Hokkaido Mathematical Journal Vol. 29 (2000) p. 553–561

The diameter of the solvable graph of a finite group

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(Received September 27, 1999)

Abstract. Let G be a finite group. We define the solvable graph $\Gamma_S(G)$ as follows: the vertices are the primes dividing the order of G and two vertices p, q are joined by an edge if there is a solvable subgroup of G of order divisible by pq. We will prove that the diameter of $\Gamma_S(G)$ is less than or equal to 4 for any finite group G. We use the classification of finite simple groups.

Key words: finite simple groups, prime graphs, solvable graphs.

1. Introduction

Let G be a finite group and $\pi(G)$ the set of primes dividing the order of G. We denote by $\pi(n)$ the set of primes dividing a natural number n.

We define the prime graph $\Gamma(G)$ as follows: the vertices are elements of $\pi(G)$, and two distinct vertices p, q are joined by an edge, we write $p \sim q$, if there is an element of order pq in G. Note that $p \sim q$ if and only if there is a cyclic subgroup of G of order pq.

We define the solvable graph $\Gamma_S(G)$ as follows: the vertices are the elements of $\pi(G)$, and two distinct vertices p, q are joined by an edge, we write $p \approx q$, if there is a solvable subgroup of G of order divisible by pq. The concept of solvable graphs was defined recently in Abe-Iiyori [1].

It has been studied about the connected components of $\Gamma(G)$ in Williams [8], Iiyori and Yamaki [5], Kondrat′ev [6]. Abe and Iiyori [1] proved that $\Gamma_S(G)$ is connected. The diameter of $\Gamma(G)$ has been determined by Lucido [7]. We denote the connected components of $\Gamma(G)$ by $\pi_1, \ldots, \pi_{n(\Gamma(G))}$, where $n(\Gamma(G))$ is the number of connected components of $\Gamma(G)$. If the order of G is even, we take π_1 to be the component containing 2. Let $d(p,q)$ (resp. $d_S(p,q)$) be the distance between two vertices p, q in $\Gamma(G)$ (resp. $\Gamma_S(G)$). We can define the diameter of $\Gamma_S(G)$ as follows:

$$\operatorname{diam}(\Gamma_S(G)) = \max\{d_S(p,q) \mid p, q \in \pi(G)\}.$$

The purpose of this paper is to prove:

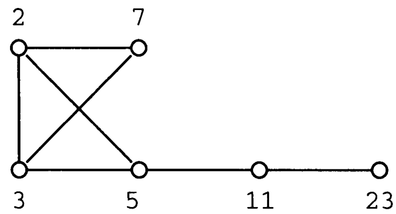
1991 Mathematics Subject Classification : 20D05.

Theorem 1 *Let G be a finite group. Then* $\mathrm{diam}(\Gamma_S(G)) \leq 4$.

Corollary *If G is a non-abelian simple group, then* $\mathrm{diam}(\Gamma_S(G)) = 2, 3$ *or* 4.

Theorem 2 *Let G be a finite group. Then $d_S(2, p) \leq 3$ for any $p \in \pi(G)$.*

Example Let G be the Mathieu simple group M_{23} of degree 23. We can draw easily $\Gamma_S(G)$ by the table of the maximal subgroups of M_{23} in Atlas [4]. Indeed, $\Gamma_S(G)$ is:



Thus $\mathrm{diam}(\Gamma_S(G)) = 4$, since $d_S(7, 23) = 4$.

2. Preliminaries

The following Lemma is fundamental to the study of solvable graphs.

Lemma 2.1 (Abe-Iiyori [1]) *We take two distinct vertices $p, q \in \pi(G)$.*

(1) *If $p \sim q$, then $p \approx q$.*
(2) *Let H be a subgroup of G. If $p \approx q$ in $\Gamma_S(H)$, then $p \approx q$ in $\Gamma_S(G)$.*
(3) *If G has a non-trivial normal subgroup K, then $p \approx q$ in $\Gamma_S(G)$ for $p \in \pi(K)$ and $q \in \pi(G/K)$.*
(4) *Let K be a normal subgroup of G. If $p \approx q$ in $\Gamma_S(G/K)$, then $p \approx q$ in $\Gamma_S(G)$.*

We will apply the following propositions.

