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Nonsingular vector fields in $\mathcal{G}^1(M^3)$ satisfy Axiom A and no cycle: a new proof of Liao's theorem

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Abstract. In 1992, Hayashi [4] proved that diffeomorphisms in $\mathcal{F}^1(M)$ satisfy Axiom A. However, there exists a vector field which does not satisfy Axiom A in $\mathcal{G}^1(M^3)$ [3]. So, we consider the following problem: Does $X \in \mathcal{G}^1(M)$ without singularity satisfy Axiom A? In 1981, Liao [7] solved this problem for the case of $\dim M = 3$, making use of, the so called, 'obstruction set' technique. But we are not familiar with the 'obstruction set' very much. So we try to prove the same theorem by a different method based on Mañé's Ergodic Closing Lemma.

Key words: $\mathcal{G}^1(M)$, Axiom A, basic set.

1. Introduction

Let M^n be a n-dimensional compact smooth manifold without boundary and let $\mathcal{X}^1(M^n)$ be the set of C^1 vector fields on M^n with the C^1 topology. We denote by X_t ($t \in \mathbb{R}$) the C^1 flow on M^n generated by $X \in \mathcal{X}^1(M^n)$. $\Omega(X)$ is the nonwandering set of X. A set $\Lambda \subset M^n$ is said to be hyperbolic set of $X \in \mathcal{X}^1(M^n)$ if it is compact, X_t-invariant for all $t \in \mathbb{R}$ and there is a continuous splitting $TM^n \mid \Lambda = E^0 \oplus E^s \oplus E^u$ ($E^0(x) = \mathbb{R} \cdot X(x), x \in \Lambda$), invariant under $D_x X_t$ such that there exist $K > 0$, $0 < \lambda < 1$, satisfying

$$\|(D_x X_t) \mid E_x^s\| \leq K\lambda^t$$

and

$$\|(D_x X_{-t}) \mid E_x^u\| \leq K\lambda^t$$

for all $t \geq 0$, $x \in \Lambda$.

When $\Omega(X)$ is hyperbolic and the periodic points are dense in $\Omega(X)$, we say that X satisfies Axiom A. Let $\mathcal{G}^1(M^n)$ denote the set of $X \in \mathcal{X}^1(M^n)$ which has a neighborhood $\mathcal{U}$ such that if $Y \in \mathcal{U}$, then all periodic orbits and singularities of Y are hyperbolic. Hayashi proved that $f \in \mathcal{F}^1(M^n)$ satisfies Axiom A in [4] where $\mathcal{F}^1(M^n)$ is the diffeomorphism version of

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$\mathcal{G}^1(M^n)$. However, for $\mathcal{G}^1(M^n)$, there exists a vector field in $\mathcal{G}^1(S^3)$ which does not satisfy Axiom A ([3]). Thus, it is quite natural for us to consider the following problem: Does $X \in \mathcal{G}^1(M^n)$ without singularity satisfy Axiom A? In 1981, Liao [7] solved this problem affirmatively for $\dim M = 3$, making use of, the so called, 'obstruction set' technique. Here we will prove the same proposition by a different method based on Mañé's Ergodic Closing Lemma.

Main Theorem *If a vector field X is in $\mathcal{G}^1(M^3)$ and has no singularities, then X satisfies Axiom* A *and no cycle condition.*

Now, we attempt to give an outline of the proof without giving precise definitions. It is known that for $X \in \mathcal{G}^1(M^n)$, the number of attracting and repelling periodic orbits is finite (Pliss [15]). $L^-(X)$ denotes the set of α-limit points of X. $L^-(X)' = L^-(X) - \{$attracting and repelling periodic orbits$\}$ and $\overline{L^-(X)'}$ is the closure of $L^-(X)'$. So attracting and repelling periodic orbits are isolated, $\overline{L^-(X)'} \cap$ (attracting and repelling periodic orbits) $= \emptyset$. For any $p \in L^-(X)'$, there exist a sequence $\{t_n\}$, $t_n \geq 0$ ($t_n \to \infty$ as $n \to \infty$) and $x \in M^3$ such that

$$p = \lim_{n\to\infty} X_{-t_n}(x).$$

Among the points $\{X_{-t_n}(x)\}$, we can find a pair $(X_{-t_{n_1}}(x), X_{-t_{n_2}}(x))$ which are arbitrarily close to each other and can be closed by Pugh's Closing Lemma. Therefore we have a sequence $\{Y^n\}$ of C^1 vector fields such that $\{Y^n\}$ converges to X and each Y^n has a periodic orbit P_n which is obtained by closing two points in $\{X_{-t_n}(x)\}$.

