we have, for any $\varepsilon>0$,

$$\begin{aligned}\langle G(x+i\varDelta 0), \varphi\rangle &=\langle f*W_*(x,\varDelta^\circ),\varphi\rangle=\langle\langle f(\xi), W_*(x-\xi,\varDelta^\circ)\rangle_\xi,\varphi(x)\rangle_x\\ &:=\langle f(\xi),\langle W_*(x-\xi,\varDelta^\circ),\varphi(x)\rangle_x\rangle_\xi\\ &\leq C_\varepsilon\sup_{|\mathrm{Im}\zeta|<\varepsilon}|\zeta||\langle W_*(x+iy-\zeta,\varDelta^\circ), \varphi(x+iy)\rangle_x|\\ &\leq C_\varepsilon\sup_{|\mathrm{Im}z|<2\varepsilon}|\mathrm{Re}z||\varphi(z)|\end{aligned}$$

by the calculus done in 1). Here it should be noted that the first line only has a symbolic meaning and the second line is the true definition of this functional. This implies by definition that $G(x+i\varDelta 0)$, interpreted in this way, becomes a Radon hyperfunction. □

Corollary A. 4 *Rodan hyperfunctions are characterized as those hyperfunctions which admit a set of defining functions $\{F_j(z)\}$ such that $F_j(z)=O(|\mathrm{Re}z|^{-1})$ and that each $F_j(x+i\Gamma_j 0)$ defines a Radon hyperfunction* (*in the above sense*).

Proof. In fact, by means of a set of polyhedral cones $\{\Gamma_j^\circ\}$ covering $\boldsymbol{R}^n$ without redundancy, we can decompose a Radon hyperfunction f as

$$f(x)=\sum F_j(x+i\Gamma_j 0),\quad F_j(z)=\langle f(\xi), W_*(z-\xi,\Gamma_j^\circ)\rangle_\xi,$$

$F_j(z)$ satisfying 2) of the above lemma with $\varDelta$ replaced by Γ_j. Note that this decomposition is assured by a more obvious one:

$$\varphi(x)=\sum W(x,\varDelta_j^\circ)*\varphi.\quad\square$$

Remark. Assume that as a consequence of calculation of Lemma A. 3, we could show the following:

$$\int_{\boldsymbol{R}^n}G(x+ib)\varphi(x+ib)dx=\int_{\boldsymbol{R}^n}\frac{G(x+ib)}{|x|+1}\cdot(|x|+1)\varphi(x+ib)dx$$

$$\leq C \sup_{x\in \boldsymbol{R}^n}(|x|+1)\varphi(x+ib).$$

Since $(|\mathrm{Re}z|+1)\mathscr{P}_*(\boldsymbol{D}^n)$ in dense in the space $c_0(\boldsymbol{R}^n)$ of continuous functions tending to zero at infinity with the supremum norm, $(G(z)/(|\mathrm{Re}z|+1))d(\mathrm{Re}z)$ would then define a continuous linear functional on $c_0(\boldsymbol{R}^n)$. Hence it would come from a measure of finite total mass, and we could conclude that $G(z)/(|\mathrm{Re}z|+1)) \in L^1(\boldsymbol{R}^n)$ as a function of $\mathrm{Re}z$, locally uniformly in y. This would give a much more pleasant characterization of Radon hyperfunctions via defining functions. But the above passage is false because as we remarked in § 3, Radon hyperfunctions do not constitute a subclass of Fourier hyperfunctions. It should be emphasized that for the component supplied by the above corollary, the representation of inner product

$$\langle F_j(x+i\Gamma_j 0),\ \varphi(x)\rangle = \int_{\boldsymbol{R}^n} F(z)\varphi(z)d(\mathrm{Re}z)$$

is valid for $\varphi\in\mathscr{P}_*(\boldsymbol{D}^n)$, but not justified for $\varphi\in\mathscr{P}_{(-1)}$. It is legal of course if $\varphi(z)=O(|\mathrm{Re}z|^{-n})$, because of the indicated estimate for $|F_j(z)|$.

