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Fully discrete approximation by Galerkin Runge-Kutta methods for quasilinear parabolic systems

(Dedicated to Professor Norio Shimakura on the occasion of his sixtieth birthday)

Etsushi NAKAGUCHI and Atsushi YAGI

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Abstract. We study fully discrete approximation of quasilinear parabolic systems. Presenting a full discretization scheme based on the Galerkin and the Runge-Kutta methods, we establish the stability and the error estimate of the scheme by means of the semigroup method. First our results are stated for a chemotaxis-growth system arising in biology, then those are generalized to quasilinear abstract parabolic evolution equations.

Key words: quasilinear parabolic systems, implicit Runge-Kutta methods, Galerkin finite element method.

1. Introduction

This paper is concerned with a numerical analysis for a quasilinear diffusion system

$$
\text{(CG)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = a\triangle u - \nabla \cdot \{u\nabla B(\rho)\} + c(u) & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial \rho}{\partial t} = d\triangle \rho + fu - g\rho & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \rho}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ \rho(x,0) = \rho_0(x) & \text{in } \Omega. \end{cases}
$$

This system was presented by Mimura and Tsujikawa [14] as a mathematical model describing aggregating patterns induced by the effects of chemotaxis and growth. $u(x,t)$ and $\rho(x,t)$ denote the population density of biological individuals and the concentration of chemical substance, respectively, at a

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position $x \in \Omega \subset \mathbb{R}^2$ and a time $t \in [0, \infty)$. $n(x)$ denotes the outer normal vector at a boundary point $x \in \partial\Omega$. $a > 0$ and $d > 0$ are diffusion rates of u and ρ, respectively. $f > 0$ and $g > 0$ are production and degradation rates of ρ, respectively. $B(\rho)$ is a sensitivity function of u with respect to ρ. Typical examples of $B(\rho)$ include ρ, ρ^2, $\log \rho$, $\frac{\rho}{1+\rho}$ and so on. $c(u)$ is a growth term of u such that $c(0) = 0$. In the case of no growth, that is $c(u) \equiv 0$, (CG) reduces to the Keller-Segel system [11] which describes the initiation of aggregating patterns of slime mold.

We shall consider a full discretization scheme based on the Galerkin and the Runge-Kutta methods, employing the semigroups of linear operators. Such a method called the semigroup method is known to be a powerful technique ever in numerical analysis. The method was founded by several mathematicians, including Fujita and Mizutani [6], Baker, Bramble and Thomée [1], Brenner and Thomée [2] and Ushijima [28] for autonomous linear equations. Then that was developed by Suzuki [23], Sammon [21] and others for non autonomous linear equations. Afterward, Keeling [10] studied semilinear equations. As a matter of fact, (CG) can be handled as a semilinear abstract equation of the form $\frac{dU}{dt} + AU = F(U)$ if we take the underlying space as $H^1(\Omega)' \times H^\varepsilon(\Omega)$, $\varepsilon > 0$. But in this setting the approximating function space must be contained in $H^1(\Omega) \times H^{2+\varepsilon}(\Omega)$ and the Galerkin method of higher order must be needed. This is the reason why we prefer a quasilinear abstract setting $\frac{dU}{dt} + A(U)U = F(U)$ in the product Hilbert space $L^2(\Omega) \times L^2(\Omega)$.

The variational method is also a very useful technique for the approximation of nonlinear parabolic equations and systems, see Thomée [26], Lubich and Ostermann [13] and so on. Some results may be applicable to (CG), but it seems very difficult to take the underlying space as $L^2(\Omega) \times L^2(\Omega)$.

Our precise assumptions are the followings:

(Ω) $\Omega \subset \mathbb{R}^2$ is a bounded and convex polygonal domain;

(B) $B(\rho)$ is a real-valued smooth function of $\rho \in (0, \infty)$;

(C) $c(u)$ is a real-valued smooth function of $u \in [0, \infty)$ such that $c(0) = 0$;

(IF) $u_0(x)$ and $\rho_0(x)$ are initial functions which satisfy

$$\begin{cases} u_0,\ \rho_0 \in H^2(\Omega) \ \text{ with } \ \dfrac{\partial u}{\partial n} = \dfrac{\partial \rho}{\partial n} = 0 \ \text{ on } \ \partial\Omega, \\ u_0(x) \geq 0,\ \rho_0(x) \geq \delta_0 \quad \text{on } \ \overline{\Omega} \end{cases} \tag{1.1}$$

with some constant $\delta_0 > 0$.

Under these conditions we shall present a full discretization scheme for (CG) and shall establish the stability in $L^2(\Omega)$ and the error estimate in $H^{1+\varepsilon}(\Omega)$, $\varepsilon > 0$. Those results have already been reported in the proceedings [18]. Since $H^{1+\varepsilon}(\Omega) \subset \mathcal{C}(\overline{\Omega})$, such an error estimate is pointwise. To prove the main results, we shall prepare some new results on the finite element method in the fractional Sobolev space $H^{1+\varepsilon}(\Omega)$. We shall also use the results in the preceding paper [16] in which the time discretization for quasilinear abstract evolution equations was studied. Indeed, those results are applicable for each approximate equation in a finite dimensional subspace.

Not only to (CG) our technique is of course applicable also to more general parabolic systems. So it may be convenient to fix our results in a general form by considering a quasilinear abstract evolution equation of parabolic type. Such generalization was partly tried in [17] also in some simple cases.

The organization of the paper is as follows. Section 2 is devoted to preparing all the necessary results in functional analysis and numerical analysis. Some of them, Propositions 2.3 and 2.4 and so on, are newly verified. In Section 3, the main results are proved but assuming various propositions without their proofs. Section 4 is devoted to verifying all the propositions used. By these the proofs of the main results are actually complete. In Section 5, we consider a quasilinear abstract evolution equation in order to fix our method in a general form.

Notations $\mathbb{R}$ (resp. $\mathbb{C}$) denotes the set of all real (resp. complex) numbers, and $\mathbb{R}^+$ denotes the set of all positive real numbers. For $0 < \theta \leq \pi$, let $S_\theta = \{z \in \mathbb{C};\ |z| > 0,\ |\arg z| < \theta\}$ be a sectorial domain.

Let Ω be a domain in $\mathbb{R}^2$. $L^p(\Omega)$, $1 \leq p \leq \infty$, denote the usual L^p spaces of complex-valued functions in Ω. $H^s(\Omega)$, $s \geq 0$, denote the usual Sobolev spaces of complex-valued functions in Ω. $\mathcal{C}(\overline{\Omega})$ denotes the space of all complex-valued continuous functions on $\overline{\Omega}$. By $\mathbb{L}^p(\Omega)$ (resp. $\mathbb{H}^s(\Omega)$) we shall denote the product space of two $L^p(\Omega)$ (resp. $H^s(\Omega)$) spaces of the form $[\,\dot{:}\,]$.

Let X be a Banach space or a Hilbert space, its norm is denoted by $\|\cdot\|_X$. $\langle\cdot,\cdot\rangle_X$ is the inner product of X when X is a Hilbert space. $\mathcal{L}(X,Y)$ denotes the space of all bounded linear operators from X into another Banach space Y which is equipped with the uniform operator norm; $\mathcal{L}(X,X)$ will be abbreviated to $\mathcal{L}(X)$.

Let I be an interval in $\mathbb{R}$. $L^p(I;X)$, $1 \le p \le \infty$, denotes the L^p space of measurable functions in I with values in a Banach space X. $\mathcal{C}(I;X)$, $\mathcal{C}^\sigma(I;X)$, $0 < \sigma < 1$, and $\mathcal{C}^m(I;X)$, $m = 1,2,3,\dots$, denote respectively the space of continuous functions, Hölder continuous functions with exponent σ, and m-times continuously differentiable functions in I with values in X.

Throughout this paper we denote by C the generic constant determined in each occurrence by the initial constants appearing in the assumptions. In a case when C depends on some parameter, say ζ, it will be denoted by C_ζ.

2. Preliminaries

In this section we shall list some known results and prepare some new results on functional analysis and numerical analysis which will be used in the subsequent sections.

2.1. Function spaces and functional analysis

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain. It is known that any bounded convex domain has a Lipschitz boundary $\partial\Omega$ (see [8, Corollary 1.2.2.3]).

Sobolev norm *For $0 \le s < \infty$, let $H^s(\Omega)$ be the Sobolev space. For $0 < \varepsilon < 1$, the norm (or equivalent norm) of $H^{1+\varepsilon}(\Omega)$ is given by*

$$\|v\|_{H^{1+\varepsilon}}^2 = \|v\|_{L^2}^2 + \sum_{j=1}^{2} \iint_{\Omega\times\Omega} \frac{|\partial_j v(x) - \partial_j v(y)|^2}{|x-y|^{2+2\varepsilon}} dxdy. \tag{2.1}$$

Interpolation space *Let $0 \le s_0 < s < s_1 \le 2$, $H^s(\Omega)$ is the interpolation space $[H^{s_0}(\Omega), H^{s_1}(\Omega)]_\theta$ between $H^{s_0}(\Omega)$ and $H^{s_1}(\Omega)$, where $s = (1-\theta)s_0 + \theta s_1$, with*

$$\|\cdot\|_{H^s} \le C\|\cdot\|_{H^{s_0}}^{1-\theta}\|\cdot\|_{H^{s_1}}^{\theta}. \tag{2.2}$$

See [27, Theorem 4.3.1/2].

Embedding theorems *When* $0 \le s < 1$, $H^s(\Omega) \subset L^p(\Omega)$, *where* $\frac{1}{p} = \frac{1-s}{2}$, *with*

$$\|\cdot\|_{L^p} \le C_s \|\cdot\|_{H^s}. \tag{2.3}$$

When $s = 1$, $H^1(\Omega) \subset L^q(\Omega)$ *for any* $1 \le q < \infty$ *with*

$$\|\cdot\|_{L^q} \le C_{q,p} \|\cdot\|_{H^1}^{1-\frac{p}{q}} \|\cdot\|_{L^p}^{\frac{p}{q}}, \tag{2.4}$$

where $1 \le p \le q < \infty$. *When* $s > 1$, $H^s(\Omega) \subset \mathcal{C}(\overline{\Omega})$ *with*

$$\|\cdot\|_{\mathcal{C}} \le C_s \|\cdot\|_{H^s}. \tag{2.5}$$

The embedding inequalities (2.3) and (2.5) are seen in [27, Theorem 4.6.1]. The inequality (2.4) holds when $\Omega = \mathbb{R}^2$ is the free space, cf. [24, Theorem 3.3]. Then in the present case, (2.4) is verified with the aid of the extension theorem of the functions in Ω to $\mathbb{R}^2$ due to Stein [22, Chap. VI, Theorem 5].

Domain of the fractional power *Consider a sesquilinear form*

$$a^0(u, v) = a_0 \int_\Omega \nabla u \cdot \nabla \bar{v} dx + c_0 \int_\Omega u \bar{v} dx, \quad u, v \in H^1(\Omega),$$

with some fixed a_0, $c_0 > 0$. *Let* A_0 *be the positive definite self-adjoint operator in* $L^2(\Omega)$ *associated with this form. According to Grisvard* [8, Theorem 3.2.1.3], A_0 *is the Laplace operator* $-a_0\triangle + c_0$ *with the domain* $\mathcal{D}(A_0) = H^2_N(\Omega) = \{u \in H^2(\Omega);\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$. *Moreover, for* $0 < \theta < \frac{3}{4}$,

$$\mathcal{D}(A_0^\theta) = H^{2\theta}(\Omega) \quad \textit{with norm equivalence.} \tag{2.6}$$

This coincidence is well-known when Ω is sufficiently smooth, see [27, Theorem 4.3.3]. But it seems to be no longer possible to apply the similar proof to the present case, especially when $\frac{1}{2} < \theta < \frac{3}{4}$. So let us describe the proof here.

Since $\mathcal{D}(A_0^i) \subset H^{2i}(\Omega)$ for $i = 0, 1$, we observe by interpolation that $\mathcal{D}(A_0^\theta) \subset H^{2\theta}(\Omega)$ for every $0 \le \theta \le 1$. On the other hand, let $\frac{1}{2} < \theta < \frac{3}{4}$ and let $u \in H^{2\theta}(\Omega)$. For $v \in \mathcal{D}(A_0)$,

$$\begin{aligned}\langle u, A_0^\theta v\rangle_{L^2} &= \frac{\sin\theta\pi}{\theta\pi} \int_0^\infty \lambda^\theta \langle u, A_0(\lambda + A_0)^{-2} v\rangle_{L^2}\, d\lambda \\ &= \frac{\sin\theta\pi}{\theta\pi} a_0 \int_0^\infty \lambda^\theta \langle \nabla u, \nabla A_0^{\theta-1} A_0^{1-\theta} (\lambda + A_0)^{-2} v\rangle_{L^2}\, d\lambda\end{aligned}$$

$$= \frac{\sin\theta\pi}{\theta\pi}\int_0^\infty \lambda^\theta \langle \tilde{u}, A_0^{1-\theta}(\lambda+A_0)^{-2}v\rangle_{L^2}\, d\lambda.$$

Here, $\tilde{u}$ is determined as follows. Since ∇ is a bounded operator from $H^{2\theta}(\Omega)$ to $H^{2\theta-1}(\Omega) = H_0^{2\theta-1}(\Omega)$ and from $H^{2(1-\theta)}(\Omega)$ to $H^{1-2\theta}(\Omega)$, we observe that $|\langle \nabla u, \nabla A_0^{\theta-1} f\rangle_{L^2}| \le C_\theta \|u\|_{H^{2\theta}} \|f\|_{L^2}$ for all $f \in \mathcal{D}(A_0^{\theta-\frac{1}{2}})$. In addition, there exists some $\tilde{u} \in L^2(\Omega)$ such that $a_0\langle \nabla u, \nabla A_0^{\theta-1} f\rangle_{L^2} = \langle \tilde{u}, f\rangle_{L^2}$ for all $f \in \mathcal{D}(A_0^{\theta-\frac{1}{2}})$ with the estimate $\|\tilde{u}\|_{L^2} \le C_\theta \|u\|_{H^{2\theta}}$. Then, with the aid of the formula

$$\int_0^\infty \lambda^\theta \|A_0^{\frac{1-\theta}{2}}(\lambda+A_0)^{-1}g\|_{L^2}^2 d\lambda = \frac{\theta\pi}{\sin\theta\pi}\|g\|_{L^2}^2, \quad g \in L^2(\Omega),$$

it follows that

$$\begin{aligned}|\langle u, A_0^\theta v\rangle_{L^2}| &\le \frac{\sin\theta\pi}{\theta\pi}\int_0^\infty \lambda^\theta \|A_0^{\frac{1-\theta}{2}}(\lambda+A_0)^{-1}\tilde{u}\|_{L^2}\|A_0^{\frac{1-\theta}{2}}(\lambda+A_0)^{-1}v\|_{L^2} d\lambda \\ &\le C_\theta \|\tilde{u}\|_{L^2}\|v\|_{L^2}.\end{aligned}$$

Hence, $u \in \mathcal{D}(A_0^\theta)$ and $\|A_0^\theta u\|_{L^2} \le C_\theta \|u\|_{H^{2\theta}}$.

Moment inequality *Let A be a densely defined closed linear operator in a Banach space X, A is of positive type in the sense that $\rho(A) \supset (-\infty, 0]$ with the estimate*

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \le \frac{M}{|\lambda|}, \quad \lambda < 0. \tag{2.7}$$

Then, for $0 \le \alpha < \beta < \gamma \le 1$,

$$\|A^\beta u\|_X \le C_{\alpha,\beta,\gamma}\|A^\alpha u\|_X^{\frac{\gamma-\beta}{\gamma-\alpha}}\|A^\gamma u\|_X^{\frac{\beta-\alpha}{\gamma-\alpha}}, \quad u \in \mathcal{D}(A^\gamma), \tag{2.8}$$

where the constant $C_{\alpha,\beta,\gamma}$ is determined by α, β, γ and M.

See [24, Chap. 2, Proposition 3.3].

Heinz-Kato type inequality *Let A (resp. B) be a densely defined closed linear operator acting in a Hilbert space X (resp. Y) of positive type in the sense above. Assume that the purely imaginary power A^{it} (resp. B^{it}) is a bounded operator on X (resp. Y) with the estimate:*

$$\|A^{it}\|_{\mathcal{L}(X)} \le N_A e^{\omega_A|t|}, \quad t \in \mathbb{R}, \tag{2.9}$$

$$\|B^{it}\|_{\mathcal{L}(Y)} \le N_B e^{\omega_B|t|}, \quad t \in \mathbb{R}. \tag{2.10}$$

Let T be a bounded operator from X to Y and at the same time from $\mathcal{D}(A)$ to $\mathcal{D}(B)$. Then, for every $0 < \theta < 1$, T is a bounded operator from $\mathcal{D}(A^\theta)$ to $\mathcal{D}(B^\theta)$ and its operator norm is estimated by

$$\|T\|_{\mathcal{L}(\mathcal{D}(A^\theta),\mathcal{D}(B^\theta))} \leq N_A N_B e^{(\omega_A+\omega_B)\sqrt{\theta(1-\theta)}} \|T\|^\theta_{\mathcal{L}(\mathcal{D}(A),\mathcal{D}(B))} \|T\|^{1-\theta}_{\mathcal{L}(X,Y)}. \tag{2.11}$$

This inequality was first proved by Heinz when A and B are self-adjoint operators, and was extended by Kato to maximal accretive operators, see [24, Chap. 2, Theorems 3.3 and 3.4]. In the proof, it is equally essential that A and B satisfy the conditions (2.9) and (2.10) respectively. Thus, (2.11) can be shown by the same argument under the conditions (2.9) and (2.10).

Let X be a Hilbert space and let A denote a densely defined closed linear operator of positive type. When does A satisfy (2.9)? It is clear that, if A is self-adjoint, then $\|A^{it}\|_{\mathcal{L}(X)} \leq 1$, $t \in \mathbb{R}$. Similarly it is known that, if A is maximal accretive, then $\|A^{it}\|_{\mathcal{L}(X)} \leq e^{\frac{\pi}{2}|t|}$, $t \in \mathbb{R}$, see [24, Chap. 2, Lemma 3.8]. More generally, several necessary and sufficient conditions for (2.9) are presented in Yagi [29]; in addition, some sufficient condition is obtained.

Purely imaginary powers *If $\mathcal{D}(A) = \mathcal{D}(A^*) = \mathcal{D}$ with the norm equivalence*

$$D^{-1}\|Au\| \leq \|A^*u\| \leq D\|Au\|, \quad u \in \mathcal{D}, \tag{2.12}$$

then A satisfies (2.9) with $\omega_A = \pi$ and some N_A depending only on D and M in (2.7).