In §2, we prove that for sufficiently large n, P_n is a saddle type hyperbolic periodic orbit. Let $\{a_n\}$ be a sequence such that $a_n \in P_n$ and $\lim_{n\to\infty} a_n = p$. Making use of this fact, we can show that $\overline{L^-(X)'}$ has a dominated splitting. In §3, using Ergodic Closing Lemma we prove that this dominated splitting over $\overline{L^-(X)'}$ is hyperbolic. Thus, we have $\overline{L^-(X)} = \overline{\mathrm{per}(X)}$ by theorem 3.1 in [12]. Hence, $\overline{L^-(x)}$ may be decomposed into a finite union of basic sets as

$$\overline{L^-(X)} = \Lambda_1 \cup \cdots \cup \Lambda_k.$$

In §4, we prove that $\overline{L^-(X)}$ has no cycles, then by theorem 4.1 in [12], we obtain

$$\overline{L^-(X)} = \overline{\mathrm{per}(X)} = \Omega(X).$$

Therefore X satisfies Axiom A and no cycle condition.

2. Dominated-Splitting

Let N^* be the normal bundle to X over M^3. Each $x \in M^3$, the fiber N_x is a subspace of $T_x M^3$ with codimension one that is perpendicular to $X(x)$. For any $u \in N_x$, let $P_t^X(u)$ be the orthogonal projection of $D_x X_t(u)$ onto $N_{X_t(x)}$. Then

$$P_t^X : N^* \to N^*$$

is a C^0 flow, which is linear on fibers. Let $\mathcal{U}_0$ be a neighborhood of X in $\mathcal{G}^1(M^3)$ such that any $Y \in \mathcal{U}_0$ has no singularities.

Theorem 2.1 *Let X belong to $\mathcal{G}^1(M^3)$ and have no singularities. Then there exist two numbers $0 < \lambda < 1$, $T > 0$ such that there is a continuous P_t^X-invariant splitting $G^s \oplus G^u$ (a dominated splitting) on $\overline{L^-(X)'}$ which satisfies the following conditions*:

(a) $G^s(x) = E^s(x)$, $G^u(x) = E^u(x)$ *if* $x \in \mathrm{per}(X)$ ($E^s \oplus E^u$ *is a hyperbolic splitting*),

(b) $\|P_t^X \mid G^s(x)\| \cdot \|P_{-t}^X \mid G^u(X_t(x))\| \le e^{-2\lambda t}$ *for any* $t \ge T$,

(c) *If τ is the period of $x \in \mathrm{per}(X)$, m is any positive integer, and $0 = t_0 < t_1 < \cdots < t_k = m\tau$ is any partition of the time interval $[0, m\tau]$ with $t_{i+1} - t_i \ge T$, then*

$$\frac{1}{m\tau} \sum_{i=0}^{k-1} \log \|P_{t_{i+1}-t_i}^X \mid E^s(X_{t_i}(x))\| \le -\lambda,$$

$$\frac{1}{m\tau} \sum_{i=0}^{k-1} \log \|P_{-(t_{i+1}-t_i)}^X \mid E^u(X_{t_{i+1}}(x))\| \le -\lambda.$$

Proof. For any $q \in L^-(X)'$, there exist a sequence t_n ($t_n \ge 0$, $t_n \to \infty$ as $n \to \infty$) and $x \in M^3$ such that

$$\lim_{n \to \infty} X_{-t_n}(x) = q.$$

We may choose a pair of points in the sequence $\{X_{-t_n}(x)\}$ and close the pair by Pugh's Closing Lemma. That is, there exist a vector field Y, C^1 close to X, and a periodic orbit P of Y through some point of $\{X_{-t_n}(x)\}$. From this, we may obtain sequences $\{Y^n\}$, $\{P_n\}$ such that

$$\lim_{n\to\infty} Y^n = X$$

and each Y^n has a periodic orbit P_n through some point of $\{X_{-t_n}(x)\}$. Let a_n be a point in P_n such that

$$\lim_{n\to\infty} a_n = q.$$

We obtain the next lemma.