Theorem A. 5 (weak decomposition theorem of analytic wave-front set) *Let $f(x)=F(x+i\Gamma 0)$ be a Fourier hyperfunction such that $F(z)=O(|\mathrm{Re}z|^{-m})$ as $|\mathrm{Re}z| \to \infty$ for some (resp. any) m. Let $\{\Gamma_j^\circ\}_{j=1}^N$ be proper closed cones such that $\{\mathrm{Int}\Gamma_j^\circ\}_{j=1}^N$ covers Γ°. Then we can find $F_j(z)\in \mathcal{O}^*(\boldsymbol{D}^n+i\Gamma_j 0)$ with the same decay property as F such that*

$$f(x)=\sum_{j=1}^N F_j(x+i\Gamma_j 0).$$

The same assertion holds also for absolutely integrable Fourier hyperfunctions or for Radon hyperfunctions under an appropriate interpretation.

Proof. Choose polyhedral cones $\Delta_j^\circ\subset\subset\Gamma_j^\circ$ intersecting only by faces to each other, such that $\bigcup_{j=1}^N \mathrm{Int}\Delta_j^\circ \supset\Gamma^\circ$. (Here and in the sequel, for two cones $\Delta\subset\subset\Gamma$ means $\Delta\cap \boldsymbol{S}^{n-1}\subset\subset\Gamma\cap\boldsymbol{S}^{n-1}$ as sets of $\boldsymbol{S}^{n-1}$.) Then decompose f as

$$f=\sum_{j=1}^N f * W_*(x, \Delta_j^\circ)+f * W_*(x, \boldsymbol{S}^{n-1}\setminus\bigcup_{j=1}^N \Delta_j^\circ).$$

Here the second convolution is calculated by decomposing the indicated subregion of $\boldsymbol{S}^{n-1}$ by convex polyhedral cones, and term by term. As remarked above, this term becomes holomorphic on a strip neighborhood of the real axis. Thus attaching this to any of the components in the sum at the right-hand side, we obtain a desired decomposition. The corresponding assertion holds for Radon hyperfunctions in view of Lemma A. 3. □

Theorem A. 6 (Martineau type edge of the wedge theorem with polynomial decay condition) *Let $f(x)$ be a Fourier hyperfunction possessing a set of defining functions $\{F_j(z)\in \mathcal{O}^*(\boldsymbol{D}^n+i\Gamma_j 0)\}_{j=1}^N$, satisfying $F_j(z)=O(|\mathrm{Re}\,z|^{-m})$ for some (resp. any) m. Assume that $f=0$ in $\mathcal{Q}(\boldsymbol{D}^n)$. Then for any choice of smaller cones $\Gamma_j'\subset\subset\Gamma_j$, we can find $F_{jk}(z)\in\mathcal{Q}^*(\boldsymbol{D}^n+i(\Gamma_j'+\Gamma_k')0)$, $j, k=1,\cdots,N$, satisfying $F_{jk}(z)=O(|\mathrm{Re}\,z|^{-m})$ for the same (resp. any) m and*

$$F_{jk}(z)=-F_{kj}(z),\quad F_j(z)=\sum_{k=1}^N F_{ik}(z)\ \text{ on } \boldsymbol{D}^n+i\Gamma_j'0.$$

Similar assertion holds also for absolutely integrable Fourier hyperfunctions, or for Radon hyperfunctions under an appropriate interpretation.

Proof. The proof is made by the induction on the number N of terms. First let $N=2$. Then $F_1(x+i\Gamma_1 0)+F_2(x+i\Gamma_2 0)=0$ implies $g(x):=F_1(x+i\Gamma_1 0)=-F_2(x+i\Gamma_2 0)$. Choose a polyhedral cone Δ° such that $\Gamma_1^\circ\cap\Gamma_2^\circ\subset\subset\Delta^\circ\subset\subset\Gamma_1'^\circ\cap\Gamma_2'^\circ$ and decompose g as

$$g=g*W_*(x,\Delta^\circ)+g*W_*(x,\boldsymbol{S}^{n-1}\backslash\Delta^\circ).\tag{A. 3}$$

Hence choosing the suitable one from $F_j(z), j=1,2$ in calculating the convolution, we conclude that the second term is holomorphic in a strip neighborhood of the real axis. Since the first term gives a function holomorphic in $\boldsymbol{D}^n+i(\Gamma_1'+\Gamma_2')0$, the sum of these two functions gives a desired function $F_{12}(z)$. Notice that in case $n=1$, this gives another proof of the three-line theorem.