2.2. Implicit Runge-Kutta method

An s-stage Runge-Kutta scheme applied to the Cauchy problem of an ordinary differential equation

$$\begin{cases} \dfrac{dy}{dt} = f(t,y), \quad 0 \leq t \leq T, \\ y(0) = y_0 \end{cases} \tag{2.13}$$

with a stepsize $h > 0$ is written as

$$Y_{n+1} = Y_n + h\sum_{j=1}^{s} b_j f(t_n + c_j h, V_{n,j}), \quad n = 0, 1, \ldots, N-1, \tag{2.14a}$$

$$V_{n,i} = Y_n + h\sum_{j=1}^{s} a_{ij} f(t_n + c_j h, V_{n,j}), \quad i = 1, 2, \ldots, s, \tag{2.14b}$$

$$Y_0 = y_0, \tag{2.14c}$$

where $t_n = nh$ and $t_N \leq T$. The approximate solution $\{Y_n\}$ is given recursively by (2.14a) with $\{V_{n,i}\}$ given by solving (2.14b) at every time step n. The parameters a_{ij}, b_j, c_j are all real numbers. Using the matrix notation

$$\mathcal{A} = \begin{bmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{s1} & \cdots & a_{ss} \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_s \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_s \end{bmatrix},$$

we can write (2.14) as

$$\begin{cases} Y_{n+1} = Y_n + h\boldsymbol{e}^T \mathcal{B} \boldsymbol{f}((t_n \mathcal{I} + h\mathcal{C})\boldsymbol{e}, \boldsymbol{V}_n), & n = 0, 1, \ldots, N-1, \\ \boldsymbol{V}_n = \boldsymbol{e} Y_n + h\mathcal{A}\boldsymbol{f}((t_n \mathcal{I} + h\mathcal{C})\boldsymbol{e}, \boldsymbol{V}_n), & \\ Y_0 = y_0, & \end{cases} \tag{2.15}$$

where $\mathcal{I}$ denotes the identity matrix, $\boldsymbol{e} = [1, \ldots, 1]^T$, $\boldsymbol{V}_n = [V_{n,1}, \ldots, V_{n,s}]^T$ and $\boldsymbol{f}(\boldsymbol{\tau}, \boldsymbol{W}) = [f(\tau_1, W_1), \ldots, f(\tau_s, W_s)]^T$ with $\boldsymbol{\tau} = [\tau_1, \ldots, \tau_s]^T$ and $\boldsymbol{W} = [W_1, \ldots, W_s]^T$. The scheme is said to be explicit if the matrix $\mathcal{A}$ is strictly lower triangular, and implicit otherwise.

The Runge-Kutta scheme is said to have an order of accuracy p if $Y_1 - y(t_1) = O(h^{p+1})$ as $h \to 0$. For an algebraic characterization of the order, see [4, Theorem 307B]. We note here only the simple identity

$$\boldsymbol{e}^T \mathcal{B} \mathcal{C}^{k-1} \boldsymbol{e} = \frac{1}{k}, \quad k = 1, \ldots, p. \tag{2.16}$$

The quadrature order is defined as the maximal number q such that

$$\mathcal{A}\mathcal{C}^{k-1}\boldsymbol{e} = \frac{1}{k}\mathcal{C}^k \boldsymbol{e}, \quad k = 1, \ldots, q. \tag{2.17}$$

The scheme is said to be strongly $A(\theta)$-stable, $0 < \theta \leq \frac{\pi}{2}$, if $(\mathcal{I} - z\mathcal{A})^{-1}$ is holomorphic on some domain containing $\mathbb{C} \setminus S_{\pi-\theta}$ and if the stability function $r(z) = 1 + z e^T \mathcal{B}(\mathcal{I} - z\mathcal{A})^{-1} e$ satisfies $|r(z)| \le 1$, $z \in \mathbb{C} \setminus S_{\pi-\theta}$, with the estimate $|r(\infty)| < 1$ (see e.g. [12, 13, 20]).

In this paper, we always assume that

(RK1) the scheme is strongly $A(\theta)$-stable with some $0 < \theta \le \frac{\pi}{2}$;

(RK2) the scheme is of order $p \ge 1$ and of quadrature order q with the relation $1 \le q \le \max\{p-1, 1\}$.

2.3. Galerkin finite element method

Let Ω be a convex polygonal domain in $\mathbb{R}^2$ and $\{\tau_\xi\}_{\xi>0}$ a family of triangulations to Ω with the size parameter $\xi = \max\{d_\sigma;\ \sigma \in \tau_\xi\} > 0$, where σ denotes a triangle appearing in τ_ξ and d_σ its diameter. It is always assumed in this paper that

(G) $\{\tau_\xi\}_{\xi>0}$ is regular and satisfies the inverse assumption, more precisely, there exists a positive number ν independent of ξ such that

$$\nu\xi \le \rho_\sigma \le d_\sigma \le r_\sigma \le \nu^{-1}\xi, \quad \sigma \in \tau_\xi, \tag{2.18}$$

where r_σ and ρ_σ denote the radii of the circumcircle and the incircle of σ, respectively.

Let

$$\mathcal{C}_\xi(\overline{\Omega}) = \big\{v \in \mathcal{C}(\overline{\Omega});\ v|_\sigma \text{ is linear in each } \sigma \in \tau_\xi\big\} \tag{2.19}$$

be the space of trial functions. We equip $\mathcal{C}_\xi(\overline{\Omega})$ with the usual L^2-inner product and consider it as a closed subspace of $L^2(\Omega)$. Let $p_\xi : L^2(\Omega) \to \mathcal{C}_\xi(\overline{\Omega})$ be the L^2-orthogonal projection and $\pi_\xi : \mathcal{C}(\overline{\Omega}) \to \mathcal{C}_\xi(\overline{\Omega})$ the interpolation operator. Let

$$\begin{cases} \mathcal{D}(A_0) = H^2_N(\Omega) = \Big\{u \in H^2(\Omega);\ \dfrac{\partial u}{\partial n} = 0 \ \text{ on } \ \partial\Omega\Big\}, \\ A_0 u = -a_0 \triangle u + c_0 u, \end{cases} \tag{2.20}$$

with some fixed a_0, $c_0 > 0$. Then the Ritz projection $R_{0\xi} : H^1(\Omega) \to \mathcal{C}_\xi(\overline{\Omega})$ with respect to A_0 is defined by

$$\alpha^0(R_{0\xi}u, \hat{v}) = \alpha^0(u, \hat{v}), \quad u \in H^1(\Omega),\ \hat{v} \in \mathcal{C}_\xi(\overline{\Omega}), \tag{2.21}$$

where $\alpha^0(\cdot,\cdot)$ is the sesquilinear form

$$\alpha^0(u, v) = a_0\langle \nabla u, \nabla v\rangle_{L^2} + c_0\langle u, v\rangle_{L^2}, \quad u, v \in H^1(\Omega), \tag{2.22}$$

on $H^1(\Omega)$ associated with A_0. Since A_0 is a positive definite self-adjoint operator in $L^2(\Omega)$, $R_{0\xi}$ satisfies also

$$\alpha^0(\hat{v}, R_{0\xi}u) = \alpha^0(\hat{v}, u), \quad u \in H^1(\Omega),\ \hat{v} \in \mathcal{C}_\xi(\overline{\Omega}). \tag{2.23}$$

We shall show here some new results on π_ξ, p_ξ and $R_{0\xi}$ in Sobolev spaces.

Proposition 2.1 *For $0 \le s < \frac{3}{2}$, $\mathcal{C}_\xi(\overline{\Omega}) \subset H^s(\Omega)$ with the estimate*

$$\|\hat{v}\|_{H^s} \le C_s \xi^{-s} \|\hat{v}\|_{L^2}, \quad \hat{v} \in \mathcal{C}_\xi(\overline{\Omega}), \tag{2.24}$$

C_s being independent of ξ.

Proof. When $s = 1$, the result is well-known (cf. [5, Theorem 3.2.6]); when $s = 0$, it is trivial. Then, for $0 \le s \le 1$, the result follows immediately from (2.2). Therefore it suffices to consider the case $s = 1+\varepsilon$, $0 < \varepsilon < \frac{1}{2}$. In this case we estimate directly the $H^{1+\varepsilon}$-norm of $\hat{v} \in \mathcal{C}_\xi(\overline{\Omega})$. By (2.1),

$$\|\hat{v}\|^2_{H^{1+\varepsilon}} = \|\hat{v}\|^2_{L^2} + \sum_{j=1}^{2} \iint_{\Omega\times\Omega} \frac{|\partial_j \hat{v}(x) - \partial_j \hat{v}(y)|^2}{|x-y|^{2+2\varepsilon}} dxdy. \tag{2.25}$$

Since $\hat{v}_{\sigma j} = \partial_j \hat{v}|_\sigma$ is constant for each $\sigma \in \tau_\xi$, it follows that

$$\begin{aligned}
&\iint_{\Omega\times\Omega} \frac{|\partial_j \hat{v}(x) - \partial_j \hat{v}(y)|^2}{|x-y|^{2+2\varepsilon}} dxdy \\
&\quad = \sum_{\substack{\sigma,\sigma' \in \tau_\xi \\ \sigma' \ne \sigma}} |\hat{v}_{\sigma j} - \hat{v}_{\sigma' j}|^2 \iint_{\sigma\times\sigma'} \frac{dxdy}{|x-y|^{2+2\varepsilon}} \\
&\quad \le \sum_{\substack{\sigma,\sigma' \in \tau_\xi \\ \sigma' \ne \sigma}} 2\left(|\hat{v}_{\sigma j}|^2 + |\hat{v}_{\sigma' j}|^2\right) \iint_{\sigma\times\sigma'} \frac{dxdy}{|x-y|^{2+2\varepsilon}} \\
&\quad \le 4 \sum_{\sigma \in \tau_\xi} |\hat{v}_{\sigma j}|^2 \iint_{\sigma\times(\Omega\setminus\sigma)} \frac{dxdy}{|x-y|^{2+2\varepsilon}}.
\end{aligned}$$

It is then sufficient to verify

$$\iint_{\sigma\times(\Omega\setminus\sigma)} \frac{dxdy}{|x-y|^{2+2\varepsilon}} \le C_\varepsilon {r_\sigma}^{2-2\varepsilon}, \tag{2.26}$$

where the constant C_ε is independent of σ. In fact, this together with (2.18) implies that

$$\|\hat{v}\|_{H^{1+\varepsilon}}^2 \le \|\hat{v}\|_{L^2}^2 + C_\varepsilon \sum_j \sum_\sigma |\hat{v}_{\sigma j}|^2 {r_\sigma}^{2-2\varepsilon}$$
$$\le \|\hat{v}\|_{L^2}^2 + C_\varepsilon \xi^{-2\varepsilon} \sum_j \sum_\sigma |\hat{v}_{\sigma j}|^2 |\sigma| \le C_\varepsilon \xi^{-2\varepsilon} \|\hat{v}\|_{H^1}^2,$$

where $|\sigma|$ denotes the area of σ; and hence (2.24) is proved.

Let us now show the estimate (2.26). Let B_σ denote the circumcirclar domain of σ with radius r_σ, and U_σ the circular domain with the same center as B_σ but with radius $2r_\sigma$. Let l_i, $i = 1, 2, 3$, be the three lines in $\mathbb{R}^2$ obtained by prolonging the sides of σ. Each l_i partitions U_σ into two subsets, so denote by S_i the one disjoint with σ. Then we obviously see that

$$\iint_{\sigma \times (\Omega \setminus \sigma)} \frac{dxdy}{|x-y|^{2+2\varepsilon}} \le \iint_{B_\sigma \times (\mathbb{R}^2 \setminus U_\sigma)} \frac{dxdy}{|x-y|^{2+2\varepsilon}}$$
$$+ \sum_{i=1}^{3} \iint_{\sigma \times S_i} \frac{dxdy}{|x-y|^{2+2\varepsilon}}.$$

For the first integral we have

$$\iint_{B_\sigma \times (\mathbb{R}^2 \setminus U_\sigma)} \frac{dxdy}{|x-y|^{2+2\varepsilon}}$$
$$= \int_0^{r_\sigma} \int_{2r_\sigma}^{\infty} \int_0^{2\pi} \int_0^{2\pi} \frac{r\rho d\theta d\varphi d\rho dr}{\{r^2 + \rho^2 - 2r\rho\cos(\varphi - \theta)\}^{1+\varepsilon}}$$
$$\le (2\pi)^2 {r_\sigma}^{2-2\varepsilon} \int_0^1 \int_2^\infty \frac{r\rho d\rho dr}{(r-\rho)^{2+2\varepsilon}} \le C_\varepsilon {r_\sigma}^{2-2\varepsilon}.$$

For the second integrals, we can assume, without loss of generality, that $\sigma \subset [0, 4r_\sigma] \times [0, 4r_\sigma]$, $S_i \subset [-4r_\sigma, 0] \times [0, 4r_\sigma]$ and $\partial\sigma \cap \partial S_i \subset \{0\} \times [0, 4r_\sigma]$. Then, changing variables $(x_2, y_2) \mapsto (z, w) = (x_2 - y_2, x_2 + y_2)$,

$$\iint_{\sigma \times S_i} \frac{dxdy}{|x-y|^{2+2\varepsilon}}$$
$$\le \int_{-4r_\sigma}^{0} \int_0^{4r_\sigma} \int_0^{4r_\sigma} \int_0^{4r_\sigma} \frac{dx_2 dy_2 dx_1 dy_1}{\{|x_1 - y_1|^2 + |x_2 - y_2|^2\}^{1+\varepsilon}}$$
$$\le (4r_\sigma)^{2-2\varepsilon} \int_{-1}^{0} \int_0^1 \left\{ \int_0^2 \int_{-1}^{1} \frac{2dzdw}{\{|x_1 - y_1|^2 + |z|^2\}^{1+\varepsilon}} \right\} dx_1 dy_1$$

$$
\begin{aligned}
&\le (4r_\sigma)^{2-2\varepsilon}\int_{-1}^{0}\int_0^1\left\{\int_{-|x_1-y_1|^{-1}}^{|x_1-y_1|^{-1}}\frac{4d\zeta}{(1+\zeta^2)^{1+\varepsilon}}\right\}\frac{dx_1dy_1}{|x_1-y_1|^{1+2\varepsilon}}\\
&\le C_\varepsilon {r_\sigma}^{2-2\varepsilon}\int_{-\infty}^{\infty}\frac{d\zeta}{(1+\zeta^2)^{1+\varepsilon}}\times\int_0^1\int_0^1\frac{dx_1dy_1}{(x_1+y_1)^{1+2\varepsilon}}\\
&\le C_\varepsilon {r_\sigma}^{2-2\varepsilon}.
\end{aligned}
$$

Hence we have verified (2.26). □

Proposition 2.2 *For* $0\le s<\frac{3}{2}$,

$$\|(1-\pi_\xi)v\|_{H^s}\le C_s\xi^{2-s}\|v\|_{H^2},\quad v\in H^2(\Omega), \tag{2.27}$$

C_s being independent of ξ.

Proof. When $s=0$ or 1, (2.27) is well-known (see e.g. [5, Theorem 3.2.1]). Then, for $0\le s\le 1$, (2.27) follows by interpolation again.

Let $v\in H^2(\Omega)$ and $\hat v=\pi_\xi v$. For $s=1+\varepsilon$, $0<\varepsilon<\frac{1}{2}$, we see that

$$
\begin{aligned}
&\iint_{\Omega\times\Omega}\frac{|\{\partial_j\hat v(x)-\partial_j v(x)\}-\{\partial_j\hat v(y)-\partial_j v(y)\}|^2}{|x-y|^{2+2\varepsilon}}dxdy\\
&\quad\le\sum_{\sigma\in\tau_\xi}\iint_{\sigma\times\sigma}\frac{|\partial_j v(x)-\partial_j v(y)|^2}{|x-y|^{2+2\varepsilon}}dxdy\\
&\quad\quad+4\sum_{\sigma\in\tau_\xi}\iint_{\sigma\times(\Omega\setminus\sigma)}\frac{|\partial_j\hat v(x)-\partial_j v(x)|^2}{|x-y|^{2+2\varepsilon}}dxdy.
\end{aligned}
$$

The first integrals are estimated by

$$
\begin{aligned}
&\sum_{\sigma\in\tau_\xi}\iint_{\sigma\times\sigma}|\partial_j v(x)-\partial_j v(y)|^2\frac{dxdy}{|x-y|^{2+2\varepsilon}}\\
&\quad\le\sum_\sigma\iiint_{\sigma\times\sigma\times(0,1)}|\nabla(\partial_j v)((1-\omega)y+\omega x)|^2\frac{dxdyd\omega}{|x-y|^{2\varepsilon}}.
\end{aligned}
$$

If we change the variables $(x,y,\omega)\mapsto(z,y,\omega)=((1-\omega)y+\omega x,y,\omega)$, the domain of integration is given by $\{(z,y,\omega);\ z,y\in\sigma,\ \omega(z,y)<\omega<1\}$, where $\omega(z,y)=\inf\{\omega;\ y+\omega^{-1}(z-y)\in\sigma\}\ge\frac{|z-y|}{2r_\sigma}$. Then,

$$\le\sum_\sigma\int_\sigma\int_\sigma\int_{\frac{|z-y|}{2r_\sigma}}^1|\nabla(\partial_j v)(z)|^2\omega^{2\varepsilon-2}\frac{d\omega dydz}{|z-y|^{2\varepsilon}}$$

$$\leq \frac{1}{1-2\varepsilon}\sum_{\sigma}\int_{\sigma}\int_{\sigma}|\nabla(\partial_j v)(z)|^2\frac{(2r_\sigma)^{1-2\varepsilon}}{|z-y|}dydz$$
$$\leq C_\varepsilon\sum_{\sigma}\|\nabla(\partial_j v)\|^2_{L^2(\sigma)}{r_\sigma}^{2-2\varepsilon}\leq C_\varepsilon\|v\|^2_{H^2}\xi^{2-2\varepsilon}.$$