Lemma 2.2 *In $\{P_n\}$ there is only a finite number of attracting and repelling periodic orbits.*

Proof. Suppose $\{P_n\}$ has an infinite number of attracting periodic orbits, for the other case follows similarly by applying the same method to X_{-t} instead of X_t. We may assume that all the P_n are attracting periodic orbits of Y^n. Since each P_n is compact, a subsequence of $\{P_n\}$, denoted also by $\{P_n\}$, converges to some X_t-invariant closed subset F in $\overline{L^-(X)'}$ with respect to Hausdorff metric. We take an individual measure μ_n corresponding to a point a_n, and we may assume that the sequence $\{\mu_n\}$ converges with weak-star topology to a probability measure μ on M. Each μ_n is invariant under Y_t^n. And μ is a measure supported on F. Let $\varphi_n : M^3 \to \mathbb{R}$ $(n \geq 1)$ be a sequence of real-valued continuous functions on M^3 such that

$$\lim_{n\to\infty} \varphi_n(a) = \varphi(a) \quad \text{uniformly on} \quad M^3.$$

Then we have

$$\lim_{n\to\infty} \int_{M^3} \varphi_n(a) d\mu_n = \int_{M^3} \varphi(a) d\mu. \tag{1}$$

Because

$$\begin{aligned}
&\left| \int_{M^3} \varphi_n(a) d\mu_n - \int_{M^3} \varphi(a) d\mu \right| \\
&\leq \left| \int_{M^3} \varphi_n(a) d\mu_n - \int_{M^3} \varphi(a) d\mu_n \right| + \left| \int_{M^3} \varphi(a) d\mu_n - \int_{M^3} \varphi(a) d\mu \right|
\end{aligned}$$

and $\varphi_n(a) \to \varphi(a)$ implies

$$\left|\int_{M^3} \varphi_n(a)d\mu_n - \int_{M^3} \varphi(a)d\mu_n\right| \leq \int_{M^3} |\varphi_n(a) - \varphi(a)|d\mu_n \to 0$$

while $\mu_n \to \mu$ implies

$$\left|\int_{M^3} \varphi(a)d\mu_n - \int_{M^3} \varphi(a)d\mu\right| \to 0.$$

Moreover, since each μ_n is Y_t^n-invariant, μ is X_t-invariant. In fact, by (1) above, for any $t \in \mathbb{R}$,

$$\begin{aligned}
&\int_{M^3} \varphi(X_t(a))d\mu \\
&= \lim_{n\to\infty} \int_{M^3} \varphi_n(X_t(a))d\mu_n - \lim_{n\to\infty} \int_{M^3} \varphi_n(Y_t^n(a))d\mu_n \\
&\quad + \lim_{n\to\infty} \int_{M^3} \varphi_n(Y_t^n(a))d\mu_n \\
&= \lim_{n\to\infty} \left(\int_{M^3} \varphi_n(X_t(a))d\mu_n - \int_{M^3} \varphi_n(Y_t^n(a))d\mu_n\right) \\
&\quad + \lim_{n\to\infty} \int_{M^3} \varphi_n(Y_t^n(a))d\mu_n \\
&= \lim_{n\to\infty} \int_{M^3} \varphi_n(Y_t^n(a))d\mu_n = \lim_{n\to\infty} \int_{M^3} \varphi_n(a)d\mu_n = \int_{M^3} \varphi(a)d\mu.
\end{aligned}$$

Remark

$$\lim_{n\to\infty} \left(\int_{M^3} \varphi_n(X_t(a))d\mu_n - \int_{M^3} \varphi_n(Y_t^n(a))d\mu_n\right) = 0.$$

In fact, since $\lim_{n\to\infty} Y^n = X$, $d(X_t(a), Y_t^n(a))$ is very small for sufficiently large n (for any $a \in M^3$ and a fixed t). On the other hand, as φ_n is uniformly continuous on M^3

$$|\varphi_n(X_t(a)) - \varphi_n(Y_t^n(a))| < \varepsilon$$

for any $a \in M^3$ and sufficiently large n.

As μ_n is an individual measure corresponding to a_n

$$\int_{M^3} \varphi_n(a)d\mu_n = \lim_{t\to\infty} \frac{1}{t} \int_0^t \varphi_n(Y_s^n(a_n))ds. \tag{2}$$

Here we have the following lemma from Liao [6].