Proposition 1 (Williams [8]) *Let G be a non-abelian simple group such that $n(\Gamma(G)) \geq 2$. Then*

(1) *G has a Hall π_i-subgroup H_i for a connected component π_i ($i \geq 2$) of $\Gamma(G)$,*
(2) *H_i is an isolated abelian subgroup of G.*

Note A subgroup H of G is called isolated if $H \cap H^g = \langle 1 \rangle$ or H for any $g \in G$, and for any $h \in H - \{1\}$, $C_G(h) \subseteq H$.

Proposition 2 (Abe-Iiyori [1]) *The following claims hold:*

(1) *Let G be a non-abelian simple group such that $n(\Gamma(G)) \geq 2$. If H_i is a Hall π_i-subgroup $(i \geq 2)$, then H_i is a proper subgroup of $N_G(H_i)$.*
(2) *$\Gamma_S(G)$ is connected.*
(3) *If G is a non-abelian simple group, then $\Gamma_S(G)$ is incomplete, i.e.,* $\mathrm{diam}(\Gamma_S(G)) \geq 2$.

Proposition 3 (Chigira-Iiyori-Yamaki [2], [3], Lucido [7]) *If G is a simple group, then $d(2,p) \leq 2$ for any $p \in \pi_1$.*

Lemma 2.2 *If $|\pi(G)| \leq 4$, then* $\mathrm{diam}(\Gamma_S(G)) \leq 3$ *and $d_S(2,p) \leq 3$ for any $p \in \pi(G)$.*

Proof. This is immediate from Proposition 2(2). □

Lemma 2.3 *Let G be a simple group with $n(\Gamma(G)) \geq 2$. Suppose that G has a subgroup H_p such that $p \in \pi(H_p)$ and $|N_G(H_p) : H_p|$ is even for any $p \in \pi(G) - \pi_1$. Then* $\mathrm{diam}(\Gamma_S(G)) \leq 4$.

Proof. As G is simple, $d(2,r) \leq 2$ for any $r \in \pi_1$. Since $|N_G(H_p)|$ is even, $2 \approx p$ for any $p \in \pi_i$ $(i \geq 2)$. Thus $\mathrm{diam}(\Gamma_S(G)) \leq 4$. □

Notation Put $\pi(C_I) = \bigcup_{t \in I(G)} \pi(C_G(t))$, where $I(G)$ is the set of all involutions in G. Put $\pi(C_J) = \bigcup_{u \in J(G)} \pi(C_G(u))$, where $J(G)$ is the set of all elements of order 3 in G.

Lemma 2.4 *Let G be a non-abelian simple group. If $n(\Gamma(G)) = 2$ and $\pi_1 = \pi(C_I)$, then* $\mathrm{diam}(\Gamma_S(G)) \leq 3$.

Proof. There is an abelian Hall π_2-subgroup H by Proposition 1. Proposition 2(1) claims the existence of $p \in \pi_1$ such that $p \,\big|\, |N_G(H)|$. For any $q \in \pi_2$, $p \approx q$. Thus $\mathrm{diam}(\Gamma_S(G)) \leq 3$. □

Lemma 2.5 *Let G be a non-abelian simple group. If $n(\Gamma(G)) = 2$ and G has an abelian subgroup H satisfying $\pi(G) - (\pi(C_I) \cup \pi_2) \subseteq \pi(H)$, then* $\mathrm{diam}(\Gamma_S(G)) \leq 4$.

Proof. There is $p \in \pi_1$ such that $p \approx q$ for any $q \in \pi_2$ by Proposition 2(1). If $p \in \pi(C_I)$, then $d_S(2,r) \leq 2$ for any $r \in \pi(G)$ by Proposition 3. If $p \in \pi(H)$, then $d_S(2,q) \leq 3$ for any $q \in \pi_2$. Since H is abelian, $\mathrm{diam}(\Gamma_S(G)) \leq 4$. □

3. Proof of Theorem 1

We will give a proof of Theorem 1.