For the second integrals, we utilize Hölder's inequality. Then,

$$\iint_{\sigma\times(\Omega\setminus\sigma)}\frac{|\partial_j\hat{v}(x)-\partial_j v(x)|^2}{|x-y|^{2+2\varepsilon}}dxdy$$
$$\leq\left\{\int_\sigma|\partial_j\hat{v}(x)-\partial_j v(x)|^{2p}dx\right\}^{\frac{1}{p}}\left\{\int_\sigma\left[\int_{\Omega\setminus\sigma}\frac{dy}{|x-y|^{2+2\varepsilon}}\right]^q dx\right\}^{\frac{1}{q}},$$

where $1<q<\frac{1}{2\varepsilon}$, and $\frac{1}{p}+\frac{1}{q}=1$. Since

$$\int_{\Omega\setminus\sigma}\frac{dy}{|x-y|^{2+2\varepsilon}}\leq\int_{|x-y|>\mathrm{dist}(x,\partial\sigma)}\frac{dy}{|x-y|^{2+2\varepsilon}}=\frac{\pi}{\varepsilon}\,\mathrm{dist}(x,\partial\sigma)^{-2\varepsilon},$$

we see that

$$\int_\sigma\left[\int_{\Omega\setminus\sigma}\frac{dy}{|x-y|^{2+2\varepsilon}}\right]^q dx\leq\int_\sigma C_\varepsilon\,\mathrm{dist}(x,\partial\sigma)^{-2\varepsilon q}dx$$
$$\leq C_\varepsilon\int_{-r_\sigma}^{r_\sigma}\int_0^{r_\sigma}\eta^{-2\varepsilon q}d\eta d\xi=C_{\varepsilon,q}{r_\sigma}^{2-2\varepsilon q}.$$

In order to estimate $\|\partial_j\hat{v}-\partial_j v\|_{L^{2p}(\sigma)}$, it is verified from the lemma below and (2.18) that

$$\|\partial_j\hat{v}-\partial_j v\|_{L^{2p}(\sigma)}$$
$$\leq C_p\Big\{\|v\|^{1-\frac{1}{p}}_{H^2(\sigma)}\|\hat{v}-v\|^{\frac{1}{p}}_{H^1(\sigma)}+{r_\sigma}^{\frac{1}{p}-1}\|\hat{v}-v\|_{H^1(\sigma)}\Big\}$$
$$\leq C_p\Big\{\|v\|^{1-\frac{1}{p}}_{H^2(\sigma)}\left(r_\sigma\|v\|_{H^2(\sigma)}\right)^{\frac{1}{p}}+{r_\sigma}^{\frac{1}{p}-1}\cdot r_\sigma\|v\|_{H^2(\sigma)}\Big\}$$
$$=C_p{r_\sigma}^{\frac{1}{p}}\|v\|_{H^2(\sigma)}.$$

Hence we have

$$\sum_{\sigma\in\tau_\xi}\iint_{\sigma\times(\Omega\setminus\sigma)}\frac{|\partial_j\hat{v}(x)-\partial_j v(x)|^2}{|x-y|^{2+2\varepsilon}}dxdy$$
$$\leq\sum_\sigma C_\varepsilon{r_\sigma}^{\frac{2}{p}}\|v\|^2_{H^2(\sigma)}\cdot{r_\sigma}^{\frac{2}{q}-2\varepsilon}\leq C_\varepsilon\xi^{2-2\varepsilon}\|v\|^2_{H^2},$$

which shows that (2.27) holds for $s = 1+\varepsilon$. Thus we have proved Proposition 2.2. $\square$

Lemma 2.1 *For* $1 \le p < \infty$,

$$\|w\|_{L^{2p}(\sigma)} \le C_p\left\{\left(\frac{r_\sigma}{\rho_\sigma}\right)^{1-\frac{1}{p}} \|\nabla w\|_{L^2(\sigma)}^{1-\frac{1}{p}} \|w\|_{L^2(\sigma)}^{\frac{1}{p}} + \rho_\sigma^{\frac{1}{p}-1}\|w\|_{L^2(\sigma)}\right\}, \qquad w \in H^1(\sigma), \tag{2.28}$$

on each triangle $\sigma \in \tau_\xi$, C_p *being independent of* σ.

Proof of the lemma. Let $\hat\sigma = \{(z_1, z_2) \in \mathbb{R}^2;\ z_1 > 0,\ z_2 > 0,\ z_1 + z_2 < 1\}$ be the reference triangle, and $\phi_\sigma : \hat\sigma \to \sigma$ an affine mapping. Then we have (cf. [5, Theorem 3.1.2])

$$C_q^{-1}|\sigma|^{\frac{1}{q}}\|w\circ\phi_\sigma\|_{L^q(\hat\sigma)} \le \|w\|_{L^q(\sigma)} \le C_q|\sigma|^{\frac{1}{q}}\|w\circ\phi_\sigma\|_{L^q(\hat\sigma)},$$

$$\|\nabla(w\circ\phi_\sigma)\|_{L^q(\hat\sigma)} \le C_q r_\sigma|\sigma|^{-\frac{1}{q}}\|\nabla w\|_{L^q(\sigma)},$$

for $1 \le q \le \infty$. Then (2.28) is readily verified by applying (2.4) on $\hat\sigma$. $\square$

Proposition 2.3 *For* $0 \le s < \frac{3}{2}$,

$$\|(1-p_\xi)v\|_{H^s} \le C_s\xi^{2-s}\|v\|_{H^2}, \quad v \in H^2(\Omega), \tag{2.29}$$

C_s *being independent of* ξ. *In addition, for* $0 \le r \le 2$,

$$\|(1-p_\xi)v\|_{L^2} \le C\xi^r\|v\|_{H^r}, \quad v \in H^r(\Omega), \tag{2.30}$$

C *being independent of* ξ *and* r.

Proof. When $s = 0$, it is easily seen from (2.27) that

$$\|(1-p_\xi)v\|_{L^2}^2 = \langle(1-p_\xi)v, (1-\pi_\xi)v\rangle_{L^2} \le C\xi^2\|(1-p_\xi)v\|_{L^2}\|v\|_{H^2}.$$

For $0 < s < \frac{3}{2}$, we verify from (2.24) and (2.27) that

$$\begin{aligned}\|(1-p_\xi)v\|_{H^s} &\le \|(1-\pi_\xi)v\|_{H^s} + \|(\pi_\xi - p_\xi)v\|_{H^s}\\ &\le C_s\xi^{2-s}\|v\|_{H^2} + C_s\xi^{-s}\|(\pi_\xi - p_\xi)v\|_{L^2} \le C_s\xi^{2-s}\|v\|_{H^2}.\end{aligned}$$

Thus we verify (2.29).

When $r = 0$ or 2, (2.30) is obvious. Then, for $0 < r < 2$, it is a direct consequence of (2.11); indeed, take $T = 1 - p_\xi$, $A = \Lambda_0$ and $B = 1$, where Λ_0 is a self-adjoint operator in $L^2(\Omega)$ with the domain $H^2(\Omega)$. $\square$

Proposition 2.4 *For $0 \le s < \frac{3}{2}$,*

$$\|(1 - R_{0\xi})v\|_{H^s} \le C_s \xi^{2-s} \|A_0 v\|_{L^2}, \quad v \in \mathcal{D}(A_0), \tag{2.31}$$

C_s being independent of ξ. In addition, for $\frac{1}{2} < r \le 2$,

$$\|(1 - R_{0\xi})v\|_{L^2} \le C_r \xi^r \|A_0^{\frac{r}{2}} v\|_{L^2}, \quad v \in D(A_0^{\frac{r}{2}}), \tag{2.32}$$

C_r also being independent of ξ. In particular, $R_{0\xi}$ can be extended over $H^r(\Omega)$ for any $r > \frac{1}{2}$.

Proof. When $s = 0$, since A_0 is a positive definite self-adjoint operator in $L^2(\Omega)$, (2.31) is well-known (see [5, Theorems 3.2.2 and 3.2.5]). Then, for general $0 < s < \frac{3}{2}$, (2.24) and (2.27) yield that

$$\begin{aligned}
\|(1 - R_{0\xi})v\|_{H^s} &\le \|(1 - \pi_\xi)v\|_{H^s} + \|(\pi_\xi - R_{0\xi})v\|_{H^s} \\
&\le C_s \xi^{2-s} \|v\|_{H^2} + C_s \xi^{-s} \|(\pi_\xi - R_{0\xi})v\|_{L^2} \\
&\le C_s \xi^{2-s} \|A_0 v\|_{L^2}.
\end{aligned}$$

In order to prove (2.32), we notice from (2.6) that $H^{2-r}(\Omega) \subset \mathcal{D}(A_0^{1-\frac{r}{2}})$ for $\frac{1}{2} < r \le 2$. Therefore, if $\frac{1}{2} < r \le 2$, it is observed by Aubin-Nitsche's trick, (2.21), (2.23) and (2.31) that

$$\begin{aligned}
\|(1 - R_{0\xi})v\|_{L^2}^2 &= a^0((1 - R_{0\xi})v, A_0^{-1}(1 - R_{0\xi})v) \\
&= a^0((1 - R_{0\xi})v, (1 - R_{0\xi})A_0^{-1}(1 - R_{0\xi})v) \\
&= a^0(v, (1 - R_{0\xi})A_0^{-1}(1 - R_{0\xi})v) \\
&= \left\langle A_0^{\frac{r}{2}} v, A_0^{1-\frac{r}{2}}(1 - R_{0\xi})A_0^{-1}(1 - R_{0\xi})v \right\rangle_{L^2} \\
&\le \|A_0^{\frac{r}{2}} v\|_{L^2} \|A_0^{1-\frac{r}{2}}(1 - R_{0\xi})A_0^{-1}(1 - R_{0\xi})v\|_{L^2} \\
&\le \|A_0^{\frac{r}{2}} v\|_{L^2} \cdot C_r \xi^r \|(1 - R_{0\xi})v\|_{L^2}, \quad v \in \mathcal{D}(A_0).
\end{aligned}$$

Since $\mathcal{D}(A_0)$ is dense in $\mathcal{D}(A_0^{\frac{r}{2}})$, (2.32) holds for $\frac{1}{2} < r \le 2$. □

3. Error estimate

In this section, under (Ω), (B), (C) and (IF), we shall establish the stability and the order estimate of error of the fully discrete approximation for the chemotaxis-growth system (CG).

We begin with writing the system as an ordinary differential equation

$$\begin{cases} \dfrac{dU}{dt} + A(U)U = F(U), \quad 0 < t \le T, \\ U(0) = U_0, \end{cases} \tag{3.1}$$

in a product L^2-space. Let $X = \mathbb{L}^2(\Omega)$ and $Z = \mathbb{H}^{1+\varepsilon}(\Omega)$ be two product spaces, where ε is an arbitrarily fixed number $0 < \varepsilon < \frac{1}{2}$. By (1.1) and (2.5), the initial function $U_0 = \begin{bmatrix} u_0 \\ \rho_0 \end{bmatrix}$ is in $Z \subset \left\{\mathcal{C}(\overline{\Omega})\right\}^2$. Let

$$K = \left\{ U = \begin{bmatrix} u \\ \rho \end{bmatrix} \in Z;\ \|U - U_0\|_Z = \sqrt{\|u - u_0\|^2_{H^{1+\varepsilon}} + \|\rho - \rho_0\|^2_{H^{1+\varepsilon}}} < r \right\}$$

be an open ball in Z with the center U_0. From (2.5), there exists $\delta > 0$ such that

$$\operatorname{Re}\rho(x) \ge \delta \quad \text{on } \overline{\Omega} \quad \text{for } U = \begin{bmatrix} u \\ \rho \end{bmatrix} \in K$$

if r is sufficiently small.

For each $U = \begin{bmatrix} u \\ \rho \end{bmatrix} \in K$, $A(U)$ is a linear operator in X defined by

$$\begin{cases} \mathcal{D}(A(U)) = \mathcal{D} = \mathbb{H}^2_N(\Omega), \\ A(U)\tilde{U} = \begin{bmatrix} A_1\tilde{u} - A_3(U)\tilde{\rho} \\ A_2\tilde{\rho} \end{bmatrix}, \quad \tilde{U} = \begin{bmatrix} \tilde{u} \\ \tilde{\rho} \end{bmatrix}, \end{cases} \tag{3.2}$$

with

$$\begin{cases} \mathcal{D}(A_1) = \mathcal{D}(A_2) = H^2_N(\Omega), \\ \mathcal{D}(A_3(U)) = H^2_N(\Omega), \\ A_1\tilde{u} = -a\Delta\tilde{u} + a\tilde{u}, \\ A_2\tilde{\rho} = -d\Delta\tilde{\rho} + d\tilde{\rho}, \\ A_3(U)\tilde{\rho} = -\nabla \cdot \{\operatorname{Re} u\, b(\operatorname{Re}\rho)\, \nabla\tilde{\rho}\}, \end{cases} \tag{3.3}$$

where $b(\rho) = B'(\rho)$. $F(U) = \begin{bmatrix} F_1(U) \\ F_2(U) \end{bmatrix}$ is a function of $U = \begin{bmatrix} u \\ \rho \end{bmatrix} \in K$ defined by

$$\begin{cases} F_1(U) = au + c(u), \\ F_2(U) = d\rho + fu - g\rho. \end{cases} \tag{3.4}$$

By (1.1), U_0 is in $\mathcal{D}$. Then (CG) is formulated as an abstract equation of the form (3.1).

Furthermore, we can also write (3.1) in the weak form

$$\begin{cases} \dfrac{d}{dt}\langle U, V\rangle_{\mathbb{L}^2} + \alpha(U; U, V) = \langle F(U), V\rangle_{\mathbb{L}^2}, \\ \qquad\qquad\qquad\qquad 0 < t \le T,\ V \in \mathbb{H}^1(\Omega), \\ U(0) = U_0. \end{cases} \tag{3.5}$$

Here $\alpha(U; \cdot, \cdot)$ is the sesquilinear form

$$\begin{aligned} \alpha(U; \widetilde{U}, \widetilde{V}) = {} & a\langle \nabla\tilde{u}, \nabla\tilde{v}\rangle_{L^2} + a\langle \tilde{u}, \tilde{v}\rangle_{L^2} \\ & - \langle \operatorname{Re} u\, b(\operatorname{Re}\rho)\nabla\tilde{\rho}, \nabla\tilde{v}\rangle_{L^2} + d\langle \nabla\tilde{\rho}, \nabla\tilde{\mu}\rangle_{L^2} + d\langle \tilde{\rho}, \tilde{\mu}\rangle_{L^2}, \end{aligned}$$

$$U = \begin{bmatrix} u \\ \rho \end{bmatrix} \in K,\ \ \widetilde{U} = \begin{bmatrix} \tilde{u} \\ \tilde{\rho} \end{bmatrix},\ \ \widetilde{V} = \begin{bmatrix} \tilde{v} \\ \tilde{\mu} \end{bmatrix} \in \mathbb{H}^1(\Omega) \tag{3.6}$$

on $\mathbb{H}^1(\Omega)$ associated with $A(U)$. It is readily verified that $\alpha(U; \cdot, \cdot)$ satisfies that

$$|\alpha(U; \widetilde{U}, \widetilde{V})| \le \alpha_0^{-1} \|\widetilde{U}\|_{\mathbb{H}^1} \|\widetilde{V}\|_{\mathbb{H}^1}, \quad U \in K, \quad \widetilde{U}, \widetilde{V} \in \mathbb{H}^1(\Omega), \tag{3.7}$$

$$\operatorname{Re}\alpha(U; \widetilde{U}, \widetilde{U}) \ge \begin{cases} \alpha_0 \|\widetilde{U}\|_{\mathbb{H}^1}^2 - \alpha_0^{-1}\|\tilde{u}\|_{H^1}^2, \\ \alpha_0 \|\widetilde{U}\|_{\mathbb{H}^1}^2 - \alpha_0^{-1}\|\tilde{\rho}\|_{H^1}^2, \end{cases}$$

$$U \in K,\ \widetilde{U} = \begin{bmatrix} \tilde{u} \\ \tilde{\rho} \end{bmatrix} \in \mathbb{H}^1(\Omega), \tag{3.8}$$

$$\begin{aligned} |\alpha(U; W_1, W_2) - \alpha(V; W_1, W_2)| &\le \alpha_1 \|U - V\|_{\mathbb{H}^{1+\varepsilon}} \|W_1\|_{\mathbb{H}^1} \|W_2\|_{\mathbb{H}^1}, \\ U, V \in \mathbb{H}^{1+\varepsilon}(\Omega),&\ \ W_1, W_2 \in \mathbb{H}^1(\Omega) \end{aligned} \tag{3.9}$$

with some positive constants α_0 and α_1.

Remark 3.1 When $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain, the existence and uniqueness of local solutions to (3.1) was shown by [30]. For any nonnegative initial data $U_0 \in Z$, (3.1) possesses a unique nonnegative local solution such that

$$U \in \mathcal{C}^{(1-\varepsilon)/2}([0,S];Z) \cap \mathcal{C}((0,S];\mathcal{D}) \cap \mathcal{C}^1((0,S];X),$$

where $S > 0$ depends on U_0. If $c(u)$ satisfies $c(u) = -c_0u^2 + c_1u$ for sufficiently large u with some constant $c_0 > 0$ and $B(\rho)$ is defined and smooth at $\rho = 0$, then (3.1) admits, for any nonnegative initial data $U_0 \in Z$, a global solution on the interval $[0,\infty)$, see [19]. In the case where $c(u) \equiv 0$ and $B(\rho) = b\rho$ with $b > 0$, however, (3.1) is known to possess some local solutions which blow up in a finite time and at the same time some global solutions depending on initial data U_0, see [9, 15].

The Galerkin finite element discretization of (CG) is obtained as follows. Assume the condition (G) in the preceding section. Let $X_\xi = \{\mathcal{C}_\xi(\overline{\Omega})\}^2$, clearly X_ξ is a finite dimensional subspace of $X = \mathbb{L}^2(\Omega)$. Let $P_\xi = p_\xi \times p_\xi : \mathbb{L}^2(\Omega) \to X_\xi$ be the L^2-orthogonal projection. The finite element approximation to (3.5) is then given by

$$\begin{cases} \dfrac{d}{dt}\langle \widehat{U}, \widehat{V}\rangle_{\mathbb{L}^2} + \alpha(\widehat{U}; \widehat{U}, \widehat{V}) = \langle F(\widehat{U}), \widehat{V}\rangle_{\mathbb{L}^2}, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 < t \le T,\ \widehat{V} \in X_\xi, \\ \widehat{U}(0) = P_\xi U_0. \end{cases} \tag{3.10}$$

For each $U \in K$, define a bounded linear operator $A_\xi(U)$ on X_ξ by

$$\langle A_\xi(U)\widehat{U}, \widehat{V}\rangle_{\mathbb{L}^2} = \alpha(U; \widehat{U}, \widehat{V}), \quad \widehat{U}, \widehat{V} \in X_\xi. \tag{3.11}$$

In matrix form $A_\xi(U)$ is written as

$$A_\xi(U) = \begin{bmatrix} A_{1\xi} & -A_{3\xi}(U) \\ 0 & A_{2\xi} \end{bmatrix}, \quad U = \begin{bmatrix} u \\ \rho \end{bmatrix}, \tag{3.12}$$

where bounded operators $A_{1\xi}$, $A_{2\xi}$ and $A_{3\xi}(U)$ on $\mathcal{C}_\xi(\overline{\Omega})$ are defined respectively by

$$\begin{cases} \langle A_{1\xi}\hat{u}, \hat{v}\rangle_{L^2} = a\langle \nabla\hat{u}, \nabla\hat{v}\rangle_{L^2} + a\langle \hat{u}, \hat{v}\rangle_{L^2}, & \hat{u}, \hat{v} \in \mathcal{C}_\xi(\overline{\Omega}), \\ \langle A_{2\xi}\hat{\rho}, \hat{\mu}\rangle_{L^2} = d\langle \nabla\hat{\rho}, \nabla\hat{\mu}\rangle_{L^2} + d\langle \hat{\rho}, \hat{\mu}\rangle_{L^2}, & \hat{\rho}, \hat{\mu} \in \mathcal{C}_\xi(\overline{\Omega}), \\ \langle A_{3\xi}(U)\hat{\rho}, \hat{v}\rangle_{L^2} = \langle \mathrm{Re}\, u\, b(\mathrm{Re}\,\rho)\nabla\hat{\rho}, \nabla\hat{v}\rangle_{L^2}, & \hat{\rho}, \hat{v} \in \mathcal{C}_\xi(\overline{\Omega}). \end{cases} \tag{3.13}$$

Therefore the approximate equation (3.10) can be written as

$$\begin{cases} \dfrac{d\widehat{U}}{dt} + A_\xi(\widehat{U})\widehat{U} = F_\xi(\widehat{U}), \quad 0 < t \le T, \\ \widehat{U}(0) = P_\xi U_0 \end{cases} \tag{3.14}$$

with $F_\xi(U) = P_\xi F(U)$ for $U \in K$.