Lemma 2.3 (Theorem 2.1 in Liao [6]). *Let X be in $\mathcal{G}^1(M^n)$. Then there exist a C^1 neighborhood $\tilde{\mathcal{U}}$ of X in $\mathcal{G}^1(M^n)$ and two numbers $0 < \lambda = \lambda(\tilde{\mathcal{U}}) < 1$ and $T = T(\tilde{\mathcal{U}}) > 0$ such that for any $Y \in \tilde{\mathcal{U}}$ and any periodic point p of Y, the following two estimates hold:*

a) $||P_t^Y \,|\, E^s(p)|| \cdot ||P_{-t}^Y \,|\, E^u(Y_t(p))|| \leq e^{-2\lambda t}$ *for any* $t \geq T$,

b) *If τ is the period of p, m is any positive integer, and if $0 = t_0 < t_1 < \cdots < t_k = m\tau$ is any partition of the interval $[0, m\tau]$ with $t_{i+1} - t_i \geq T$, then*

$$\frac{1}{m\tau} \sum_{i=0}^{k-1} \log ||P_{t_{i+1}-t_i}^Y \,|\, E^s(Y_{t_i}(p))|| < -\lambda,$$
$$\frac{1}{m\tau} \sum_{i=0}^{k-1} \log ||P_{-(t_{i+1}-t_i)}^Y \,|\, E^u(Y_{t_{i+1}}(p))|| < -\lambda.$$

Assuming this lemma, we have a neighborhood $\mathcal{U}$ of X in $\mathcal{U}_0$ and $\lambda > 0$, $T > 0$ satisfying above theorem. Since Y^n converges to X, we may assume that Y^n is in $\mathcal{U}$ for all n. For each n, let T_n be the period of P_n of Y^n, the periodic orbit with which we have been dealing. Since $Y^n \to X$ and we may assume that q is not a periodic point of X, we have

$$\lim_{n\to\infty} T_n = +\infty.$$

Now let

$$\xi_T^n(a) = \frac{1}{T} \log ||P_T^{Y^n}(a)||,$$
$$\xi_T(a) = \frac{1}{T} \log ||P_T^X(a)||.$$

Then $\lim_{n\to\infty} \xi_T^n(a) = \xi_T(a)$ uniformly on M^3 for the number $T = T(\mathcal{U}) > 0$. From Lemma 2.3 b), for sufficiently large n,

$$\frac{T}{T_n} \left\{ \sum_{k=1}^{m_n} \xi_T^n(Y_{(k-1)T}^n(a_n)) + \frac{1}{T} \log ||P_{T_n - m_n T}^{Y^n}(Y_{m_n T}^n(a_n))|| \right\} \leq -\lambda$$

where m_n is the greatest integer with $T_n - m_n T \geq T$ which certainly exists for sufficiently large n. For sufficiently large n, we have

$$\frac{T}{m_n T}\sum_{k=1}^{m_n}\xi_T^n(Y_{(k-1)T}^n(x)) < -\frac{\lambda}{2} \quad \text{for all} \quad x \in P_n.$$

Thus

$$\frac{1}{lm_n}\left(\sum_{k=1}^{lm_n}\xi_T^n(Y_{(k-1)T}^n(a_n))\right) < -\frac{\lambda}{2} \quad l = 1, 2 \dots .$$

Therefore

$$\begin{aligned}
&\frac{1}{lm_n T}\int_0^{lm_n T} \xi_T^n(Y_s^n(a_n))ds \\
&= \frac{1}{lm_n T}\sum_{k=0}^{lm_n-1}\int_{kT}^{(k+1)T} \xi_T^n(Y_s^n(a_n))ds \\
&= \frac{1}{T}\int_0^T \frac{1}{lm_n}\sum_{k=0}^{lm_n-1} \xi_T^n(Y_{kT}^n(Y_s^n(a_n)))ds < -\frac{\lambda}{2} \quad l = 1, 2, \dots .
\end{aligned}$$

Thus from (2)

$$\int_{P_n}\xi_T^n(a)d\mu_n < -\frac{\lambda}{2}.$$

Thus, from (1) we have

$$\int_F \xi_T(a)d\mu = \lim_{n\to\infty}\int_{P_n}\xi_T^n(a)d\mu_n \leq -\frac{\lambda}{2} < 0.$$

Now we need a lemma from Liao [6] to proceed.

Lemma 2.4 (Liao [6], Lemma 3.2). *Let F be a closed subset of M^n, invariant under X_t. Assume that for a certain $\tilde{T} \in (0, \infty)$, there is a probability measure μ on F, invariant under X_t such that:*

$$\int_F \xi_{\tilde{T}}(a)d\mu < 0 \quad \textit{or} \quad \int_F \xi_{-\tilde{T}}(a)d\mu > 0 \tag{$*$}$$

Then, F contains a periodic orbit of X attracting or repelling corresponding to the first inequality or the second of $(*)$.

Using this lemma, we obtain attracting periodic orbit in F but this is a contradiction because attracting periodic orbit is isolated from $\overline{L^-(X)'}$.