Lemma 3.1 *If G is not a simple group, then* $\operatorname{diam}(\Gamma_S(G)) = 1$ *or* 2.

Proof. Suppose G has a non-trivial proper normal subgroup N. It follows from Lemma 2.1(3) that there is $q \in \pi(N)$ such that $p_1 \approx q \approx p_2$ for any $p_1, p_2 \in \pi(G/N)$. Similarly, there is $p \in \pi(G/N)$ such that $q_1 \approx p \approx q_2$ for any $q_1, q_2 \in \pi(N)$. Since $\pi(G) = \pi(G/N) \cup \pi(N)$, $\operatorname{diam}(\Gamma_S(G)) \leq 2$. $\square$

Lemma 3.2 *If G is the alternating group, then* $\operatorname{diam}(\Gamma_S(G)) \leq 3$.

Proof. Suppose that G is the alternating group A_n of degree n ($n \geq 5$). If $n = 5,\ 6$, then $\operatorname{diam}(\Gamma_S(G)) = 2$. Suppose $n \geq 7$. If there is a prime p such that $n - 2 \leq p \leq n$, then Sylow p-subgroups of G are cyclic. There is $q \in \pi((p-1)/2) \cup \{2\}$ such that $p \approx q$. Thus $\operatorname{diam}(\Gamma_S(G)) \leq 3$. $\square$

Remark Let A_n be the alternating group of degree n ($n \geq 5$). If $\operatorname{diam}(\Gamma_S(A_n)) = 3$, then either n or $n-1$ is a prime p such that $p \equiv 3 \bmod 4$.

Proof. Suppose n and $n-1$ are not prime $p \equiv 3 \bmod 4$. Since A_n has a subgroup which is isomorphic to a symmetric group of degree $n-2$, $2 \approx p$ for any prime $p \leq n-2$. If n or $n-1$ is a prime p such that $p \equiv 1 \bmod 4$, then A_n has a dihedral subgroup of order $2p$, and so $2 \approx p$. Thus $2 \approx p$ for any $p \in \pi(A_n)$. It follows from Proposition 2(3) that $\operatorname{diam}\Gamma_S(A_n) = 2$. $\square$

Lemma 3.3 *If G is a sporadic simple group, then* $\operatorname{diam}(\Gamma_S(G)) \leq 4$.

Proof. We can draw $\Gamma_S(G)$ for a sporadic simple group G by tables of maximal subgroups and p-local subgroups in Atlas [4] using Lemma 2.1. It have been completely classified that the maximal subgroups of the Baby Monster simple group B by Wilson [9].

For example, let M be the Monster simple group. Then $\pi(M) = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}$. M has a 2-local subgroup isomorphic to $2^{\cdot}B$, $\pi(B) = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 31, 47\}$. For any $p \in \pi(B)$, $p \approx 2$. M has p-local subgroups isomorphic to $71 : 35$, $59 : 29$, $41 : 40$, $29 : 14$ for $p = 71,\ 59,\ 41,\ 29$. It follows that $59 \approx 29 \approx 7 \approx 71$. Since $d_S(59, 71) = 3$, $\operatorname{diam}(\Gamma_S(M)) = 3$.

The following table shows the diameter of $\Gamma_S(G)$ for a sporadic simple group G.

Sporadic simple group	Diameter
J_1, J_2, He, Ru, Suz, $O'N$, Fi_{22}, Ly, Th, Fi_{23}, Co_1, J_4, Fi'_{24}	2
M_{11}, M_{12}, M_{22}, HS, J_3, M_{24}, M^cL, Co_3, Co_2, HN, B, M	3
M_{23}	4

□

Lemma 3.4 *If G is either $E_6(q)$ or ${}^2E_6(q^2)$, then* $\mathrm{diam}(\Gamma_S(G)) \leq 3$.