Next we consider the Runge-Kutta approximation to (3.14). For the scheme assume the conditions (RK1–2) in the preceding section. Apply the scheme with stepsize $h > 0$. Then the following fully discrete approximation to (3.1)

$$\begin{cases} \widehat{U}_{n+1} = \widehat{U}_n + h\boldsymbol{e}^T\mathcal{B}\{-\boldsymbol{A}_\xi(\widehat{\boldsymbol{V}}_n)\widehat{\boldsymbol{V}}_n + \boldsymbol{F}_\xi(\widehat{\boldsymbol{V}}_n)\}, \quad n = 0, 1, \dots, N-1, \\ \widehat{\boldsymbol{V}}_n = \boldsymbol{e}\widehat{U}_n + h\mathcal{A}\{-\boldsymbol{A}_\xi(\widehat{\boldsymbol{V}}_n)\widehat{\boldsymbol{V}}_n + \boldsymbol{F}_\xi(\widehat{\boldsymbol{V}}_n)\}, \\ \widehat{U}_0 = P_\xi U_0 \end{cases} \tag{3.15}$$

is obtained, where $\widehat{\boldsymbol{V}}_n = [\widehat{V}_{n,1}, \dots, \widehat{V}_{n,s}]^T$, $\boldsymbol{A}_\xi(\boldsymbol{V}) = \mathrm{diag}[A_\xi(V_1), \dots, A_\xi(V_s)]$ and $\boldsymbol{F}_\xi(\boldsymbol{V}) = [F_\xi(V_1), \dots, F_\xi(V_s)]^T$ for $\boldsymbol{V} = [V_1, \dots, V_s]^T$, and N is a positive integer such that $N \cdot h \le T$.

In the preceding paper [16], we have already handled the approximate equation (3.15) in each fixed space X_ξ. Theorem 5.6 of [16] indeed provides the existence and uniqueness of solution to (3.15) as well as some estimate on $A_\xi(\widehat{U}_n)\widehat{U}_n$. Applying those results, we first prove the stability theorem.

Theorem 3.1 *Under* (RK1–2) *and* (G) *let* $h_0 > 0$, $\xi_0 > 0$ *and* $S \in (0, T]$ *be sufficiently small. Then, for any* $0 < h < h_0$ *and* $0 < \xi < \xi_0$, *the equation* (3.15) *possesses a unique solution* $\widehat{\mathcal{U}} = [\widehat{U}_0, \dots, \widehat{U}_N, \widehat{\boldsymbol{V}}_0, \dots, \widehat{\boldsymbol{V}}_{N-1}]$ *on the subinterval* $[0, S]$, *where* $N \le S/h$. *Moreover,* $\widehat{\mathcal{U}}$ *satisfies*

$$\begin{cases} \displaystyle\max_{n=0,1,\dots,N} \left\{ \|\widehat{U}_n\|_{X_\xi} + \|A_\xi(\widehat{U}_n)\widehat{U}_n\|_{X_\xi} \right\} \le C, \\ \displaystyle\max_{n=0,1,\dots,N-1} \left\{ \|\widehat{\boldsymbol{V}}_n\|_{X_\xi^s} + \|\boldsymbol{A}_\xi(\widehat{\boldsymbol{V}}_n)\widehat{\boldsymbol{V}}_n\|_{X_\xi^s} \right\} \le C, \end{cases} \tag{3.16}$$

$C > 0$ *being independent of* h *and* ξ.

Proof. In this section we shall describe only the trunk of the proof of the theorem assuming all the branch Propositions 3.1–3.10 below. The proof of these propositions will be collected in the next section.

The following Propositions 3.1–3.7 are verified.

Proposition 3.1 *For $\xi > 0$, $X_\xi \subset Z$.*

Proposition 3.2 *There is a $\xi_0 > 0$ such that $P_\xi U_0 \in K$ for all $0 < \xi < \xi_0$.*

Proposition 3.3 *Let $\hat{\alpha} = \frac{1+\varepsilon}{2}$. The inequality $\|\cdot\|_{Z_\xi} \le \hat{D}\|A_\xi(P_\xi U_0)^{\hat{\alpha}}\cdot\|_{X_\xi}$ holds uniformly in ξ, where Z_ξ is the space X_ξ equipped with the induced norm of Z.*

Proposition 3.4 *The norms $\|P_\xi\|_{\mathcal{L}(X)}$ and $\|A_\xi(P_\xi U_0)P_\xi A(U_0)^{-1}\|_{\mathcal{L}(X)}$ are bounded uniformly in ξ.*

Proposition 3.5 *The resolvent sets $\rho(A_\xi(U))$, $U \in K$, contain a sectorial domain $\mathbb{C} \setminus \overline{S_{\hat{\varphi}}}$, $0 < \hat{\varphi} < \frac{\pi}{2}$, and the resolvents satisfy*

$$\|(\lambda - A_\xi(U))^{-1}\|_{\mathcal{L}(X_\xi)} \le \frac{\hat{M}_A}{|\lambda| + 1}, \quad \lambda \notin \overline{S_{\hat{\varphi}}},\ U \in K$$

with some constant $\hat{M}_A$ independent of ξ.

Proposition 3.6 *$A_\xi(\cdot)$ satisfies the Lipschitz conditions*

$$\|\{A_\xi(U) - A_\xi(V)\}A_\xi(V)^{-1}\|_{\mathcal{L}(X_\xi)} \le \hat{L}_A\|U - V\|_Z, \quad U, V \in K,$$
$$\|A_\xi(U)^{-1}\{A_\xi(U) - A_\xi(V)\}\|_{\mathcal{L}(X_\xi)} \le \hat{L}_A\|U - V\|_Z, \quad U, V \in K$$

with some constant $\hat{L}_A$ independent of ξ.

Proposition 3.7 *$F_\xi(\cdot)$ satisfies the Lipschitz condition*

$$\|F_\xi(U) - F_\xi(V)\|_{X_\xi} \le \hat{L}_F\|U - V\|_Z, \quad U, V \in K$$

with some constant $\hat{L}_F$ independent of ξ.

By Proposition 3.1, X_ξ can be equipped also with the induced norm of Z, and the space is denoted by Z_ξ. Propositions 3.2–3.7 then show that one can apply Theorem 5.6 of [16]. In fact, we see that the assumptions (A4–5), (Sp), (F3) and (I3) in [16, Section 5] are satisfied, by substituting X, Z, K, A and f by X_ξ, Z_ξ, K_ξ, A_ξ and F_ξ, respectively, where $K_\xi = \{\widehat{U} \in X_\xi;\ \|\widehat{U} - P_\xi U_0\|_Z \le \hat{r}\}$ and $\hat{r} = r - \|(1 - P_\xi)U_0\|_Z > 0$. We must notice that, since all constants in Propositions 3.1–3.7 are uniform in ξ, the constant h_0 appearing in [16, Theorem 5.6] is independent of ξ. Therefore we conclude that, if ξ_0 is as in Proposition 3.2, h_0 is as above, and S is sufficiently small, then, for any $0 < h < h_0$ and $0 < \xi < \xi_0$, (3.15) has a unique local solution $\widehat{\mathcal{U}}$ in the closed subset

$$\begin{aligned}\mathcal{K}_{\xi,h}(S) = \{\widehat{\mathcal{U}} = [\widehat{U}_0, \ldots, \widehat{U}_N, \widehat{\boldsymbol{V}}_0, \ldots, \widehat{\boldsymbol{V}}_{N-1}] \in Z_\xi^{N+1} \times (Z_\xi^s)^N;\\ \widehat{U}_0 = P_\xi U_0,\ \|\widehat{U}_n - \widehat{U}_m\|_{Z_\xi} \le |(n-m)h|^\eta,\ 0 \le m \le n \le N,\\ \|\widehat{\boldsymbol{V}}_n - \boldsymbol{e}\widehat{U}_n\|_{Z_\xi^s} \le h^\eta,\ 0 \le n \le N-1\}\end{aligned}$$

of the space $\mathcal{X}_{\xi,h}(S) = Z_\xi^{N+1} \times (Z_\xi^s)^N$, η being a fixed number $0 < \eta < 1 - \hat{\alpha}$. $\widehat{\mathcal{U}}$ satisfies the estimate (3.16). Moreover, we have the representation formula

$$\begin{cases} \widehat{U}_n = \Phi_{\xi,h}(\widehat{\mathcal{U}}; n, 0) P_\xi U_0 \\ \qquad + h \displaystyle\sum_{\ell=0}^{n-1} \Phi_{\xi,h}(\widehat{\mathcal{U}}; n, \ell+1) \boldsymbol{b}^T \boldsymbol{J}_{\xi,h}(\widehat{\boldsymbol{V}}_\ell) \mathcal{A} \boldsymbol{F}_\xi(\widehat{\boldsymbol{V}}_\ell), \\ \qquad\qquad n = 0, 1, \ldots, N, \\ \widehat{\boldsymbol{V}}_n = \boldsymbol{J}_{\xi,h}(\widehat{\boldsymbol{V}}_n) \boldsymbol{e} \Phi_{\xi,h}(\widehat{\mathcal{U}}; n, 0) P_\xi U_0 + h \boldsymbol{J}_{\xi,h}(\widehat{\boldsymbol{V}}_n) \mathcal{A} \boldsymbol{F}_\xi(\widehat{\boldsymbol{V}}_n) \\ \qquad + h \displaystyle\sum_{\ell=0}^{n-1} \boldsymbol{J}_{\xi,h}(\widehat{\boldsymbol{V}}_n) \boldsymbol{e} \Phi_{\xi,h}(\widehat{\mathcal{U}}; n, \ell+1) \boldsymbol{b}^T \boldsymbol{J}_{\xi,h}(\widehat{\boldsymbol{V}}_\ell) \mathcal{A} \boldsymbol{F}_\xi(\widehat{\boldsymbol{V}}_\ell), \\ \qquad\qquad n = 0, 1, \ldots, N-1, \end{cases} \tag{3.17}$$

where

$$\Phi_{\xi,h}(\widehat{\mathcal{U}}; n, m) = \begin{cases} 1, & n = m, \\ \left\{1 - h\boldsymbol{e}^T \mathcal{B} \boldsymbol{A}_\xi(\widehat{\boldsymbol{V}}_{n-1}) \boldsymbol{J}_{\xi,h}(\widehat{\boldsymbol{V}}_{n-1}) \boldsymbol{e}\right\} & \\ \quad \times \cdots \times \left\{1 - h\boldsymbol{e}^T \mathcal{B} \boldsymbol{A}_\xi(\widehat{\boldsymbol{V}}_m) \boldsymbol{J}_{\xi,h}(\widehat{\boldsymbol{V}}_m) \boldsymbol{e}\right\}, & n > m, \end{cases} \tag{3.18}$$

is the discrete evolution operator, and $\boldsymbol{J}_{\xi,h}(\widehat{\boldsymbol{V}}_n) = (\mathcal{I} + h\mathcal{A}\boldsymbol{A}_\xi(\widehat{\boldsymbol{V}}_n))^{-1}$.

□

We are now in a position to establish the error estimate. For $n = 0, 1, \ldots, N$, set $E_n = \widehat{U}_n - U(t_n)$, where $t_n = n \cdot h$.

Theorem 3.2 *Let U be a solution to* (3.1) *such that $U \in \mathcal{C}^{p+1}([0,S]; \mathbb{L}^2(\Omega)) \cap \mathcal{C}^{q'+1}([0,S]; \mathbb{H}^2(\Omega))$ with $q' = \min\{q, p-1\}$. Under* (RK1–2) *and* (G) *let $h_0 > 0$, $\xi_0 > 0$ and $S \in (0,T]$ be sufficiently small. Then, for any*

$0 < h < h_0$ *and* $0 < \xi < \xi_0$, *the errors are estimated by*

$$\begin{aligned}&\max_{n=0,1,\dots,N} \|\widehat{U}_n - U(t_n)\|_{\mathbb{H}^{1+\varepsilon}} \\ &\le C_U \Big[\sigma^{-1}\xi^{1-\varepsilon}\|U\|_{\mathcal{C}^\sigma([0,S];\mathbb{H}^2)} \\ &\qquad + h^p\|U^{(p+1)}\|_{\mathcal{C}([0,S];\mathbb{L}^2)} + h^{q'+1}\|U^{(q'+1)}\|_{\mathcal{C}([0,S];\mathbb{H}^2)}\Big], \quad (3.19)\end{aligned}$$

where $N \le S/h$. *Here,* $\sigma > 0$ *is any exponent and the constant* $C_U > 0$ *depends on* $\|U'\|_{\mathcal{C}([0,S];\mathbb{L}^2)}$ *and* $\|U\|_{\mathcal{C}([0,S];\mathbb{H}^2)}$ *but is independent of* h *and* ξ.

Proof. Write $E_n = P_\xi E_n + (1-P_\xi)E_n$, it is clear that $P_\xi E_n = \widehat{U}_n - P_\xi U(t_n)$ and $(1-P_\xi)E_n = -(1-P_\xi)U(t_n)$. The term $(1-P_\xi)U(t_n)$ was already estimated in Proposition 2.3, so we have only to estimate the first term $P_\xi E_n$.

The proof consists of five steps. In this proof, we shall use the following notations: $U_n = U(t_n)$, $V_{n,i} = U(t_n + c_i h)$, $\boldsymbol{V}_n = [V_{n,1}, \dots, V_{n,s}]$, $\widetilde{U}_n = P_\xi U_n$, $\widetilde{\boldsymbol{V}}_n = P_\xi \boldsymbol{V}_n$, $\widetilde{\mathcal{U}} = [\widetilde{U}_0, \dots, \widetilde{U}_N, \widetilde{\boldsymbol{V}}_0, \dots, \widetilde{\boldsymbol{V}}_{N-1}]$, $\widetilde{E}_n = P_\xi E_n = \widehat{U}_n - \widetilde{U}_n$, and $\widetilde{\boldsymbol{D}}_n = \widehat{\boldsymbol{V}}_n - \widetilde{\boldsymbol{V}}_n$.

Step 1. Representation formula for $\widetilde{E}_n$ and $\widetilde{\boldsymbol{D}}_n$.

We introduce the errors e_n and $\boldsymbol{d}_n$, $0 \le n \le N-1$, defined by

$$\begin{cases} U_{n+1} = U_n + h\boldsymbol{e}^T\mathcal{B}\{-\boldsymbol{A}(\boldsymbol{V}_n)\boldsymbol{V}_n + \boldsymbol{F}(\boldsymbol{V}_n)\} + e_n, \\ \boldsymbol{V}_n = \boldsymbol{e}U_n + h\mathcal{A}\{-\boldsymbol{A}(\boldsymbol{V}_n)\boldsymbol{V}_n + \boldsymbol{F}(\boldsymbol{V}_n)\} + \boldsymbol{d}_n. \end{cases} \tag{3.20}$$

Operating P_ξ, we observe that

$$\begin{cases} \widetilde{U}_{n+1} = \widetilde{U}_n + h\boldsymbol{e}^T\mathcal{B}\Big\{-\boldsymbol{A}_\xi(\widetilde{\boldsymbol{V}}_n)\widetilde{\boldsymbol{V}}_n + \boldsymbol{F}_\xi(\widetilde{\boldsymbol{V}}_n)\Big\} + h\boldsymbol{e}^T\mathcal{B}\tilde{\boldsymbol{c}}_n + P_\xi e_n, \\ \widetilde{\boldsymbol{V}}_n = \boldsymbol{e}\widetilde{U}_n + h\mathcal{A}\Big\{-\boldsymbol{A}_\xi(\widetilde{\boldsymbol{V}}_n)\widetilde{\boldsymbol{V}}_n + \boldsymbol{F}_\xi(\widetilde{\boldsymbol{V}}_n)\Big\} + h\mathcal{A}\tilde{\boldsymbol{c}}_n + P_\xi \boldsymbol{d}_n, \\ \widetilde{U}_0 = P_\xi U_0, \end{cases} \tag{3.21}$$

where

$$\tilde{\boldsymbol{c}}_n = P_\xi\Big\{-\boldsymbol{A}(\boldsymbol{V}_n)\boldsymbol{V}_n + \boldsymbol{F}(\boldsymbol{V}_n)\Big\} - \Big\{-\boldsymbol{A}_\xi(\widetilde{\boldsymbol{V}}_n)\widetilde{\boldsymbol{V}}_n + \boldsymbol{F}_\xi(\widetilde{\boldsymbol{V}}_n)\Big\}. \tag{3.22}$$