Thus we have completed the proof of Lemma 2.2. □

For any $q \in L^-(X)'$, we can take sequences $\{Y^n\}$, $\{P_n\}$, $\{a_n\}$, such that

$$\lim_{n\to\infty} Y^n = X, \quad \lim_{n\to\infty} a_n = q \quad (a_n \in P_n)$$

and each P_n is a hyperbolic saddle. By Lemma 2.3 we may assume that for any P_n

$$||P_t^{Y^n}(a_n) \mid E^s(a_n)|| \cdot ||P_{-t}^{Y^n}(Y_t^n(a_n)) \mid E^u(Y_t^n(a_n))|| \le e^{-2\lambda t}, \ \ t \ge T.$$

And for any $t \in \mathbb{R}$, the sequences $E^s(Y_t^n(a_n)), E^u(Y_t^n(a_n))$ converge to subspaces of $N_{X_t(q)}$ which we define as $G^s(X_t(q))$ and $G^u(X_t(q))$ respectively. Hence we attach to every $q \in L^-(X)'$ two subspaces $G^s(X_t(q))$ and $G^u(X_t(q))$ with the following properties:

$$\dim G_q^s + \dim G_q^u = 2, \tag{3}$$

$$(P_t^X(q))G_x^\sigma = G_{X_t(q)}^\sigma \quad \sigma = s, u \ \text{ for any } \ t \in \mathbb{R}, \tag{4}$$

$$||P_t^X \mid G_x^s|| \cdot ||P_{-t}^X \mid G_{X_t(x)}^u|| \le e^{-2\lambda t}, \quad t \ge T \tag{5}$$

and if $x \in \mathrm{per}(X)$, then $G^s(x) = E^s(x)$, $G^u(x) = E^u(x)$. Moreover, as in p. 526 of Mañé [8],

$$G^s(x) \oplus G^u(x) = N_x \quad \text{for any } \ x \in L^-(X)'. \tag{6}$$

By Proposition I.3 of Mañé [10], we can extend this splitting to $\overline{L^-(X)'}$. Properties (3)–(6) imply that the subspaces $G^s(x)$ and $G^u(x)$ depend continuously on $x \in \overline{L^-(X)'}$. □

3. Hyperbolicity of $\overline{L^-(X)'}$

In §2 we obtained dominated splitting $N^* \mid \overline{L^-(X)'} = G^s \oplus G^u$. Now we show that this splitting is hyperbolic. By compactness of $\overline{L^-(X)'}$, it is easy to see that if

$$\liminf_{n\to\infty} ||P_n^X(a) \mid G^s(a)|| = 0 \tag{7}$$

and

$$\liminf_{n\to\infty} ||P_{-n}^X(a) \mid G^u(a)|| = 0$$

hold for all $a \in \overline{L^-(X)'}$, then the splitting $N^* \mid \overline{L^-(X)'} = G^s \oplus G^u$ is hyperbolic. We shall prove only the first one, because the second one follows if we apply the same method to X_{-t} instead of X_t. If (7) does not hold for all $a \in \overline{L^-(X)'}$, we can find $x \in \overline{L^-(X)'}$ and a sequence $j_n \to +\infty$ such that

$$\lim_{n\to\infty} \frac{1}{j_n} \log ||(P^X_{mj_n}(x)) \mid G^s(x)|| \geq 0$$

where m is the smallest integer satisfying $m \geq T$ (T is given in Theorem 2.1). Without loss of generality we may suppose that the sequence $\{j_n\}$ is such that there exists an X_m-invariant probability measure μ on $\overline{L^-(X)'}$ such that

$$\int_{\overline{L^-(X)'}} \varphi d\mu = \lim_{n\to\infty} \frac{1}{j_n} \sum_{i=0}^{j_n-1} \varphi(X_{mi}(x))$$

for every continuous $\varphi : M^3 \to \mathbb{R}$. Setting

$$\varphi(a) = \log ||(P^X_m(a)) \mid G^s(a)||$$

we have

$$\begin{aligned}\int_{\overline{L^-(X)'}} \varphi d\mu &= \lim_{n\to\infty} \frac{1}{j_n} \sum_{i=0}^{j_n-1} \log ||(P^X_m(X_{mi}(x)) \mid G^s(X_{mi}(x))|| \\ &\geq \lim_{n\to\infty} \frac{1}{j_n} \log ||(P^X_{mj_n}(x)) \mid G^s(x)|| \geq 0. \end{aligned} \tag{8}$$

On the other hand, by Birkhoff's theorem

$$\begin{aligned}&\int_{\overline{L^-(X)'}} \varphi d\mu \\ &= \int_{\overline{L^-(X)'}} \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log ||(P^X_m(X_{mi}(a)) \mid G^s(X_{mi}(a))|| d\mu. \end{aligned} \tag{9}$$

Now we need the following lemma.