Now assume that the assertion is proved up to $N-1$ terms. Consider

$$g(x):=F_N(x+i\Gamma_N 0)=-\sum_{j=1}^{N-1}F_j(x+i\Gamma_j 0).$$

By Teorem A. 5 we have a decompositon

$$g=F_N(x+i\Gamma_N 0)=\sum_{j=1}^{N-1}G_j(x+i\Delta_j 0)+\sum_{k=1}^{M}H_k(x+iE_k 0),$$

where Δ_j°, $j=1,\cdots,N-1$, E_k°, $k=1,\cdots,M$ constitutes a covering of Γ_N° such that $\Gamma_j^\circ\cap\Gamma_N^\circ\subset\Delta_j^\circ\subset\subset\Gamma_j'^\circ\cap\Gamma_N'^\circ$, $E_k^\circ\cap\bigcup_{j=1}^{N-1}\Gamma_j^\circ=0$. The calculation of $H_k=g*W_*(x,E_k^\circ)$ employing the other expression of g shows that it is holomorphic in a strip neighborhood of $\boldsymbol{R}^n$, with estimate. Thus replacing F_j by F_j+G_j and attaching E_k to one of them, we obtain a situation in $N-1$

terms. Thus by the induction hypothesis, we can find $F_{jk}(z)\in\mathcal{O}^*(\boldsymbol{D}^n+i(\Gamma_j'+I_k')0)$, j, $k=1,\cdots,N-1$ satisfying the required properties. Put

$$F_{Nj}(z) = -F_{jN}(z) = G_j(z), \quad j=1,\cdots,N-1.$$

These, together with the already found F_{jk}'s satisfy the required properties in the case of N terms. □

Theorem A. 7 (softness) *Let f be a Fourier hyperfunction possessing a set of defining functions $\{F_j(z)\in\mathcal{O}^*(\boldsymbol{D}^n+i\Gamma_j0)\}_{j=1}^N$, satisfying $F_j(z)=O(|\mathrm{Re}\,z|^{-m})$ for some* (*resp. any*) *m. Let $\{\Delta_k^\circ\}_{k=1}^M$ be a set of closed cones which cover $\boldsymbol{R}^n$ without redundancy* (*i.e. with superposition only by faces*). *Then for any choice of $\Delta_k'^\circ\supset\supset\Delta_k^\circ$, we can find $\{f_k\}_{k=1}^M$ with the same decay property and with*

$$\operatorname{supp} f_k\subset\Delta_k'^\circ, \quad f=\sum_{k=1}^M f_k.$$

Similar assertion holds for an absolutely integrable Fourier hyperfunction, or for a Radon hyperfunction.

Proof. We first show that f can be decomposed in such a way with respect to the singular support, allowing a flow out of support by exponentially decreasing real analytic functions. (If the sheaf theory is introduced for the classes of Fourier hyperfunctions we are treating, this can be precisely stated as a decomposition in the meaning of sections of the corresponding quotient sheaves.) For this we can assume that $f(x)=F(x+i\Gamma 0)$. With a fixed point $b\in\Gamma$, set

$$G_k(z)=\int_{\Delta_k^\circ}\mathrm{W}_*(z-\xi-ib,\Delta^\circ)F(\xi+ib)d\xi, \quad k=1,\cdots,M.$$

By the standard argument we can see that $G_k(x+i\Delta 0)$ defines a Fourier hyperfunction of the same class as f, and that it is an exponentially decreasing real analytic function outside Δ_k°. The difference

$$f(x)-\sum_{k=1}^M G_k(x+i\Delta 0) = \int_{\boldsymbol{R}^n} W_*(z-\xi-ib)F(\xi+ib)d\xi$$

which may be symbolically written as $W(x-ib, \boldsymbol{S}^{n-1}\backslash\Delta^\circ)*f(x)$ (but its exact meaning is given via the sum of complex integrals of above type with Δ° replaced by decomposed parts Δ_j° of $\boldsymbol{S}^{n-1}\backslash\Delta^\circ$), is seen to be an exponentially decreasing real analytic function on the whole space just in the same way as in the proof of Theorem A. 5. Combining this term to any one of G_k, we thus obtained a decomposition by singular support.

Now it suffices to cut off the support of each $G_k(x+i\Delta 0)$, using the characteristic functions, with a small merge on Δ_k° and exactly on the border of Δ_l°, $l \neq k$. (It should be noted that on any compact set, precise decomposition of support is possible in view of the flabbiness of the sheaf $\mathscr{B}$ of usual hyperfunctions. Hence this theorem is meaningful only in the neighborhood of points at infinity.) □

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Department of Mathematics
Defense Academy
1-10-20 Hashirimizu
Yokosuka,
Kanagawa 239 Japan

Department of Mathematical Sciences
University of Tokyo
3-8-1, Komaba, Meguro-ku
Tokyo 153 Japan