Subtracting (3.21) from (3.15), we obtain that

$$\begin{cases} \widetilde{E}_{n+1} = \widetilde{E}_n + h\boldsymbol{e}^T\mathcal{B}\Big\{-\boldsymbol{A}_\xi(\widetilde{\boldsymbol{V}}_n)\widetilde{\boldsymbol{D}}_n + \boldsymbol{G}_\xi(\widetilde{\boldsymbol{V}}_n + \widetilde{\boldsymbol{D}}_n, \widetilde{\boldsymbol{V}}_n)\Big\} \\ \qquad\qquad - h\boldsymbol{e}^T\mathcal{B}\tilde{\boldsymbol{c}}_n - P_\xi e_n, \\ \widetilde{\boldsymbol{D}}_n = \boldsymbol{e}\widetilde{E}_n + h\mathcal{A}\Big\{-\boldsymbol{A}_\xi(\widetilde{\boldsymbol{V}}_n)\widetilde{\boldsymbol{D}}_n + \boldsymbol{G}_\xi(\widetilde{\boldsymbol{V}}_n + \widetilde{\boldsymbol{D}}_n, \widetilde{\boldsymbol{V}}_n)\Big\} \\ \qquad\qquad - h\mathcal{A}\tilde{\boldsymbol{c}}_n - P_\xi \boldsymbol{d}_n, \\ \widetilde{E}_0 = 0, \end{cases} \tag{3.23}$$

where

$$\boldsymbol{G}_\xi(\widehat{\boldsymbol{V}}, \widehat{\boldsymbol{W}}) = -\Big\{\boldsymbol{A}_\xi(\widehat{\boldsymbol{V}}) - \boldsymbol{A}_\xi(\widehat{\boldsymbol{W}})\Big\}\widehat{\boldsymbol{V}} + \Big\{\boldsymbol{F}_\xi(\widehat{\boldsymbol{V}}) - \boldsymbol{F}_\xi(\widehat{\boldsymbol{W}})\Big\}.$$

The second equality of (3.23) can then be written as

$$\widetilde{\boldsymbol{D}}_n = \boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_n)\boldsymbol{e}\widetilde{E}_n + h\boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_n)\mathcal{A}\boldsymbol{G}_\xi(\widetilde{\boldsymbol{V}}_n + \widetilde{\boldsymbol{D}}_n, \widetilde{\boldsymbol{V}}_n) - \boldsymbol{\delta}^{(1)}_{\xi,h}(n) - \boldsymbol{\delta}^{(2)}_{\xi,h}(n), \tag{3.24}$$

where

$$\boldsymbol{\delta}^{(1)}_{\xi,h}(n) = h\boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_n)\mathcal{A}\tilde{\boldsymbol{c}}_n, \tag{3.25}$$

$$\boldsymbol{\delta}^{(2)}_{\xi,h}(n) = \boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_n)P_\xi\boldsymbol{d}_n. \tag{3.26}$$

Moreover, utilizing the discrete evolution operator $\Phi_{\xi,h}(\widetilde{\mathcal{U}};\cdot,\cdot)$, as in the proof of Theorem 3.1, we can represent $\widetilde{E}_n$ by

$$\widetilde{E}_n = h\sum_{\ell=0}^{n-1}\Phi_{\xi,h}(\widetilde{\mathcal{U}};n,\ell+1)\boldsymbol{b}^T\boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_\ell)\mathcal{A}\boldsymbol{G}_\xi(\widetilde{\boldsymbol{V}}_\ell + \widetilde{\boldsymbol{D}}_\ell, \widetilde{\boldsymbol{V}}_\ell) - \epsilon^{(1)}_{\xi,h}(n) + \epsilon^{(2)}_{\xi,h}(n), \tag{3.27}$$

where

$$\epsilon^{(1)}_{\xi,h}(n) = h\sum_{\ell=0}^{n-1}\Phi_{\xi,h}(\widetilde{\mathcal{U}};n,\ell+1)\boldsymbol{b}^T\boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_\ell)\mathcal{A}\tilde{\boldsymbol{c}}_\ell, \tag{3.28}$$

$$\epsilon^{(2)}_{\xi,h}(n) = \sum_{\ell=0}^{n-1}\Phi_{\xi,h}(\widetilde{\mathcal{U}};n,\ell+1)\Big\{h\boldsymbol{b}^T\boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_\ell)\mathcal{A}\boldsymbol{A}_\xi(\widetilde{\boldsymbol{V}}_\ell)P_\xi\boldsymbol{d}_\ell - P_\xi e_\ell\Big\}. \tag{3.29}$$

Step 2. Estimates of $\widetilde{E}_n$ and $\widetilde{\boldsymbol{D}}_n$ in terms of $\epsilon^{(i)}_{\xi,h}(n)$, $\boldsymbol{\delta}^{(i)}_{\xi,h}(n)$, $i = 1, 2$.

If ξ is sufficiently small, $\widetilde{U}(t) = P_\xi U(t)$ is in K_ξ for every $0 \le t \le S$. The Hölder continuity of $\widetilde{U}(\cdot)$ is also verified from Propositions 3.3, 3.4 and the assumption that $U \in \mathcal{C}^1([0,S]; X) \cap \mathcal{C}([0,S]; \mathcal{D})$; indeed, with the aid of (2.8),

$$\begin{aligned}
&\|\widetilde{U}(t) - \widetilde{U}(s)\|_{Z_\xi} \\
&\quad \le \hat{D}\|A_\xi(P_\xi U_0)^{\hat{\alpha}} P_\xi(U(t) - U(s))\|_X \\
&\quad \le C\|P_\xi(U(t) - U(s))\|_X^{1-\hat{\alpha}} \|A_\xi(P_\xi U_0) P_\xi(U(t) - U(s))\|_X^{\hat{\alpha}} \\
&\quad \le C_U |t - s|^{1-\hat{\alpha}}, \quad t, s \in [0, S].
\end{aligned}$$

Here $C_U > 0$ depends on $\|U'\|_{\mathcal{C}([0,S];X)}$ and $\|A(U)U\|_{\mathcal{C}([0,S];X)}$. This means that the Hölder conditions (4.16) and (4.17) in [16, Remark 4.1] hold for the family $\{A_\xi(\widetilde{U}_0), \dots, A_\xi(\widetilde{U}_N), \boldsymbol{A}_\xi(\widetilde{\boldsymbol{V}}_0), \dots, \boldsymbol{A}_\xi(\widetilde{\boldsymbol{V}}_{N-1})\}$; consequently, $\boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_n)$ and $\Phi_{\xi,h}(\widetilde{\mathcal{U}}; \cdot, \cdot)$ are defined as bounded operators in X_ξ and satisfy the same estimates established in Lemma 4.1, Propositions 4.2, 4.3 and 4.7 of [16]. In addition, the following estimates hold:

$$\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}} \boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_n)\|_{\mathcal{L}(X_\xi^s)} \le C_U h^{-\hat{\alpha}}, \tag{3.30}$$

$$\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}} \boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_n) A_\xi(\widetilde{U}_0)^{-\hat{\alpha}}\|_{\mathcal{L}(X_\xi^s)} \le C_U, \tag{3.31}$$

$$\begin{aligned}
&\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}} \Phi_{\xi,h}(\widetilde{\mathcal{U}}; n, m+1) \boldsymbol{b}^T \boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_m)\|_{\mathcal{L}(X_\xi^s, X_\xi)} \\
&\quad \le C_U((n-m)h)^{-\hat{\alpha}},
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
&\|\Phi_{\xi,h}(\widetilde{\mathcal{U}}; n, m+1) \boldsymbol{b}^T \mathcal{A} \boldsymbol{A}_\xi(\widetilde{\boldsymbol{V}}_m) \boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_m)\|_{\mathcal{L}(X_\xi^s, X_\xi)} \\
&\quad \le C_U((n-m)h)^{-1}.
\end{aligned} \tag{3.33}$$

The first and the second estimates are given by [16, Lemma 4.1] if we notice that

$$\begin{aligned}
A_\xi(\widetilde{U}_0)^{\hat{\alpha}} \boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_n) A_\xi(\widetilde{U}_0)^{-\hat{\alpha}} &= \boldsymbol{J}_{\xi,h}(\widetilde{U}_0) \\
&+ h A_\xi(\widetilde{U}_0)^{\hat{\alpha}} \boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_n) \mathcal{A}\{\mathcal{I} - \boldsymbol{A}_\xi(\widetilde{\boldsymbol{V}}_n) A_\xi(\widetilde{U}_0)^{-1}\} A_\xi(\widetilde{U}_0)^{1-\hat{\alpha}} \boldsymbol{J}_{\xi,h}(\widetilde{U}_0).
\end{aligned}$$

Applying (2.8), we verify (3.32) from [16, Propositions 4.2 and 4.7]. Finally (3.33) is obtained by the dual argument of [16, Proposition 4.7] in view of the second condition of Proposition 3.6.

Operating $A_\xi(\widetilde{U}_0)^{\hat{\alpha}}$ to (3.24) and using (3.30) and (3.31), we observe that

$$\begin{aligned}\|\widetilde{\boldsymbol{D}}_n\|_{Z^s} \le \hat{D}\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\widetilde{\boldsymbol{D}}_n\|_{X^s} &\le C_U\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\widetilde{E}_n\|_X \\ &\quad + C_U h^{1-\hat{\alpha}}\|\widetilde{\boldsymbol{D}}_n\|_{Z^s} + C_U\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\boldsymbol{\delta}^{(1)}_{\xi,h}(n)\|_{X^s} \\ &\quad + C_U\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\boldsymbol{\delta}^{(2)}_{\xi,h}(n)\|_{X^s}. \end{aligned}\tag{3.34}$$

Therefore, if h is sufficiently small, then

$$\begin{aligned}\|\widetilde{\boldsymbol{D}}_n\|_{Z^s} &\le C_U\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\widetilde{E}_n\|_X + C_U\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\boldsymbol{\delta}^{(1)}_{\xi,h}(n)\|_{X^s} \\ &\quad + C_U\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\boldsymbol{\delta}^{(2)}_{\xi,h}(n)\|_{X^s}. \end{aligned}\tag{3.35}$$

This shows also that $\widetilde{\boldsymbol{D}}_n$ is estimated by $\widetilde{E}_n$ as the solution of the equation (3.24).

On the other hand, using (3.32), we observe from (3.27) that

$$\begin{aligned}\|\widetilde{E}_n\|_Z &\le \hat{D}\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\widetilde{E}_n\|_X \\ &\le C_U h\sum_{\ell=0}^{n-1}((n-\ell)h)^{-\hat{\alpha}}\|\widetilde{\boldsymbol{D}}_\ell\|_{Z^s} \\ &\quad + C_U\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\epsilon^{(1)}_{\xi,h}(n)\|_X + C_U\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\epsilon^{(2)}_{\xi,h}(n)\|_X. \end{aligned}\tag{3.36}$$

This together with (3.35) yields a discrete integral inequality of Volterra type

$$\begin{aligned}\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\widetilde{E}_n\|_X &\le C_U h\sum_{\ell=0}^{n-1}((n-\ell)h)^{-\hat{\alpha}}\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\widetilde{E}_\ell\|_X \\ &\quad + C_U\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\epsilon^{(1)}_{\xi,h}(n)\|_X + C_U\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\epsilon^{(2)}_{\xi,h}(n)\|_X \\ &\quad + C_U h\sum_{\ell=0}^{n-1}((n-\ell)h)^{-\hat{\alpha}}\Big\{\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\boldsymbol{\delta}^{(1)}_{\xi,h}(\ell)\|_{X^s} + \|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\boldsymbol{\delta}^{(2)}_{\xi,h}(\ell)\|_{X^s}\Big\} \end{aligned}\tag{3.37}$$

with respect to $A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\widetilde{E}_n$. Thus we obtain (by [16, Proposition A.1]) that

$$\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\widetilde{E}_n\|_X \le C_U \max_{\ell=1,\dots,n}\Big\{\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\epsilon^{(1)}_{\xi,h}(\ell)\|_X + \|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\epsilon^{(2)}_{\xi,h}(\ell)\|_X\Big\}$$

$$+ C_U h \sum_{\ell=0}^{n-1} ((n-\ell)h)^{-\hat{\alpha}} \Big\{ \|A_\xi(\widetilde{U}_0)^{\hat{\alpha}} \boldsymbol{\delta}_{\xi,h}^{(1)}(\ell)\|_{X^s} + \|A_\xi(\widetilde{U}_0)^{\hat{\alpha}} \boldsymbol{\delta}_{\xi,h}^{(2)}(\ell)\|_{X^s} \Big\}$$
$$\leq C_U(\gamma_{\xi,h}^{(1)}(n) + \gamma_{\xi,h}^{(2)}(n)),$$

where

$$\gamma_{\xi,h}^{(i)}(n) = \max_{\ell=1,\dots,n} \|A_\xi(\widetilde{U}_0)^{\hat{\alpha}} \epsilon_{\xi,h}^{(i)}(\ell)\|_X + \max_{\ell=0,\dots,n-1} \|A_\xi(\widetilde{U}_0)^{\hat{\alpha}} \boldsymbol{\delta}_{\xi,h}^{(i)}(\ell)\|_{X^s}, \quad i = 1, 2. \tag{3.38}$$

Hence we conclude that

$$\|\widetilde{E}_n\|_Z \leq \hat{D} \|A_\xi(\widetilde{U}_0)^{\hat{\alpha}} \widetilde{E}_n\|_X \leq C_U(\gamma_{\xi,h}^{(1)}(n) + \gamma_{\xi,h}^{(2)}(n)). \tag{3.39}$$

Step 3. Estimate of $\gamma_{\xi,h}^{(1)}(n)$.

First let us estimate $\epsilon_{\xi,h}^{(1)}(n)$. Inserting a term $\boldsymbol{A}_\xi(\widetilde{\boldsymbol{V}}_n)\boldsymbol{A}_\xi(\boldsymbol{V}_n)^{-1} P_\xi \boldsymbol{A}(\boldsymbol{V}_n)\boldsymbol{V}_n$, we write $\tilde{\boldsymbol{c}}_n$ in (3.22) in the form

$$\tilde{\boldsymbol{c}}_n = \boldsymbol{H}_\xi(\boldsymbol{V}_n) + \boldsymbol{A}_\xi(\widetilde{\boldsymbol{V}}_n)\boldsymbol{L}_\xi(\boldsymbol{V}_n),$$

where

$$\begin{aligned} \boldsymbol{H}_\xi(\boldsymbol{V}) &= \left\{ \boldsymbol{A}_\xi(P_\xi \boldsymbol{V})\boldsymbol{A}_\xi(\boldsymbol{V})^{-1} - \mathcal{I} \right\} P_\xi \boldsymbol{A}(\boldsymbol{V})\boldsymbol{V} \\ &\qquad - \left\{ \boldsymbol{F}_\xi(P_\xi \boldsymbol{V}) - \boldsymbol{F}_\xi(\boldsymbol{V}) \right\}, \\ \boldsymbol{L}_\xi(\boldsymbol{V}) &= \left\{ P_\xi - \boldsymbol{A}_\xi(\boldsymbol{V})^{-1} P_\xi \boldsymbol{A}(\boldsymbol{V}) \right\} \boldsymbol{V} = P_\xi \left\{ \mathcal{I} - \boldsymbol{R}_\xi(\boldsymbol{V}) \right\} \boldsymbol{V}, \end{aligned}$$

and $\boldsymbol{R}_\xi(\boldsymbol{V}) = [R_\xi(V_1), \dots, R_\xi(V_s)]^T$ for $\boldsymbol{V} = [V_1, \dots, V_s]^T$,

$$R_\xi(U) = A_\xi(U)^{-1} P_\xi A(U) \tag{3.40}$$

being the so-called Ritz projection defined on $\mathcal{D}(A(U))$. Accordingly $\epsilon_{\xi,h}^{(1)}(n)$ in (3.28) is divided into $\epsilon_{\xi,h}^{(1)}(n) = \epsilon_{\xi,h}^{(1a)}(n) + \epsilon_{\xi,h}^{(1b)}(n)$ with

$$\epsilon_{\xi,h}^{(1a)}(n) = h \sum_{\ell=0}^{n-1} \varPhi_{\xi,h}(\widetilde{\mathcal{U}}; n, \ell+1) \boldsymbol{b}^T \boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_\ell) \mathcal{A} \boldsymbol{H}_\xi(\boldsymbol{V}_\ell),$$
$$\epsilon_{\xi,h}^{(1b)}(n) = h \sum_{\ell=0}^{n-1} \varPhi_{\xi,h}(\widetilde{\mathcal{U}}; n, \ell+1) \boldsymbol{b}^T \boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_\ell) \mathcal{A} \boldsymbol{A}_\xi(\widetilde{\boldsymbol{V}}_\ell) \boldsymbol{L}_\xi(\boldsymbol{V}_\ell).$$

Then, using (3.32) and Propositions 3.6, 3.7, 3.4 and 2.3, we verify that

$$
\begin{aligned}
&\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\epsilon^{(1a)}_{\xi,h}(n)\|_X \\
&\le \left\| h\sum_{\ell=0}^{n-1} A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\varPhi_{\xi,h}(\widetilde{\mathcal{U}};n,\ell+1)\boldsymbol{b}^T\boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_\ell)\mathcal{A}\{\boldsymbol{F}_\xi(\widetilde{\boldsymbol{V}}_\ell)-\boldsymbol{F}_\xi(\boldsymbol{V}_\ell)\}\right\|_X \\
&\quad + \left\| h\sum_{\ell=0}^{n-1} A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\varPhi_{\xi,h}(\widetilde{\mathcal{U}};n,\ell+1)\boldsymbol{b}^T\boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_\ell)\mathcal{A}\right. \\
&\qquad\qquad \left. \times\{\boldsymbol{A}_\xi(\widetilde{\boldsymbol{V}}_\ell)\boldsymbol{A}_\xi(\boldsymbol{V}_\ell)^{-1}-\mathcal{I}\}P_\xi\boldsymbol{A}(\boldsymbol{V}_\ell)\boldsymbol{V}_\ell\right\|_X \\
&\le C_U h\sum_{\ell=0}^{n-1}((n-\ell)h)^{-\hat{\alpha}}\|\widetilde{\boldsymbol{V}}_\ell-\boldsymbol{V}_\ell\|_{Z^s}\{\hat{L}_F+\hat{L}_A\|P_\xi\boldsymbol{A}(\boldsymbol{V}_\ell)\boldsymbol{V}_\ell\|_{X^s}\} \\
&\le C_U\|(1-P_\xi)U\|_{C([0,S];Z)} \le C_U\xi^{2-2\hat{\alpha}}\|A(U)U\|_{C([0,S];X)}.
\end{aligned}
$$

On the other hand, it is seen that

$$
\begin{aligned}
\epsilon^{(1b)}_{\xi,h}(n) &= h\sum_{\ell=0}^{n-1}\varPhi_{\xi,h}(\widetilde{\mathcal{U}};n,\ell+1)\boldsymbol{b}^T\boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_\ell)\mathcal{A}\boldsymbol{A}_\xi(\widetilde{\boldsymbol{V}}_\ell)P_\xi\{\mathcal{I}-\boldsymbol{R}_\xi(\boldsymbol{V}_\ell)\}\boldsymbol{V}_\ell \\
&= h\sum_{\ell=0}^{n-1}\varPhi_{\xi,h}(\widetilde{\mathcal{U}};n,\ell+1)\boldsymbol{b}^T\boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_\ell)\mathcal{A}\boldsymbol{A}_\xi(\widetilde{\boldsymbol{V}}_\ell) \\
&\qquad\qquad \times P_\xi\Big[\{\mathcal{I}-\boldsymbol{R}_\xi(\boldsymbol{V}_\ell)\}\boldsymbol{V}_\ell-\boldsymbol{e}\{1-R_\xi(U_n)\}U_n\Big] \\
&\quad + \{1-\varPhi_{\xi,h}(\widetilde{\mathcal{U}};n,0)\}P_\xi\{1-R_\xi(U_n)\}U_n,
\end{aligned}
$$

here we used the equality

$$
1-\varPhi_{\xi,h}(\widetilde{\mathcal{U}};n,0) = h\sum_{\ell=0}^{n-1}\varPhi_{\xi,h}(\widetilde{\mathcal{U}};n,\ell+1)\boldsymbol{b}^T\boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_\ell)\mathcal{A}\boldsymbol{A}_\xi(\widetilde{\boldsymbol{V}}_\ell)\boldsymbol{e}.
$$

Now we need the following three propositions, the proof of which will be given in the next section.