Lemma 3.1 (Ergodic Closing Lemma, Lemma VII.6 in Hayashi [5]).

$$\mu(\Sigma(X) \cup \mathrm{Sing}(X)) = 1$$

for every X_1-invariant probability measure μ on the borel sets of M^n, where $\mathrm{Sing}(X)$ *denotes the set of singularities of X.*

We define $\Sigma(X)$ as the set of points $x \in M^3$ such that for every neighborhood $\mathcal{U}(X)$ and every $\varepsilon > 0$, there exist $Y \in \mathcal{U}(X)$ $y \in \mathrm{per}(Y)$, $T_0 > 0$ and $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$ such that $Y_{T_0}(y) = y$, $X = Y$ on $M^3 - B_\varepsilon(X, x)$ $(B_\varepsilon(X,x) = \{y \in M^3 : d(X_t(x), y) \leq \varepsilon$ for some $t \in \mathbb{R}\})$, $\{X_t(x) : t_0 \leq t \leq t_1\} \subset \{Y_t(y) : t \geq 0\}$, $(t_1 - t_0)/T_0 > 1 - \varepsilon$ and $d(Y_t(y), X_t(x)) \leq \varepsilon$ for all $0 \leq t \leq T_0$. Note that $\Sigma(X)$ is X_1-invariant. From the above Lemma 3.1 and (8), (9), there exists $p \in \Sigma(X) \cap \overline{L^-(X)'}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \| P_m^X(X_{mi}(p)) \mid G^s(X_{mi}(p)) \| \geq 0. \tag{10}$$

Now take $-\lambda < -\lambda_0 < 0$, $n_0 >$ such that

$$\frac{1}{n} \sum_{i=0}^{n-1} \log \| P_m^X(X_{mi}(p)) \mid G^s(X_{mi}(p)) \| \geq -\frac{\lambda_0}{2} m \tag{11}$$

where $n \geq n_0$. Observe that the point p is not periodic because if it were, it should be hyperbolic with $E^s(p) = G^s(p)$ and then (11) would contradict the first inequality of (b) of Lemma 2.3. Since $p \in \Sigma(X) \cap \overline{L^-(X)'}$, we can find Y arbitrarily near X and $\bar{p} \in \mathrm{per}(Y)$ such that $X = Y$ on $M^3 - B_\varepsilon(X, p)$ with sufficiently small $\varepsilon > 0$ and such that the distance between $X_t(p)$ and $Y_t(\bar{p})$ is small for all $0 \leq t \leq T_0$, where T_0 denotes the minimum Y-period of $\bar{p}$. Since p is not X_t-periodic, the period T_0 goes to ∞ when Y approaches X. Using the same technique as that in p. 349 of Wen [17], we may take $\bar{Y}$ arbitrarily close to Y and a periodic point $\bar{p}$ of $\bar{Y}$ with period T_0. That is, N^* restricted to the $\bar{Y}$-orbit $\bar{p}$ has an $P_t^{\bar{Y}}$-invariant splitting $\bar{G}^s \oplus \bar{G}^u$ such that

$$\begin{aligned} &\| P_{-m}^{\bar{Y}}(\bar{Y}_{m(j+1)}(\bar{p})) \mid \bar{G}^u(\bar{Y}_{m(j+1)}(\bar{p})) \| \\ &\qquad = \| P_{-m}^X(X_{m(j+1)}(p)) \mid G^u(X_{m(j+1)}(p)) \|, \\ &\| P_m^{\bar{Y}}(\bar{Y}_{mj}(\bar{p})) \mid \bar{G}^s(\bar{Y}_{mj}(\bar{p})) \| \\ &\qquad = \| P_m^X(X_{mj}(p)) \mid G^s(X_{mj}(p)) \| \end{aligned}$$

for all $0 \leq j \leq [T_0/m] - 2$,

$$\begin{aligned} &\| P_{-(T_0 - m[T_0/m] + m)}^{\bar{Y}}(\bar{Y}_{T_0}(\bar{p})) \mid \bar{G}^u(\bar{Y}_{T_0}(\bar{p})) \| \\ &\qquad = \| P_{-(T_0 - m[T_0/m] + m)}^X(X_{T_0}(p)) \mid G^u(X_{T_0}(p)) \|, \end{aligned}$$

and

$$\begin{aligned}&\|P^{\bar{Y}}_{T_0-m[T_0/m]+m}(\bar{Y}_{m[T_0/m]-m}(\bar{p}))\mid \bar{G}^s(\bar{Y}_{m[T_0/m]-m}(\bar{p}))\|\\&\quad=\|P^{X}_{T_0-m[T_0/m]+m}(X_{m[T_0/m]-m}(p))\mid G^s(X_{m[T_0/m]-m}(p))\|.\end{aligned}$$