Proof. Let $G = E_6(q)$. Suppose that q is odd. Then $\pi((q^6+q^3+1)(q^4-q^2+1)/(3,q-1))$ contains $\pi(G)-\pi(C_I)$ by [8]. $\pi(C_I) \ni 2,3$ and $p_1 \approx 3$ for any $p_1 \in \pi((q^6+q^3+1)/(3,q-1))$, $p_2 \approx 2$ for any $p_2 \in \pi(q^4-q^2+1)$ by [1]. Thus $\mathrm{diam}(\Gamma_S(G)) \leq 3$. Suppose that q is even. Then $\pi((q^6+q^3+1)(q^4+1)(q^4-q^2+1)/(3,q-1))$ contains $\pi(G)-\pi(C_I)$ by [5]. $\pi(C_I) \ni 2,3$ and $p_1 \approx 3$ for any $p_1 \in \pi((q^6+q^3+1)/(3,q-1))$, $p_2 \approx 2$ for any $p_2 \in \pi(q^4+1)$, $p_3 \approx 2$ for any $p_3 \in \pi(q^4-q^2+1)$ by [1]. Thus $\mathrm{diam}(\Gamma_S(G)) \leq 3$.

Let $G = {}^2E_6(q)$. Suppose that q is odd. Then $\pi((q^6-q^3+1)(q^4-q^2+1)/(3,q+1))$ contains $\pi(G)-\pi(C_I)$ by [8]. $\pi(C_I) \ni 2,3$ and $p_1 \approx 3$ for any $p_1 \in \pi((q^6-q^3+1)/(3,q+1))$, $p_2 \approx 2$ for any $p_2 \in \pi(q^4-q^2+1)$ by [1]. Thus $\mathrm{diam}(\Gamma_S(G)) \leq 3$. Suppose that q is even. Then $\pi((q^6-q^3+1)(q^4-q^2+1)(q^4+1)/(3,q+1))$ contains $\pi(G)-\pi(C_I)$ by [5]. $\pi(C_I) \ni 2,3$ and $p_1 \approx 3$ for any $p_1 \in \pi((q^6-q^3+1)/(3,q+1))$, $p_2 \approx 2$ for any $p_2 \in \pi(q^4-q^2+1)$, $p_3 \approx 2$ for any $p_3 \in \pi(q^4+1)$ by [1]. Thus $\mathrm{diam}(\Gamma_S(G)) \leq 3$. □

Lemma 3.5 *If G is a simple group of Lie type such that $n(\Gamma(G)) = 1$, then* $\mathrm{diam}(\Gamma_S(G)) \leq 4$.

Proof. For any $p \in \pi(G)$, $d(2,p) \leq 2$ by Proposition 3.
Thus $\mathrm{diam}(\Gamma_S(G)) \leq 4$. □

Lemma 3.6 *If G is a simple group of Lie type such that $n(\Gamma(G)) = 3, 4$ or 5, then* $\mathrm{diam}(\Gamma_S(G)) \leq 4$.

Proof. If G satisfies the hypotheses of Lemma 2.2 or Lemma 2.3, then

$\mathrm{diam}(\Gamma_S(G)) \leq 4$. Thus we can assume that G is one of the following groups by the tables of [8], [5] and [1].

$$\begin{array}{lll} A_1(q), & q \equiv -1 \pmod 4), & q \geq 19, \\ {}^2A_5(2), & & \\ {}^2D_p(3^2), & p = 2^n + 1, & n \geq 2, \\ {}^2E_6(2^2). & & \end{array}$$