Proposition 3.8 *$A(\cdot)$ satisfies the Lipschitz condition*

$$
\|\{A(U)-A(V)\}A(V)^{-1}\|_{\mathcal{L}(X)} \le L_A\|U-V\|_Z, \quad U,V\in K
$$

with some constant L_A.

Proposition 3.9

$$\|A_\xi(U)\|_{\mathcal{L}(X_\xi)} \le \hat{N}_A\,\xi^{-2}, \quad U \in K$$

with some constant $\hat{N}_A$ independent of ξ.

Proposition 3.10 *$R_\xi(U) = A_\xi(U)^{-1}P_\xi A(U)$ satisfies that*

$$\|\{1 - R_\xi(U)\}A(U)^{-1}\|_{\mathcal{L}(X)} \le \hat{M}_R\,\xi^2, \quad U \in K,$$

$$\|\{R_\xi(U) - R_\xi(V)\}A(V)^{-1}\|_{\mathcal{L}(X)} \le \hat{L}_R\,\xi^2\|U - V\|_Z, \quad U, V \in K$$

with some constants $\hat{M}_R$ and $\hat{L}_R$ independent of ξ.

Proposition 3.9 implies that

$$\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\|_{\mathcal{L}(X)} \le C\xi^{-2\hat{\alpha}}.$$

Similarly, Proposition 3.10 joined with Proposition 3.4 implies that

$$\|P_\xi\{1 - R_\xi(U_n)\}U_n\|_X \le C\xi^2\|A(U_n)U_n\|_X.$$

Moreover,

$$\begin{aligned}
&\big\|P_\xi\big[\{1 - \boldsymbol{R}_\xi(\boldsymbol{V}_\ell)\}\boldsymbol{V}_\ell - \boldsymbol{e}\{1 - R_\xi(U_n)\}U_n\big]\big\|_{X^s}\\
&\le \|P_\xi\{1 - R_\xi(U_n)\}A(U_n)^{-1}\{\boldsymbol{A}(\boldsymbol{V}_\ell)\boldsymbol{V}_\ell - \boldsymbol{e}A(U_n)U_n\}\|_{X^s}\\
&\quad + \|P_\xi\{1 - R_\xi(U_n)\}\boldsymbol{A}(\boldsymbol{V}_\ell)^{-1}\{\boldsymbol{A}(\boldsymbol{V}_\ell)A(U_n)^{-1} - \mathcal{I}\}\boldsymbol{A}(\boldsymbol{V}_\ell)\boldsymbol{V}_\ell\|_{X^s}\\
&\quad + \big\|P_\xi\{\boldsymbol{R}_\xi(\boldsymbol{V}_\ell) - \mathcal{I}R_\xi(U_n)\}\boldsymbol{V}_\ell\|_{X^s}\\
&\le C\xi^2\Big[((n-\ell)h)^\sigma\|A(U)U\|_{\mathcal{C}^\sigma([0,S];X)}\\
&\qquad\qquad + ((n-\ell)h)^{\frac{1-\varepsilon}{2}}\|U\|_{\mathcal{C}^{\frac{1-\varepsilon}{2}}([0,S];Z)}\|A(U)U\|_{\mathcal{C}([0,S];X)}\Big].
\end{aligned}$$

Here, the fact $U \in \mathcal{C}^{\frac{1-\varepsilon}{2}}([0,S];Z)$ and $A(U)U \in \mathcal{C}^\sigma([0,S];X)$ are verified from $U \in \mathcal{C}^1([0,S];X)\cap\mathcal{C}^\sigma([0,S];\mathcal{D})$, $Z = \mathbb{H}^{1+\varepsilon}(\Omega) = [X,\mathcal{D}]_{\frac{1+\varepsilon}{2}}$ and Proposition 3.8. Utilizing these estimates and (3.33), we readily verify that

$$\begin{aligned}
&\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\epsilon_{\xi,h}^{(1b)}(n)\|_X\\
&\quad\le C_U h\sum_{\ell=0}^{n-1}\xi^{-2\hat{\alpha}}((n-\ell)h)^{-1}\xi^2((n-\ell)h)^\sigma\|A(U)U\|_{\mathcal{C}^\sigma([0,S];X)}\\
&\qquad + C_U\xi^{-2\hat{\alpha}}\xi^2\|A(U_n)U_n\|_X \le C_U\sigma^{-1}\xi^{2-2\hat{\alpha}}\|A(U)U\|_{\mathcal{C}^\sigma([0,S];X)},
\end{aligned}$$

where $\sigma \in \left(0, \frac{1-\varepsilon}{2}\right]$ is an arbitrary exponent. Therefore, since $\hat{\alpha} = \frac{1+\varepsilon}{2}$, we conclude that

$$\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}} \epsilon^{(1)}_{\xi,h}(n)\|_X \leq C_U \sigma^{-1} \xi^{1-\varepsilon} \|A(U)U\|_{\mathcal{C}^\sigma([0,S];X)}.$$

The term $\boldsymbol{\delta}^{(1)}_{\xi,h}(n)$ is also estimated in the same way as for $\epsilon^{(1)}_{\xi,h}(n)$. Indeed, it is divided into $\boldsymbol{\delta}^{(1)}_{\xi,h}(n) = \boldsymbol{\delta}^{(1a)}_{\xi,h}(n) + \boldsymbol{\delta}^{(1b)}_{\xi,h}(n)$ with

$$\begin{aligned}
\boldsymbol{\delta}^{(1a)}_{\xi,h}(n) &= h \boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_n) \mathcal{A} \boldsymbol{H}_\xi(\boldsymbol{V}_n), \\
\boldsymbol{\delta}^{(1b)}_{\xi,h}(n) &= h \boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_n) \mathcal{A} \boldsymbol{A}_\xi(\widetilde{\boldsymbol{V}}_n) \boldsymbol{L}_\xi(\boldsymbol{V}_n).
\end{aligned}$$

Then the two terms are estimated by

$$\begin{aligned}
\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}} \boldsymbol{\delta}^{(1a)}_{\xi,h}(n)\|_{X^s} &\leq C_U h^{1-\hat{\alpha}} \|(1-P_\xi)U\|_{\mathcal{C}([0,S];Z)} \\
&\leq C_U h^{1-\hat{\alpha}} \xi^{2-2\hat{\alpha}} \|A(U)U\|_{\mathcal{C}([0,S];X)},
\end{aligned}$$

and by

$$\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}} \boldsymbol{\delta}^{(1b)}_{\xi,h}(n)\|_{X^s} \leq C_U \xi^{2-2\hat{\alpha}} \|A(U)U\|_{\mathcal{C}([0,S];X)},$$

respectively. Hence,

$$\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}} \boldsymbol{\delta}^{(1)}_{\xi,h}(n)\|_{X^s} \leq C_U \xi^{1-\varepsilon} \|A(U)U\|_{\mathcal{C}([0,S];X)}.$$

We have thus proved that

$$\gamma^{(1)}_{\xi,h}(n) \leq C_U \sigma^{-1} \xi^{1-\varepsilon} \|A(U)U\|_{\mathcal{C}^\sigma([0,S];X)}. \tag{3.41}$$

Step 4. Estimate of $\gamma^{(2)}_{\xi,h}(n)$.

We have from (3.29)

$$\begin{aligned}
\epsilon^{(2)}_{\xi,h}(n) = h \sum_{\ell=0}^{n-1} & \Phi_{\xi,h}(\widetilde{\mathcal{U}}; n, \ell+1) \boldsymbol{b}^T \boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_\ell) \mathcal{A} \boldsymbol{A}_\xi(\widetilde{\boldsymbol{V}}_\ell) P_\xi \boldsymbol{d}_\ell \\
& - \sum_{\ell=0}^{n-1} r(\infty)^{n-\ell-1} P_\xi e_\ell \\
& - \sum_{\ell=0}^{n-2} \sum_{\ell'=\ell+1}^{n-1} \Phi_{\xi,h}(\widetilde{\mathcal{U}}; n, \ell'+1) \boldsymbol{b}^T \boldsymbol{J}_{\xi,h}(\widetilde{\boldsymbol{V}}_{\ell'}) \boldsymbol{e} r(\infty)^{\ell'-\ell-1} P_\xi e_\ell.
\end{aligned}$$

Thus we verify from (3.32) and Proposition 3.4 that

$$\begin{aligned}
&\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\epsilon^{(2)}_{\xi,h}(n)\|_X \\
&\le C_U h\sum_{\ell=0}^{n-1}((n-\ell+1)h)^{-\hat{\alpha}}\|\boldsymbol{A}_\xi(\widetilde{\boldsymbol{V}}_\ell)P_\xi \boldsymbol{d}_\ell\|_{X^s} \\
&\quad+\sum_{\ell=0}^{n-1}|r(\infty)|^{n-\ell-1}\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}P_\xi e_\ell\|_X \\
&\quad+C_U h\sum_{\ell'=0}^{n-1}((n-\ell')h)^{-\hat{\alpha}}\sum_{\ell=0}^{\ell'-1}|r(\infty)|^{\ell'-\ell-1}\|P_\xi e_\ell/h\|_X \\
&\le C_U\max_\ell\|A(U_0)\boldsymbol{d}_\ell\|_{X^s}+C_U\max_\ell\|A(U_0)e_\ell\|_X+C_U\max_\ell\|e_\ell/h\|_X,
\end{aligned}$$

since $|r(\infty)|<1$. In the same way the norm of $\boldsymbol{\delta}^{(2)}_{\xi,h}(n)$ defined by (3.26) is estimated by

$$\|A_\xi(\widetilde{U}_0)^{\hat{\alpha}}\boldsymbol{\delta}^{(2)}_{\xi,h}(n)\|_{X^s}\le C_U\|A_\xi(\widetilde{U}_0)P_\xi\boldsymbol{d}_n\|_{X^s}\le C_U\|A(U_0)\boldsymbol{d}_n\|_{X^s}.$$

So it is enough to estimate e_n and $\boldsymbol{d}_n$. But it is already known (cf. [16, (3.16) and (3.17)]) that e_n and $\boldsymbol{d}_n$ are represented by

$$\begin{aligned}
e_n&=\int_0^h\Big(\frac{(h-t)^q}{q!}U^{(q+1)}(t_n+t)\\
&\qquad-\frac{h(h-t)^{q-1}}{(q-1)!}\boldsymbol{e}^T\mathcal{B}\mathcal{C}^q\boldsymbol{V}^{(q+1)}((t_n\mathcal{I}+t\mathcal{C})\boldsymbol{e})\Big)dt\\
&=\int_0^h\Big(\frac{(h-t)^p}{p!}U^{(p+1)}(t_n+t)\\
&\qquad-\frac{h(h-t)^{p-1}}{(p-1)!}\boldsymbol{e}^T\mathcal{B}\mathcal{C}^p\boldsymbol{V}^{(p+1)}((t_n\mathcal{I}+t\mathcal{C})\boldsymbol{e})\Big)dt,\\
\boldsymbol{d}_n&=\int_0^h\Big(\frac{(h-t)^q}{q!}\mathcal{C}^{q+1}-\frac{h(h-t)^{q-1}}{(q-1)!}\mathcal{A}\mathcal{C}^q\Big)\boldsymbol{V}^{(q+1)}((t_n\mathcal{I}+t\mathcal{C})\boldsymbol{e})dt,
\end{aligned}$$

respectively. As a consequence,

$$\|e_n/h\|_X\le Ch^p\|U^{(p+1)}\|_{\mathcal{C}([0,S];X)},$$

$$\|A(U_0)e_n\|_X,\ \|A(U_0)\boldsymbol{d}_n\|_{X^s}\le Ch^{q'+1}\|A(U_0)U^{(q'+1)}\|_{\mathcal{C}([0,S];X)}.$$

Therefore we obtain the estimate

$$\gamma_{\xi,h}^{(2)}(n) \leq C_U h^p \|U^{(p+1)}\|_{C([0,S];X)} + C_U h^{q'+1} \|A(U_0)U^{(q'+1)}\|_{C([0,S];X)}. \tag{3.42}$$

Step 5. Completion of the proof.

From (3.39), (3.41) and (3.42) we have

$$\|\widetilde{E}_n\|_Z \leq C_U \sigma^{-1} \xi^{1-\varepsilon} \|A(U)U\|_{C^\sigma([0,S];X)} + C_U h^p \|U^{(p+1)}\|_{C([0,S];X)} + C_U h^{q'+1} \|A(U_0)U^{(q'+1)}\|_{C([0,S];X)}.$$

Since $\|E_n\|_Z \leq \|\widetilde{E}_n\|_Z + \|(1-P_\xi)U(t_n)\|_Z$, the desired estimate (3.19) then follows from Proposition 2.3. □

Remark 3.2 When $c(u) \equiv 0$, it is easily seen that, the function u of the solution $U = \begin{bmatrix} u \\ \rho \end{bmatrix}$ to (3.1) has the conservation law of L^1-norm, that is, $\int_\Omega u(t)dx \equiv \int_\Omega u_0 dx$ for all $0 \leq t \leq S$. Accordingly, also $\hat{u}_n$, where $\widehat{U}_n = \begin{bmatrix} \hat{u}_n \\ \hat{\rho}_n \end{bmatrix}$, has the same law. Indeed, from the first equation of (3.15) we see that

$$\hat{u}_{n+1} = \hat{u}_n + h \sum_{j=1}^{s} b_j \left\{ -A_{1\xi} \hat{v}_{n,j} + A_{3\xi}(\hat{v}_{n,j}, \hat{\mu}_{n,j}) \hat{\mu}_{n,j} + F_{1\xi}(\hat{v}_{n,j}, \hat{\mu}_{n,j}) \right\}.$$

Then, since $c(u) \equiv 0$, (3.13) and (3.4) imply that

$$\begin{aligned} \int_\Omega \hat{u}_{n+1} dx - \int_\Omega \hat{u}_n dx &= \langle \hat{u}_{n+1} - \hat{u}_n, 1 \rangle_{L^2} \\ &= h \sum_{j=1}^{s} b_j \left\{ a \langle \nabla \hat{v}_{n,j}, \nabla 1 \rangle_{L^2} - \langle \operatorname{Re} \hat{v}_{n,j} b(\operatorname{Re} \hat{\mu}_{n,j}) \nabla \hat{\mu}_{n,j}, \nabla 1 \rangle_{L^2} \right\} \\ &= 0, \end{aligned}$$

therefore $\int_\Omega \hat{u}_n dx \equiv \int_\Omega \hat{u}_0 dx$ for all n.

Remark 3.3 From (2.5) the estimate (3.19) provides automatically the pointwise error estimate. Since the positivity of solution to (3.1) is known (cf. [30, Theorem 2.1]), this shows that also the approximate solution $\hat{u}_n$, $\hat{\rho}_n$ can be positive if ξ and h are sufficiently small.

Remark 3.4 The backward Euler scheme, that is a Runge-Kutta scheme satisfying (RK1–2) with $s = p = q = 1$ and $q' = 0$, is seen to have the conservation law of positivity; in fact this can be proved by the truncation method as in [30, Theorem 2.1].

Remark 3.5 The order of convergence in spatial discretization is $2(1 - \hat{\alpha}) = 1 - \varepsilon$. In view of Propositions 2.3 and 2.4, this exponent is optimal.

Remark 3.6 The constant S in Theorems 3.1 and 3.2, which gives the interval of existence of the solution $\widehat{U}_n$ to (3.15), is determined by the initial data and constants appearing in Propositions 3.1–3.7 and conditions (G) and (RK1). To study the prolongation of $\widehat{U}_n$ beyond the time S, we have to establish some a priori estimates for $\widehat{U}_n$.

4. Proofs of propositions

This section is devoted to describing the proof of all the propositions announced in the preceding section.

First we note that Propositions 3.1 and 3.2 are direct consequences of Propositions 2.1, 2.3 and the assumption that $U_0 \in \mathbb{H}^2(\Omega)$.