Then we have

$$\begin{aligned}&\|P^{\bar{Y}}_{-T_0}(\bar{Y}_{T_0}(\bar{p}))\mid \bar{G}^u(\bar{Y}_{T_0}(\bar{p}))\| \quad (k=[T_0/m]-1)\\&\le\prod_{i=0}^{k-1}\Big(\|P^{\bar{Y}}_{-m}(\bar{Y}_{mk-mi}(\bar{p}))\mid \bar{G}^u(\bar{Y}_{mk-mi}(\bar{p}))\|\\&\qquad\qquad\times\|P^{\bar{Y}}_{m}(\bar{Y}_{m(k-1)-mi}(\bar{p}))\mid \bar{G}^s(\bar{Y}_{m(k-1)-mi}(\bar{p}))\|\Big)\\&\quad\times\left(\prod_{i=0}^{k-1}\|P^{X}_{m}(X_{m(k-1)-mi}(p))\mid G^s(X_{m(k-1)-mi}(p))\|\right)^{-1}\\&\qquad\qquad\times\|P^{\bar{Y}}_{-(T_0-mk)}(\bar{Y}_{T_0}(\bar{p}))\mid \bar{G}^u(\bar{Y}_{T_0}(\bar{p}))\|\\&\le e^{-2\lambda mk}\times e^{\frac{\lambda_0}{2}mk}\times\|P^{\bar{Y}}_{-(T_0-mk)}(\bar{Y}_{T_0}(\bar{p}))\mid \bar{G}^u(\bar{Y}_{T_0}(\bar{p}))\|.\end{aligned}$$

Note that we used (11) to induced the last inequality above. If T_0 is very large, then k can be also large so that:

$$e^{-2\lambda mk}\times e^{\frac{\lambda_0}{2}mk}\times\|P^{\bar{Y}}_{-(T_0-mk)}(\bar{Y}_{T_0}(\bar{p}))\mid \bar{G}^u(\bar{Y}_{T_0}(\bar{p}))\|<1.$$

Thus, $\bar{G}^u\subset E^u$ (where $E^s\oplus E^u$ is hyperbolic splitting of orbit $\bar{p}$ for $\bar{Y}$). Therefore $\dim E^u\ge\dim\bar{G}^u=1$. That is, $\dim E^s\le 1$. If $\dim E^s=1$, then $\bar{G}^u=E^u$, $\bar{G}^s=E^s$. By Lemma 2.3 b) and (11) above,we have,

$$\begin{aligned}e^{-\frac{\lambda}{2}mk}&>\prod_{i=0}^{k-1}\|P^{\bar{Y}}_{m}(\bar{Y}_{mi}(\bar{p}))\mid E^s(\bar{Y}_{mi}(\bar{p}))\|\\&=\prod_{i=0}^{k-1}\|P^{X}_{m}(X_{mi}(p))\mid G^s(X_{mi}(p))\| \quad \text{(from (11))}\\&>e^{-\frac{\lambda_0}{2}mk}.\end{aligned}$$

This is a contradiction. Thus, $\dim E^s=0$ whenever we create $\bar{Y}$ C^1-close to Y. Therefore, we obtain a sequence $\{\bar{Y}^n\}$, $\bar{Y}^n\to X$ such that $\bar{Y}^n$ has a periodic orbit with index 0 and which converges to some closed set in $\overline{L^-(X)'}$. Then as in the argument of Lemma 2.2, $\overline{L^-(X)'}$ has a periodic orbit with index 0, contradicting the definition of $\overline{L^-(X)'}$. Thus, we can conclude that G^s is contracting and that $\overline{L^-(X)'}$ is a hyperbolic set for X.

4. Proof of Main Theorem

In §3 we proved that if X is in $\mathcal{G}^1(M^3)$ and has no singularities, then $\overline{L^-(X)}$ is hyperbolic. In this section we show that X satisfies Axiom A.