It follows that $\mathrm{diam}(\Gamma_S(A_1(q))) \leq 3$. Indeed, $\mathrm{diam}(\Gamma_S({}^2A_5(2))) = 3$ by Atlas [4]. We showed $\mathrm{diam}(\Gamma_S({}^2E_6(2^2))) \leq 3$ in Lemma 3.4. Let G be ${}^2D_p(3^2)$. G has an abelian subgroups H_2 such that $\pi_2 \subseteq \pi(H_2)$ and $|N_G(H_2) : H_2|$ is a power of 2. Since we can know that $\pi_1 = \pi(C_I)$ from [8], $2 \approx r$ for any prime r in $\pi_1 \cup \pi_2$. There is $r \in \pi_1 \cup \pi_2$ such that $r \approx s$ for any $s \in \pi_3$ from Proposition 1. Thus $\mathrm{diam}(\Gamma_S(G)) \leq 3$. □

Lemma 3.7 *If G is a simple group of Lie type such that $n(\Gamma(G)) = 2$, then* $\mathrm{diam}(\Gamma_S(G)) \leq 4$.

Proof. If G satisfies the hypotheses of Lemma 2.2 or Lemma 2.3, then the result is trivial. Groups in the following list satisfy the hypotheses of Lemma 2.4.

$$\begin{array}{ll} A_{p-1}(q), & \\ B_{2^n}(q), & n \geq 2, \\ B_p(3), & \\ C_{2^n}(q), & n \geq 1, \\ C_p(3), & \\ D_p(3), & p \geq 5, \\ D_{p+1}(3), & p \geq 3, \\ {}^2A_{p-1}(q^2), & \\ {}^2A_p(q^2), & q+1 \mid p+1, \\ {}^2D_{2^n}(q^2), & \end{array}$$

where q is odd and p is an odd prime.

$$C_{2^n}(q), \quad n \geq 1,$$

where q is even.

Groups in the following list satisfy the hypotheses of Lemma 2.5.

$$\begin{array}{lll} A_p(q), & q-1\,|\,p+1 & \\ D_p(5), & p \geq 5 & \\ {}^2D_l(3^2), & l \neq 2^n+1, & l=p \\ {}^2D_l(3^2), & l = 2^n+1, & l \neq p, \end{array}$$

where q is odd and p is an odd prime.

$$\begin{array}{ll} A_{p-1}(q), & p \geq 5, \\ A_p(q), & q-1\,|\,p+1, \\ C_p(2), & \\ {}^2A_{p-1}(q^2), & \\ {}^2A_p(q^2), & q+1\,|\,p+1, \end{array}$$

where q is even and p is an odd prime.
Lemma 3.7 holds for these groups by [8] and [5].
Thus we can assume that G is one of the following groups:

$$\begin{array}{ll} D_p(2), & \\ D_{p+1}(2), & \\ {}^2D_{2^n}(q^2), & n \geq 2, \\ {}^2D_{2^n+1}(q^2), & n \geq 2, \\ A_2(2^n), & n \geq 3, \end{array}$$

where q is even and p is an odd prime.

Suppose that G is $D_k(2)$. G contains a subgroup isomorphic to $D_{k-1}(2) \times Z_3$. We have $|D_k(2) : D_{k-1}(2)| = 2^{2(k-1)}(2^k+1)(2^{k-1}+1)$. There is a maximal torus $T(D_k)$, of order $3(2^{k-1}+1)$. $\pi(D_k(2)) = \pi(C_J) \cup \pi_2$. Thus $\mathrm{diam}(\Gamma_S(G)) \leq 3$.

Suppose that G is ${}^2D_k(q)$. G contains subgroups isomorphic to $D_{k-1}(q) \times Z_{q+1}$, ${}^2D_{k-1}(q) \times Z_{q-1}$. We have $|{}^2D_k(q) : D_{k-1}(q)| = q^{2(k-1)}(q^k+1)(q^{k-1}+1)$, $|{}^2D_k(q) : {}^2D_{k-1}(q)| = q^{2(k-1)}(q^k+1)(q^{k-1}-1)$. There are maximal tori T_1, T_2, T_3 such that $|T_1| = q^k+1$, $|T_2| = (q^{k-1}+1)(q-1)$, $|T_3| = (q+1)(q^{k-1}-1)$. Suppose $q = 2^\alpha$ (α is even), then $3\,\big|\,|T_2|$, $3\,\big|\,|T_3|$. Suppose $q = 2^\alpha$ (α is odd) and $k = 2^n$, then $3\,\big|\,|T_2|$, $3\,\big|\,|T_3|$. Suppose $q = 2$, $k = 2^n+1$, then $3\,\big|\,|T_1|$, $3\,\big|\,|T_3|$. Thus $\pi(G) = \pi(C_J) \cup \pi_2$, $\mathrm{diam}(\Gamma_S(G)) \leq 3$.