Proof of Proposition 3.7. The Lipschitz condition on $F_\xi(\cdot)$ is verified directly from

$$\|F(U) - F(V)\|_X \leq L_F \|U - V\|_Z, \quad U, V \in K.$$

Since $c(u)$ is locally Lipschitz and satisfies

$$\begin{aligned}\|c(u) - c(v)\|_{L^2} &= \left\| \int_0^1 c'((1-\omega)u + \omega v)d\omega\,(u - v) \right\|_{L^2} \\ &\leq C\|u - v\|_{H^{1+\varepsilon}},\end{aligned}$$

this condition is obvious from (3.4). □

Proof of Proposition 3.8. The Lipschitz condition on $A(\cdot)$ is also verified by a direct calculation in view of $\mathcal{D}(A(U)) \subset \mathbb{H}^2(\Omega)$, see [30, p.245]. □

Proof of Proposition 3.5. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}^+$ and $U = \begin{bmatrix} u \\ \rho \end{bmatrix} \in K$. Since $A_{1\xi}$ and $A_{2\xi}$ are positive definite self-adjoint operators on $\mathcal{C}_\xi(\overline{\Omega})$, it is easily observed that $\lambda - A_\xi(U)$ has a bounded inverse on X_ξ

$$(\lambda - A_\xi(U))^{-1} = \begin{bmatrix} (\lambda - A_{1\xi})^{-1} & -(\lambda - A_{1\xi})^{-1}A_{3\xi}(U)(\lambda - A_{2\xi})^{-1} \\ 0 & (\lambda - A_{2\xi})^{-1} \end{bmatrix}.$$

For a $0 < \hat{\varphi} < \frac{\pi}{2}$ fixed arbitrarily, it holds that

$$\|(\lambda - A_{1\xi})^{-1}\|_{\mathcal{L}(\mathcal{C}_\xi(\overline{\Omega}))} + \|(\lambda - A_{2\xi})^{-1}\|_{\mathcal{L}(\mathcal{C}_\xi(\overline{\Omega}))} \leq \frac{C}{|\lambda| + 1}, \quad \lambda \notin S_{\hat{\varphi}}.$$

On the other hand, we verify that

$$\|A_{3\xi}(U)A_{2\xi}^{-1}\|_{\mathcal{L}(\mathcal{C}_\xi(\overline{\Omega}))} \leq C$$

uniformly in U and ξ. In fact, using $R_{2\xi} = A_{2\xi}^{-1}P_\xi A_2$, it is seen that

$$\begin{aligned} \langle A_{3\xi}(U)A_{2\xi}^{-1}\hat{f}, \hat{v}\rangle_{L^2} &= \langle \operatorname{Re} u\, b(\operatorname{Re}\rho)\nabla\{R_{2\xi} - 1\}A_2^{-1}\hat{f}, \nabla\hat{v}\rangle_{L^2} \\ &\quad + \langle \nabla \cdot \{\operatorname{Re} u\, b(\operatorname{Re}\rho)\nabla A_2^{-1}\hat{f}\}, \hat{v}\rangle_{L^2}. \end{aligned}$$

By (2.3), (2.5), (2.24) and (2.31),

$$\begin{aligned} |\langle A_{3\xi}(U)A_{2\xi}^{-1}\hat{f}, \hat{v}\rangle_{L^2}| &\leq C\xi\|\operatorname{Re} u\, b(\operatorname{Re}\rho)\|_{\mathcal{C}}\|\hat{f}\|_{L^2}\|\hat{v}\|_{H^1} \\ &\quad + \Big\{\|\nabla(\operatorname{Re} u\, b(\operatorname{Re}\rho))\|_{L^{\frac{2}{1-\varepsilon}}}\|\nabla A_2^{-1}\hat{f}\|_{L^{\frac{2}{\varepsilon}}} \\ &\qquad + \|\operatorname{Re} u\, b(\operatorname{Re}\rho)\|_{\mathcal{C}}\|\nabla A_2^{-1}\hat{f}\|_{H^1}\Big\}\|\hat{v}\|_{L^2} \\ &\leq C\|\operatorname{Re} u\, b(\operatorname{Re}\rho)\|_{H^{1+\varepsilon}}\|\hat{f}\|_{L^2}\|\hat{v}\|_{L^2}, \quad \hat{f}, \hat{v} \in \mathcal{C}_\xi(\overline{\Omega}). \end{aligned}$$

Hence Proposition 3.5 is proved. $\square$

Proof of Proposition 3.6. Let $\widehat{F} = \begin{bmatrix} \hat{f} \\ \hat{g} \end{bmatrix}$, $\widehat{F}' = \begin{bmatrix} \hat{f}' \\ \hat{g}' \end{bmatrix} \in X_\xi$. We see that

$$\begin{aligned} &\langle\{A_\xi(U) - A_\xi(V)\}A_\xi(V)^{-1}\widehat{F}, \widehat{F}'\rangle_{\mathbb{L}^2} \\ &\quad = -\langle\{A_{3\xi}(U) - A_{3\xi}(V)\}A_{2\xi}^{-1}\hat{g}, \hat{f}'\rangle_{L^2} \\ &\quad = -\langle\{\operatorname{Re} u\, b(\operatorname{Re}\rho) - \operatorname{Re} v\, b(\operatorname{Re}\mu)\}\nabla R_{2\xi}A_2^{-1}\hat{g}, \nabla\hat{f}'\rangle_{L^2}. \end{aligned}$$

Arguing in the same way as above, we verify that

$$\begin{aligned} &|\langle\{A_\xi(U) - A_\xi(V)\}A_\xi(V)^{-1}\widehat{F}, \widehat{F}'\rangle_{\mathbb{L}^2}| \\ &\quad \leq C\|\operatorname{Re} u\, b(\operatorname{Re}\rho) - \operatorname{Re} v\, b(\operatorname{Re}\mu)\|_{H^{1+\varepsilon}}\|\hat{g}\|_{L^2}\|\hat{f}'\|_{L^2} \\ &\quad \leq C\|U - V\|_{\mathbb{H}^{1+\varepsilon}}\|\widehat{F}\|_{\mathbb{L}^2}\|\widehat{F}'\|_{\mathbb{L}^2}. \end{aligned}$$

Similarly,

$$\begin{aligned}
&|\langle A_\xi(U)^{-1}\{A_\xi(U)-A_\xi(V)\}\widehat{F},\widehat{F}'\rangle_{\mathbb{L}^2}| \\
&\quad = |\langle \hat{g},\{A_{3\xi}(U)^*-A_{3\xi}(V)^*\}A_{1\xi}^{-1}\hat{f}'\rangle_{L^2}| \\
&\quad \le C\|\operatorname{Re} u\, b(\operatorname{Re}\rho)-\operatorname{Re} v\, b(\operatorname{Re}\mu)\|_{H^{1+\epsilon}}\|\hat{g}\|_{L^2}\|\hat{f}'\|_{L^2} \\
&\quad \le C\|U-V\|_{\mathbb{H}^{1+\epsilon}}\|\widehat{F}\|_{\mathbb{L}^2}\|\widehat{F}'\|_{\mathbb{L}^2}.
\end{aligned}$$

These complete the proof. □

Proof of Proposition 3.9. It is easily verified by (2.24) that

$$\begin{aligned}
\langle A_\xi(U)\widehat{V},\widehat{W}\rangle_{\mathbb{L}^2} &= a(U;\widehat{V},\widehat{W}) \le C\|\widehat{V}\|_{\mathbb{H}^1}\|\widehat{W}\|_{\mathbb{H}^1} \\
&\le C\xi^{-2}\|\widehat{V}\|_{\mathbb{L}^2}\|\widehat{W}\|_{\mathbb{L}^2}, \quad \widehat{V},\widehat{W}\in X_\xi
\end{aligned}$$

(cf. [7, Proposition 3.1]). □

By Proposition 3.5 proved above, $A_\xi(U)$ is an operator of positive type; therefore, the fractional powers of $A_\xi(U)$ are defined. With the aid of (2.8) it is seen from Proposition 3.9 that

$$\|A_\xi(U)^\theta\|_{\mathcal{L}(X_\xi)} \le C_\theta \xi^{-2\theta}, \quad 0\le\theta\le 1, \tag{4.1}$$

C_θ being independent of $U\in K$ and ξ. This has already been used in the proof of Theorem 3.2.

Before proving other propositions, we shall prepare two lemmas.

Lemma 4.1 *$A(U)$, $U\in K$, is of positive type and satisfies*

$$\|(\lambda-A(U))^{-1}\|_{\mathcal{L}(X)} \le \frac{M_A}{|\lambda|+1}, \quad \lambda\le 0, \tag{4.2}$$

with some constant M_A independent of U. $A(U)$ and its adjoint $A(U)^*$ have the same domain

$$\mathcal{D}(A(U)) = \mathcal{D}(A(U)^*) = \mathbb{H}^2_N(\Omega) \tag{4.3}$$

with norm equivalence which is uniform in U. Therefore,

$$\|A(U)^{it}\|_{\mathcal{L}(X)} \le \mu_A e^{\pi|t|}, \quad t\in\mathbb{R}, \tag{4.4}$$

with some constant $\mu_A>0$ independent of U. Finally, for $0\le\theta<\frac{3}{4}$,

$$\mathcal{D}(A(U)^\theta) = \mathcal{D}(A(U)^{*\theta}) = \mathbb{H}^{2\theta}(\Omega) \tag{4.5}$$

with norm equivalence which is uniform in U.

Proof of the lemma. Let $\lambda \le 0$ and $U \in K$. It is easily seen that

$$(\lambda - A(U))^{-1} = \begin{bmatrix} (\lambda - A_1)^{-1} & -(\lambda - A_1)^{-1} A_3(U)(\lambda - A_2)^{-1} \\ 0 & (\lambda - A_2)^{-1} \end{bmatrix}.$$

Then the estimate (4.2) is verified in the same way as in [30, p.249], and hence $A(U)$ is of positive type.

For each $U \in K$, $A(U)^*$ is given by

$$\begin{cases} \mathcal{D}(A(U)^*) = \mathcal{D} = H_N^2(\Omega) \times H_N^2(\Omega), \\ A(U)^* \widetilde{U} = \begin{bmatrix} A_1 \tilde{u} \\ -A_3(U)^* \tilde{u} + A_2 \tilde{\rho} \end{bmatrix}, \quad \widetilde{U} = \begin{bmatrix} \tilde{u} \\ \tilde{\rho} \end{bmatrix}. \end{cases} \tag{4.6}$$

Therefore we easily see that

$$\|A(U)A(U)^{*-1}\|_{\mathcal{L}(X)}, \; \|A(U)^* A(U)^{-1}\|_{\mathcal{L}(X)} \le C, \quad U \in K. \tag{4.7}$$

Hence (4.3) is verified. As mentioned in Section 2, (4.7) implies (4.4) generally. On the other hand, let $B = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ be a self-adjoint operator in X with the domain $\mathcal{D}(B) = \mathbb{H}_N^2(\Omega)$. Then, by virtue of the Heinz-Kato type inequality (2.11) (with $T =$ identity), it follows that $\mathcal{D}(A(U)^\theta) = \mathcal{D}(B^\theta)$ for $0 \le \theta \le 1$. But, from (2.6), $\mathcal{D}(B^\theta) = \mathbb{H}^{2\theta}(\Omega)$ for $0 \le \theta < \frac{3}{4}$. Hence (4.5) is proved. □

Lemma 4.2 *For* $0 \le s < \frac{3}{2}$,

$$\|(1 - R_\xi(U))V\|_{\mathbb{H}^s} \le C_s \xi^{2-s} \|A(U)V\|_{\mathbb{L}^2}, \quad V \in \mathcal{D}(A(U)), \tag{4.8}$$

C_s *being independent of* $U \in K$ *and* ξ. *In addition, for* $\frac{1}{2} < r \le 2$,

$$\|(1 - R_\xi(U))V\|_{\mathbb{L}^2} \le C_r \xi^r \|A(U)^{\frac{r}{2}} V\|_{\mathbb{L}^2}, \quad V \in \mathcal{D}(A(U)^{\frac{r}{2}}), \tag{4.9}$$

C_r *also being independent of* $U \in K$ *and* ξ.

Proof of the lemma. First we notice from Proposition 2.3 that the projection $P_\xi = \begin{bmatrix} p_\xi & 0 \\ 0 & p_\xi \end{bmatrix}$ satisfies

$$\|(1 - P_\xi)V\|_{\mathbb{H}^s} \le C_s \xi^{2-s} \|V\|_{\mathbb{H}^2}, \quad V \in \mathbb{H}^2(\Omega), \; 0 \le s < \frac{3}{2}. \tag{4.10}$$

$R_\xi(U)$ introduced by (3.40) is written in the form

$$R_\xi(U)V = \begin{bmatrix} R_{1\xi}v + \{-R_{1\xi}A_1^{-1}A_3(U) + A_{1\xi}^{-1}A_{3\xi}(U)R_{2\xi}\}\mu \\ R_{2\xi}\mu \end{bmatrix},$$

$$U \in K, \quad V = \begin{bmatrix} v \\ \mu \end{bmatrix} \in \mathcal{D}(A(U)),$$

where $R_{i\xi}$ are the Ritz projections with respect to A_i, $i = 1, 2$. Moreover $R_\xi(U)$ satisfies

$$\alpha(U; R_\xi(U)V, \widehat{F}) = \alpha(U; V, \widehat{F}), \quad V \in \mathcal{D}(A(U)), \ \widehat{F} \in X_\xi. \tag{4.11}$$

Then, it is verified in view of (3.8), (4.3) and (4.10) that

$$\begin{aligned} &\|(1 - R_\xi(U))V\|_{\mathbb{H}^1}^2 - \alpha_0^{-2}\|(1 - R_{2\xi})\mu\|_{H^1}^2 \\ &\quad \le \alpha_0^{-1} \operatorname{Re} \alpha\big(U; (1 - R_\xi(U))V, (1 - R_\xi(U))V\big) \\ &\quad = \alpha_0^{-1} \operatorname{Re} \alpha\big(U; (1 - R_\xi(U))V, (1 - P_\xi)A(U)^{-1}A(U)V\big) \\ &\quad \le C\xi\|(1 - R_\xi(U))V\|_{\mathbb{H}^1} \|A(U)V\|_{\mathbb{L}^2}. \end{aligned}$$

We have also from (2.31) that

$$\begin{aligned} \|(1 - R_{2\xi})\mu\|_{H^1}^2 &\le \|(1 - R_{2\xi})\mu\|_{H^1} \cdot C\xi\|A_2\mu\|_{L^2} \\ &\le C\xi\|(1 - R_\xi(U))V\|_{\mathbb{H}^1} \|A(U)V\|_{\mathbb{L}^2}. \end{aligned}$$

Therefore (4.8) is proved for $s = 1$. For $s = 0$, we use Aubin-Nitsche's trick [5, Theorem 3.2.5]. Indeed, in view of (4.3) and (4.11),

$$\begin{aligned} &\|(1 - R_\xi(U))V\|_{\mathbb{L}^2}^2 \\ &\quad = \alpha(U; (1 - R_\xi(U))V, A(U)^{*-1}(1 - R_\xi(U))V) \\ &\quad = \alpha(U; (1 - R_\xi(U))V, (1 - R_\xi(U))A(U)^{*-1}(1 - R_\xi(U))V) \\ &\quad \le C\|(1 - R_\xi(U))V\|_{\mathbb{H}^1} \|(1 - R_\xi(U))A(U)^{*-1}(1 - R_\xi(U))V)\|_{\mathbb{H}^1} \\ &\quad \le C\xi^2\|A(U)V\|_{\mathbb{L}^2}\|(1 - R_\xi(U))V\|_{\mathbb{L}^2}. \end{aligned}$$

Now we can prove (4.8) for general $0 \le s < \frac{3}{2}$. Indeed, from (2.24) and (4.10),

$$\begin{aligned} \|(1 - R_\xi(U))V\|_{\mathbb{H}^s} &\le \|(1 - P_\xi)V\|_{\mathbb{H}^s} + \|(P_\xi - R_\xi(U))V\|_{\mathbb{H}^s} \\ &\le C_s\xi^{2-s}\|V\|_{\mathbb{H}^2} + C_s\xi^{-s}\|(P_\xi - R_\xi(U))V\|_{\mathbb{L}^2} \\ &\le C_s\xi^{2-s}\|A(U)V\|_{\mathbb{L}^2}. \end{aligned}$$

In order to prove (4.9) we use the Ritz projection $\widetilde{R}_\xi(U) = A_\xi(U)^{*-1}P_\xi A(U)^*$ with respect to the adjoint operator $A(U)^*$. Clearly,

$$\alpha(U;\widehat{F},\widetilde{R}_\xi(U)V) = \alpha(U;\widehat{F},V), \quad V \in \mathcal{D}(A(U)^*),\ \widehat{F} \in X_\xi.$$

By the same argument as above, $\widetilde{R}_\xi(U)$ is shown to satisfy (4.8) for $0 \le s < \frac{3}{2}$. Let $\frac{1}{2} < r \le 2$ and $V \in \mathcal{D}(A(U))$. Then, in view of (4.5) and (4.8),

$$\begin{aligned}
&\|(1-R_\xi(U))V\|_{\mathbb{L}^2}^2 \\
&= \alpha(U;(1-R_\xi(U))V, A(U)^{*-1}(1-R_\xi(U))V) \\
&= \alpha(U;(1-R_\xi(U))V, (1-\widetilde{R}_\xi(U))A(U)^{*-1}(1-R_\xi(U))V) \\
&= \alpha(U;V, (1-\widetilde{R}_\xi(U))A(U)^{*-1}(1-R_\xi(U))V) \\
&= \left\langle A(U)^{\frac{r}{2}}V, A(U)^{*1-\frac{r}{2}}(1-\widetilde{R}_\xi(U))A(U)^{*-1}(1-R_\xi(U))V\right\rangle_{\mathbb{L}^2} \\
&\le \|A(U)^{\frac{r}{2}}V\|_{\mathbb{L}^2}\|A(U)^{*1-\frac{r}{2}}(1-\widetilde{R}_\xi(U))A(U)^{*-1}(1-R_\xi(U))V\|_{\mathbb{L}^2} \\
&\le \|A(U)^{\frac{r}{2}}V\|_{\mathbb{L}^2}\cdot C_r\xi^r\|(1-R_\xi(U))V\|_{\mathbb{L}^2}.
\end{aligned}$$

Since $\mathcal{D}(A(U))$ is dense in $\mathcal{D}(A(U)^{\frac{r}{2}})$, (4.9) is proved for $\frac{1}{2} < r \le 2$. □

Proof of Proposition 3.10. The first estimate is nothing more than (4.9) with $r = 2$. The second one is proved as follows (cf. [23]). Let $U, V \in K$, $W \in \mathcal{D}(A(U))$ and $\widehat{F} = (R_\xi(U) - R_\xi(V))W$. Then,

$$\|\widehat{F}\|_{\mathbb{L}^2}^2 = \langle(1-R_\xi(V))W,\widehat{F}\rangle_{\mathbb{L}^2} - \langle(1-R_\xi(U))W,\widehat{F}\rangle_{\mathbb{L}^2},$$

and

$$\begin{aligned}
\langle(1-R_\xi(V))W,\widehat{F}\rangle_{\mathbb{L}^2} &= \alpha\big(V;(1-R_\xi(V))W, A(V)^{*-1}\widehat{F}\big) \\
&= \alpha\big(V;(1-R_\xi(V))W, (1-\widetilde{R}_\xi(U))A(V)^{*-1}\widehat{F}\big), \\
\langle(1-R_\xi(U))W,\widehat{F}\rangle_{\mathbb{L}^2} &= \alpha\big(U;(1-R_\xi(U))W, A(U)^{*-1}\widehat{F}\big) \\
&= \alpha\big(U;(1-R_\xi(U))W, (1-\widetilde{R}_\xi(U))A(U)^{*-1}\widehat{F}\big) \\
&= \alpha\big(U;(1-R_\xi(V))W, (1-\widetilde{R}_\xi(U))A(U)^{*-1}\widehat{F}\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\widehat{F}\|_{\mathbb{L}^2}^2 &= \alpha\big(V;(1-R_\xi(V))W, (1-\widetilde{R}_\xi(U))\{A(V)^{*-1} - A(U)^{*-1}\}\widehat{F}\big) \\
&\quad + \{\alpha(V;\cdot,\cdot) - \alpha(U;\cdot,\cdot)\}\big((1-R_\xi(V))W, (1-\widetilde{R}_\xi(U))A(U)^{*-1}\widehat{F}\big).
\end{aligned}$$