Theorem 4.1 (Theorem 3.1 in Newhouse [12]). *If $\overline{L^-(X)}$ is hyperbolic, then $\overline{L^-(X)} = \overline{\mathrm{per}(X)}$.*

Proof. The same method as in Theorem 3.1 of [12]. □

Therefore, we can conclude that $\overline{L^-(X)} = \overline{\mathrm{per}(X)}$ and $\overline{L^-(X)}$ is a hyperbolic set for X. Now, $\overline{L^-(X)}$ is decomposable to finite union of basic sets.

$$\overline{L^-(X)} = \Lambda_1 \cup \Lambda_2 \cdots \cup \Lambda_k.$$

A basic set means an isolated, transitive hyperbolic set. If $\overline{L^-(X)}$ has no cycles, we obtain the next theorem from [12].

Theorem 4.2 (Theorem 4.1 in [12]). *If $\overline{L^-(X)}$ is hyperbolic and X has no cycles, then $\overline{L^-(X)} = \overline{\mathrm{per}(X)} = \Omega(X)$.*

Proof. In this theorem if $\overline{L^-(X)}$ is hyperbolic and X has no c-cycle, then the conditions are the same as in Theorem 4.1 in [12]. But by Proposition 3.10 in [12], we have that no cycle implies no c-cycle. Therefore, we may apply the Theorem 4.1 in [12]. And as the proof of Theorem 4.1 in [12] depends on a filtration theorem to basic sets which have no c-cycle, we can apply the proof of Theorem 4.1 in [12] to flows. □

By this theorem, it suffices to prove that $\overline{L^-(X)}$ has no cycles, to conclude that X satisfies Axiom A and no cycle condition.

Theorem 4.3 *If X is in $\mathcal{G}^1(M^3)$ and has no singularities, then $\overline{L^-(X)} = \Lambda_1 \cup \cdots \cup \Lambda_k$ has no cycles.*

Proof. Suppose that there is a cycle $\Lambda_{i_1}, \ldots, \Lambda_{i_s}$ of basic sets with $\Lambda_{i_j} \neq \Lambda_{i_k}$ $(0 \leq j < k \leq s)$. Let b_j $(j = 1, \ldots, s)$ be points of M^3 such that

$$b_j \in W^u(\Lambda_{i_j}) \cap W^s(\Lambda_{i_{j+1}}), \quad j = 1, \ldots, s-1$$

and

$$b_s \in W^u(\Lambda_{i_s}) \cap W^s(\Lambda_{i_1}).$$

Then at least one of them is not transversal intersection. We obtain next lemma which is the flow version of Lemma II.9 in [8]. For the next lemma we shall need the concept of angle between subspaces of the Euclidean space. If E_1, E_2 are subspaces of $\mathbb{R}^2$ such that $E_1 \oplus E_2 = \mathbb{R}^2$ we define the *angle* $\alpha(E_1, E_2)$ as $\alpha(E_1, E_2) = ||L||^{-1}$, where $L : E_1^{\perp} \to E_1$ is the linear map such that $E_2 = \{v + Lv \mid v \in E_1^{\perp}\}$; in particular, $\alpha(E_1, E_1^{\perp}) = +\infty$.

Lemma 4.4 *If X is in $\mathcal{G}^1(M^n)$ then there exist $\alpha' > 0$, neighborhood $\mathcal{U}'$ of X and $T' > 0$ such that if $Y \in \mathcal{U}'$ and P is a periodic orbit of Y with period $T_Y(> T')$ then $\alpha(E^s(p), E^u(p)) > \alpha'$ $(p \in P,\ E^s(p), E^u(p) \subset N^*)$.*

Now we assume that only b_1 is not transversal intersection. Then we perturb X at b_1 and obtain Y C^1-close to X such that b_1 is transversal intersection and $v_1, v_2 \in N^*(b_1)$ are tangent to $W^s(\Lambda_{i_2}), W^u(\Lambda_{i_1})$ respectively such that $\alpha(V_1, V_2) < \alpha'$, where V_1 and V_2 are subspaces in $N^*(b_1)$ spanned by v_1, v_2 respectively. Then we can find a periodic point p arbitrarily close to b_1 whose period T_p is sufficiently large. Since p is very close to b_1, $W^s(p)$ and $W^u(p)$ are C^1 close to $W^s(\Lambda_{i_2})$, $W^u(\Lambda_{i_1})$ respectively at b_1. So, $\alpha(E^s(p), E^u(p)) < \alpha'$. But this contradicts Lemma 4.4. We have completed the proof of Theorem 4.3. □

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