Suppose that G is $A_2(2^n)$, $n \geq 3$. Then $\pi(C_I) = 2(q-1)/(3, q-1)$. There are tori of order $(q-1)^2/(3, q-1)$, $(q^2-1)/(3, q-1)$ and $(q^2+q+$

$1)/(3, q-1)$ in G. It follows that $r \sim s$ for any $r \in \pi(q-1/(3, q-1))$, $s \in \pi_1$. Thus $\mathrm{diam}(\Gamma_S(G)) \leq 3$. □

Proof of Theorem 1. If G is not simple group, then $\mathrm{diam}(\Gamma_S(G)) = 1$ or 2 from Lemma 3.1. We may assume that G is isomorphic to one of the following simple groups:

(1) an alternating group A_n with $n \geq 5$,

(2) one of the 26 sporadic simple groups,

(3) a simple group of Lie type.

Thus we have proved that $\mathrm{diam}(\Gamma_S(G)) \leq 4$ for a non-abelian simple group G. This completes the proof of Theorem 1. □

Proof of Corollary. If G is a non-abelian simple group, then $\mathrm{diam}(\Gamma_S(G)) \geq 2$ from Proposition 2(3). Thus this is trivial from Theorem 1. □

4. Proof of Theorem 2

We can assume that G is a simple group by Lemma 3.1 and G is not an alternating group by Lemma 3.2. The diameter of any sporadic simple group is 2 or 3, except for M_{23}. If G is M_{23}, then $d_S(2, p) \leq 3$ for any $p \in \pi(G)$ by $\Gamma_S(M_{23})$.

Proof of Theorem 2. We can assume that G is a simple group of Lie type. Actually, we have also proved that $d_S(2, p) \leq 3$ for $p \in \pi(G)$ in Lemmas 3.4, 3.5, 3.6 and 3.7. The proof of Theorem 2 is complete. □

Acknowledgment The author would like to thank her supervisor, Professor Hiroyoshi Yamaki. He took time off to help her and gave her a lot of suggestions.

References

[1] Abe S. and Iiyori N., *A generalization of prime graphs of finite groups.* Hokkaido Math. J. **29** No. 2 (2000), 391–407.

[2] Chigira N., Iiyori N. and Yamaki H., *Nonabelian Sylow subgroups of finite groups of even order.* ERA Amer. Math. Soc. **4** (1998), 88–90.

[3] Chigira N., Iiyori N. and Yamaki H., *Non-abelian Sylow subgroups of finite groups of even order.* Invent. Math. **139** (2000), 525–539.

[4] Conway J.H., Curtis R.T., Norton S.P., Parker R.A. and Wilson R.A., *"Atlas of finite groups."*. Clarendon Press, Oxford, (1985).

[5] Iiyori N. and Yamaki H., *Prime graph components of the simple groups of Lie type over the field of even characteristic.* J. Algebra **155** (1993), 335–343; Corrigenda **181** No. 2 (1996), 659.

[6] Kondrat′ev A.S., *Prime graph components of finite simple groups.* Math. USSR Sbornik **67** (1990), 235–247.

[7] Lucido M.S., *The diameter of the prime graph of a finite group.* J. Group Theory **2** (1999), 157–172.

[8] Williams J., *Prime graph components of finite groups.* J. Algebra **69** (1981), 487–513.

[9] Wilson R.A., *The maximal subgroups of the Baby Monster, I.* J. Algebra **211** (1999), 1–14.

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