Using (3.9) and Proposition 3.8, we conclude that

$$\|\widehat{F}\|_{\mathbb{L}^2}^2 \le C\xi^2 \|U-V\|_{\mathbb{H}^{1+\varepsilon}} \|A(V)W\|_{\mathbb{L}^2} \|\widehat{F}\|_{\mathbb{L}^2},$$

and hence the desired estimate is obtained. □

Proof of Proposition 3.4. It is clear that $\|P_\xi V\|_{\mathbb{L}^2} \le \|V\|_{\mathbb{L}^2}$, $V \in \mathbb{L}^2(\Omega)$; hence, $\|P_\xi\|_{\mathcal{L}(X)} \le 1$. By definition,

$$\|A_\xi(U)R_\xi(U)V\|_{\mathbb{L}^2} \le C\|A(U)V\|_{\mathbb{L}^2}, \quad V \in \mathcal{D}(A(U)).$$

Then, from (4.1), (4.3) and (4.8),

$$\begin{aligned}\|A_\xi(U)P_\xi V\|_{\mathbb{L}^2} &\le \|A_\xi(U)R_\xi(U)V\|_{\mathbb{L}^2} + \|A_\xi(U)(P_\xi - R_\xi(U))V\|_{\mathbb{L}^2}\\ &\le C\|A(U)V\|_{\mathbb{L}^2} + C\xi^{-2}\|(P_\xi - R_\xi(U))V\|_{\mathbb{L}^2}\\ &\le C\|A(U)V\|_{\mathbb{L}^2}, \quad V \in \mathcal{D}(A(U)).\end{aligned}$$

Moreover, we can verify that, for $0 \le \theta \le 1$,

$$\|A_\xi(U)^\theta P_\xi V\|_{\mathbb{L}^2} \le C_\theta \|A(U)^\theta V\|_{\mathbb{L}^2}, \quad V \in \mathcal{D}(A(U)^\theta), \tag{4.12}$$

and, for $\frac{1}{4} < \theta \le 1$,

$$\|A_\xi(U)^\theta R_\xi(U) V\|_{\mathbb{L}^2} \le C_\theta \|A(U)^\theta V\|_{\mathbb{L}^2}, \quad V \in \mathcal{D}(A(U)^\theta), \tag{4.13}$$

C_θ being independent of ξ and $U \in K$. Indeed, we easily see that

$$\|A_\xi(U)A_\xi(U)^{*-1}\|_{\mathcal{L}(X_\xi)},\ \|A_\xi(U)^* A_\xi(U)^{-1}\|_{\mathcal{L}(X_\xi)} \le C \tag{4.14}$$

uniformly in ξ and U, since $\|A_{3\xi}(U)A_{2\xi}^{-1}\|_{\mathcal{C}_\xi(\overline{\Omega})}$ and $\|A_{3\xi}(U)^* A_{1\xi}^{-1}\|_{\mathcal{C}_\xi(\overline{\Omega})}$ are bounded uniformly in ξ and U. Then, as mentioned in Section 2,

$$\|A_\xi(U)^{it}\|_{\mathcal{L}(X_\xi)} \le \hat{\mu}_A e^{\pi|t|}, \quad t \in \mathbb{R},\ U \in K \tag{4.15}$$

with some constant $\hat{\mu}_A > 0$ independent of ξ and U. Therefore, in view of (4.4) and (4.15), we can apply (2.11) with $T = P_\xi$, $A = A(U)$ and $B = A_\xi(U)$ to verify (4.12) for $0 \le \theta \le 1$. From (4.1), (4.9) and (4.12) it is seen that, for $\frac{1}{4} < \theta < 1$,

$$\begin{aligned}\|A_\xi(U)^\theta R_\xi(U)V\|_{\mathbb{L}^2} &\le \|A_\xi(U)^\theta P_\xi V\|_{\mathbb{L}^2} + \|A_\xi(U)^\theta (P_\xi - R_\xi(U))V\|_{\mathbb{L}^2}\\ &\le C_\theta\|A(U)^\theta V\|_{\mathbb{L}^2} + C_\theta \xi^{-2\theta}\|(P_\xi - R_\xi(U))V\|_{\mathbb{L}^2}\\ &\le C_\theta\|A(U)^\theta V\|_{\mathbb{L}^2}, \quad V \in \mathcal{D}(A(U)^\theta),\end{aligned}$$

so that we prove (4.13). □

Proof of Proposition 3.3. By (4.5), $\|\cdot\|_Z \le C\|A(U)^{\hat{\alpha}}\cdot\|_X$ uniformly in $U \in K$. Therefore the assertion of proposition is an immediate consequence of the following lemma. □

Lemma 4.3 *For* $0 \le \theta < \frac{3}{4}$,

$$C_\theta^{-1}\|A(U)^\theta \widehat{F}\|_{\mathbb{L}^2} \le \|A_\xi(U)^\theta \widehat{F}\|_{\mathbb{L}^2} \le C_\theta\|A(U)^\theta \widehat{F}\|_{\mathbb{L}^2}, \quad \widehat{F} \in X_\xi, \tag{4.16}$$

C_θ *being independent of* ξ *and* $U \in K$.

Proof of the lemma. The second inequality has already been proved by (4.12). On the other hand,

$$\begin{aligned}\langle A(U)^\theta \widehat{F}, V\rangle_{\mathbb{L}^2} &= a(U; \widehat{F}, \widetilde{R}_\xi(U)A(U)^{*\theta-1}V) \\ &= \langle A_\xi(U)^\theta \widehat{F}, A_\xi(U)^{*1-\theta}\widetilde{R}_\xi(U)A(U)^{*\theta-1}V\rangle_{\mathbb{L}^2} \\ &\le \|A_\xi(U)^\theta \widehat{F}\|_{\mathbb{L}^2}\|A_\xi(U)^{*1-\theta}\widetilde{R}_\xi(U)A(U)^{*\theta-1}V\|_{\mathbb{L}^2} \\ &\le C_\theta\|A_\xi(U)^\theta \widehat{F}\|_{\mathbb{L}^2}\|V\|_{\mathbb{L}^2}, \quad V \in \mathbb{L}^2(\Omega).\end{aligned}$$

Here we used the dual estimate

$$\|A_\xi(U)^{*\theta}\widetilde{R}_\xi(U)V\|_{\mathbb{L}^2} \le C_\theta\|A(U)^{*\theta}V\|_{\mathbb{L}^2}, \quad V \in \mathcal{D}(A(U)^{*\theta}),$$

for $\frac{1}{4} < \theta \le 1$, of (4.13) which is verified by an analogous argument. Hence (4.16) is verified. □

5. Full discretization for quasilinear abstract evolution equations

Let us consider the Cauchy problem of a quasilinear equation

$$\text{(E)} \qquad \begin{cases} \dfrac{dU}{dt} + A(U)U = F(U), & 0 < t \le T, \\ U(0) = U_0 \end{cases}$$

in a Banach space X. Here, $A(U)$ are densely defined, closed linear operators acting in X defined for all $U \in K = \{U \in Z; \|U - U_0\|_Z \le r\}$, $r > 0$, Z being another Banach space continuously embedded in X. $F(U)$ is a function from K to X. $U_0 \in Z$ is an initial value. $U = U(t)$ is the unknown function.

We make the following assumptions:

(A_U1) $\rho(A(U))$, $U \in K$, contain $\mathbb{C} \setminus \overline{S_\varphi}$, $0 < \varphi < \frac{\pi}{2}$, and the resolvents satisfy

$$\|(\lambda - A(U))^{-1}\|_{\mathcal{L}(X)} \le \frac{M_A}{|\lambda| + 1}, \quad \lambda \notin \overline{S_\varphi},\ U \in K$$

with some constant M_A.

(A_U2) The domains $\mathcal{D}(A(U)) \equiv \mathcal{D}$ are constant, and $A(\cdot)$ satisfies

$$\|\{A(U) - A(V)\}A(V)^{-1}\|_{\mathcal{L}(X)} \le L_A\|U - V\|_Z, \quad U, V \in K$$

with some constant L_A.

(F_U) $F(\cdot)$ satisfies

$$\|F(U) - F(V)\|_X \le L_F\|U - V\|_Z, \quad U, V \in K$$

with some constant L_F.

(Sp) For some $0 < \alpha < 1$, $\mathcal{D}(A(U_0)^\alpha) \subset Z$ and $\|\cdot\|_Z \le D\|A(U_0)^\alpha \cdot\|_X$ with some constant D.

(In) U_0 is in $\mathcal{D}(A(U_0))$.

Under the assumptions (A_U1–2), (F_U), (Sp) and (In), the equation (E) possesses a unique local solution

$$U \in \mathcal{C}^\eta([0, S]; Z) \cap \mathcal{C}^1([0, S]; X) \cap \mathcal{C}([0, S]; \mathcal{D}), \quad 0 < S \le T,$$

where η is any exponent such that $0 < \eta < 1 - \alpha$, see [30].

Let $\{X_\xi\}_{\xi>0}$ be a family of finite dimensional subspaces of X such that $X_\xi \subset Z$. Denote by Z_ξ the space X_ξ equipped with the induced norm of Z. For $\xi > 0$, $P_\xi : X \to X_\xi$ is a projection operator; and, as $\xi \to 0$, $P_\xi \to I$ strongly on X. Let $A_\xi(U)$, $U \in K$, be an approximate operator of $A(U)$ such that $A_\xi(U)$ is a bounded linear operator on X_ξ. Then the approximate equation in X_ξ is given by

$$(\mathrm{E}_\xi) \qquad \begin{cases} \dfrac{d\widehat{U}}{dt} + A_\xi(\widehat{U})\widehat{U} = F_\xi(\widehat{U}), & 0 < t \le T, \\ \widehat{U}(0) = P_\xi U_0, \end{cases}$$

where $F_\xi(U) = P_\xi F(U)$, $U \in K$.

On (E_ξ) we assume the following conditions:

($\mathrm{A}_{U\xi}1$) $\rho(A_\xi(U))$, $U \in K$, contain $\mathbb{C} \setminus \overline{S_{\hat\varphi}}$, $0 < \hat\varphi < \frac{\pi}{2}$, and the resolvents satisfy

$$\|(\lambda - A_\xi(U))^{-1}\|_{\mathcal{L}(X_\xi)} \le \frac{\hat M_A}{|\lambda| + 1}, \quad \lambda \notin \overline{S_{\hat\varphi}},\ U \in K$$

with some constant $\hat M_A$ independent of ξ.

($\mathrm{A}_{U\xi}2$) $A_\xi(\cdot)$ satisfies

$$\|\{A_\xi(U) - A_\xi(V)\}A_\xi(V)^{-1}\|_{\mathcal{L}(X_\xi)} \le \hat L_A \|U - V\|_Z, \quad U, V \in K,$$
$$\|A_\xi(U)^{-1}\{A_\xi(U) - A_\xi(V)\}\|_{\mathcal{L}(X_\xi)} \le \hat L_A \|U - V\|_Z, \quad U, V \in K$$

with some constant $\hat L_A$ independent of ξ.

($\mathrm{A}_{U\xi}3$) The norms of $A_\xi(U)$ are estimated by

$$\|A_\xi(U)\|_{\mathcal{L}(X_\xi)} \le \hat N_A Q_\xi^{-1}, \quad U \in K$$

with some constant $\hat N_A$ independent of ξ, where Q_ξ denotes a function of ξ such that $Q_\xi \to 0$ as $\xi \to 0$.

($\mathrm{A}_{U\xi}4$) The operators $R_\xi(U) = A_\xi(U)^{-1} P_\xi A(U)$, $U \in K$, satisfy

$$\|\{1 - R_\xi(U)\}A(U)^{-1}\|_{\mathcal{L}(X)} \le \hat M_R Q_\xi, \quad U \in K,$$
$$\|\{R_\xi(U) - R_\xi(V)\}A(V)^{-1}\|_{\mathcal{L}(X)} \le \hat L_R Q_\xi \|U - V\|_Z, \quad U, V \in K$$

with some constants $\hat M_R$ and $\hat L_R$ independent of ξ.

(In_ξ) There exists $\xi_0 > 0$ such that $P_\xi U_0$ is in K for all $0 < \xi < \xi_0$.

($\mathrm{Sp}_\xi 1$) The norms $\|P_\xi\|_{\mathcal{L}(X)}$ and $\|A_\xi(P_\xi U_0) P_\xi A(U_0)^{-1}\|_{\mathcal{L}(X)}$ are bounded uniformly in ξ.

($\mathrm{Sp}_\xi 2$) For some $\hat\alpha \in [\alpha, 1)$, $\|\cdot\|_{Z_\xi} \le \hat D \|A_\xi(P_\xi U_0)^{\hat\alpha} \cdot\|_{X_\xi}$ with some constant $\hat D$ independent of ξ.

Utilizing an s-stage Runge-Kutta scheme $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ with the stepsize $h > 0$, we obtain the fully discrete approximation of (E)

$$(\mathrm{E}_{\xi,h}) \qquad \begin{cases} \widehat U_{n+1} = \widehat U_n + h\boldsymbol{e}^T \mathcal{B}\Big\{-\boldsymbol{A}_\xi(\widehat{\boldsymbol{V}}_n)\widehat{\boldsymbol{V}}_n + \boldsymbol{F}_\xi(\widehat{\boldsymbol{V}}_n)\Big\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad n = 0, 1, \dots, N-1, \\ \widehat{\boldsymbol{V}}_n = \boldsymbol{e}\widehat U_n + h\mathcal{A}\Big\{-\boldsymbol{A}_\xi(\widehat{\boldsymbol{V}}_n)\widehat{\boldsymbol{V}}_n + \boldsymbol{F}_\xi(\widehat{\boldsymbol{V}}_n)\Big\}, \\ \widehat U_0 = P_\xi U_0, \end{cases}$$

where $\widehat{\boldsymbol{V}}_n = [\widehat{V}_{n,1}, \dots, \widehat{V}_{n,s}]^T$, $\boldsymbol{A}_\xi(\boldsymbol{V}) = \mathrm{diag}[A_\xi(V_1), \dots, A_\xi(V_s)]$ and $\boldsymbol{F}_\xi(\boldsymbol{V}) = [F_\xi(V_1), \dots, F_\xi(V_s)]^T$ for $\boldsymbol{V} = [V_1, \dots, V_s]^T$, and N is a positive integer such that $Nh \le T$.

$(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is assumed to satisfy (RK1–2) in Section 2 with $\theta > \hat{\varphi}$.

Our first result is the existence and uniqueness of solution to $(\mathrm{E}_{\xi,h})$ and the stability of the approximate solution.

Theorem 5.1 *Assume* $(\mathrm{A}_{U\xi}1\text{–}2)$, (In_ξ), $(\mathrm{Sp}_\xi 1\text{–}2)$, (F_U) *and* (RK1) *with* $\theta > \hat{\varphi}$. *Let* $h_0 > 0$, $\xi_0 > 0$ *and* $S \in (0, T]$ *be sufficiently small. Then, for any* $0 < h < h_0$ *and* $0 < \xi < \xi_0$, $(\mathrm{E}_{\xi,h})$ *possesses a unique solution* $\widehat{\mathcal{U}} = [\widehat{U}_0, \dots, \widehat{U}_N, \widehat{\boldsymbol{V}}_0, \dots, \widehat{\boldsymbol{V}}_{N-1}]$ *on the subinterval* $[0, S]$, *where* $N \le S/h$. *Moreover,* $\widehat{\mathcal{U}}$ *satisfies*

$$\begin{cases} \max\limits_{n=0,1,\dots,N} \left\{ \|\widehat{U}_n\|_{X_\xi} + \|A_\xi(\widehat{U}_n)\widehat{U}_n\|_{X_\xi} \right\} \le C, \\ \max\limits_{n=0,1,\dots,N-1} \left\{ \|\widehat{\boldsymbol{V}}_n\|_{X_\xi^s} + \|\boldsymbol{A}_\xi(\widehat{\boldsymbol{V}}_n)\widehat{\boldsymbol{V}}_n\|_{X_\xi^s} \right\} \le C, \end{cases}$$

$C > 0$ *being independent of* h *and* ξ.

Proof. We remember that the proof of Theorem 3.1 was carried out straightforward under Propositions 3.1–3.7. This fact naturally shows us that all the conditions announced in those propositions can imply the same assertion as in Theorem 3.1. □

We next verify the error estimate.

Theorem 5.2 *Assume* $(\mathrm{A}_U 1\text{–}2)$, (F_U), (Sp), (In), $(\mathrm{A}_{U\xi}1\text{–}4)$, (In_ξ), $(\mathrm{Sp}_\xi 1\text{–}2)$ *and* (RK1–2) *with* $\theta > \hat{\varphi}$. *Let* U *be a solution to* (E) *such that* $U \in \mathcal{C}^{p+1}([0, S]; X) \cap \mathcal{C}^{q'+1}([0, S]; \mathcal{D})$ *with* $q' = \min\{q, p-1\}$. *Let* $h_0 > 0$, $\xi_0 > 0$ *and* $S \in (0, T]$ *be sufficiently small. Then, for any* $0 < h < h_0$ *and* $0 < \xi < \xi_0$, *the errors are estimated by*

$$\begin{aligned} &\max_{n=0,1,\dots,N} \|\widehat{U}_n - U(t_n)\|_Z \\ &\quad \le C_U \Big[\|(1 - P_\xi)U\|_{\mathcal{C}([0,S];Z)} + \sigma^{-1} Q_\xi^{1-\hat{\alpha}} \|U\|_{\mathcal{C}^\sigma([0,S];\mathcal{D})} \\ &\qquad\qquad + h^p \|U^{(p+1)}\|_{\mathcal{C}([0,S];X)} + h^{q'+1} \|U^{(q'+1)}\|_{\mathcal{C}([0,S];\mathcal{D})} \Big], \end{aligned}$$

where $N \le S/h$. *Here,* $\sigma > 0$ *is any exponent and the constant* $C_U > 0$ *depends on* $\|U'\|_{\mathcal{C}([0,S];X)}$ *and* $\|U\|_{\mathcal{C}([0,S];\mathcal{D})}$ *but is independent of* h *and* ξ.

Proof. Similarly to the proof of Theorem 5.1, we notice the fact that Theorem 3.2 was proved straightforward from Propositions 3.1–3.10. □

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Etsushi Nakaguchi
Department of Applied Physics
Graduate School of Engineering
Osaka University, 2-1 Yamadaoka
Suita, Osaka 565-0871, Japan
E-mail: nakaguti@ap.eng.osaka-u.ac.jp

Atsushi Yagi
Department of Applied Physics
Graduate School of Engineering
Osaka University, 2-1 Yamadaoka
Suita, Osaka 565-0871, Japan
E-mail: yagi@ap.eng.osaka-u.ac.jp